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ON THE EXCHANGE PROPERTY IN A DIRECTSUM OF INDECOMPOSABLE MODULES

Dedicated to Professor Kiiti Morita on his 60th birthday

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Throughout R will denote a ring with identity and every R-modules considered in this note are unitary R-modules. Let M be an R-module. If $\mathrm{End}_R(M)$ is a local ring, we call M a *completely indecomposable module*. We take a set of completely indecomposable modules $\{M_\alpha\}_I$ and put $M=\sum_I \oplus M_\alpha$. Then we know several properties of M with respect to this decomposition. For instance, let $M=\sum_J \oplus N_\beta$ be another decomposition and I' a finite subset of I, then $M=\sum_{I'} \oplus M_{\alpha'} \oplus \sum_{J-\varphi(I')} \oplus N_\beta$, where $\varphi\colon I' \to J$ is a one-to-one into mapping [1]. H. Kanbara [8] shows that the above fact is true for any subset I' of I if and only if $\{M_\alpha\}_I$ is a locally semi-T-nilpotent (see the definition below).

In this note, we fix a subset I' (not necessarily finite) and give criteria for $\sum_{I'} \oplus M_{\alpha'}$ to satisfy the above property. If $\{M_{\alpha'}\}_{I'}$ is locally semi-T-nilpotent, $\sum_{I'} \oplus M_{\alpha'}$ satisfies it, however the converse is not true [4]. When we fix the subset I', the above property does depend not only on $\sum_{I'} \oplus M_{\alpha'}$ but also on $\sum_{I-I'} \oplus M_{\alpha''}$. On the other hand, the concept of semi-T-nilpotency of $\{M_{\alpha'}\}_{I'}$ does depend only on $\sum_{I'} \oplus M_{\alpha'}$. Hence, we shall define a new concept in this note, namely relative semi-T-nilpotency (see the definition below) and give a relation between relative semi-T-nilpotency and the property above.

In the final saction (Appendix), we shall generalize [6], Lemma 5 as Theorem A.1 by virtue of K. Yamagata's idea [12], [13] and [14] (Lemma A.1). That theorem gives the complete proof of [5], Lemma 2 (Corollary 2) and a generalization of [14], Theorem (Theorem A.2).

1. Definition

Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules. We shall recall definitions of locally semi-T-nilpotency and the induced category $\mathfrak{A}$ from

$\{M_\alpha\}_I$. Let $\{M_{\alpha_i}\}_1^\infty$ be a countable subset of $\{M_\alpha\}_I$ and $\{f_i: M_{\alpha_i}\to M_{\alpha_{i+1}}\}$ a set of non-isomorphisms. If for any m in M_{α_1} there exists a natural number n, which depends on m, such that $f_n f_{n-1}\cdots f_1(m)=0$, then the set $\{f_i\}$ is called *locally T-nilpotent*. If every sets $\{f_i\}$ of non-isomorphisms for every countable subsets $\{M_{\alpha_i}\}$ of $\{M_\alpha\}_I$ are locally T-nilpotent, then the set $\{M_\alpha\}_I$ is called *locally semi-T-nilpotent* (cf. [3] and [4]). In the definition above, if we allow that $\alpha_i=\alpha_j$ for some $i\neq i$, we call $\{M_\alpha\}_I$ *locally T-nilpotent*. We shall generalize this concept as follows: Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be sets of completely indecomposable modules. We take countable subsets $\{M_{\alpha_i}\}_1^\infty$, $\{N_{\beta_j}\}_1^\infty$ of $\{M_\alpha\}_I$ and $\{N_\beta\}_J$, respectively. We take sets $\{f_i\}$ and $\{g_i\}$ of non-isomorphisms, where $f_i: M_{\alpha_i}\to N_{\beta_i}$ and $g_i: N_{\beta_i}\to M_{\alpha_{i+1}}$. If for any element m in M_{α_1} there exists n, which depends on m, such that $g_n f_n\cdots g_1 f_1(m)=0$, then we call $\{f_i, g_i\}$ *(locally and) relatively T-nilpotent.* If for every countable subsets $\{M_{\alpha_i}\}_1^\infty$ and $\{N_{\beta_j}\}_1^\infty$, every sets $\{f_i, g_i\}$ of non-isomorphisms are relatively T-nilpotent, then we call $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ *(locally and) relatively semi-T-nilpotent* (see Remark 3 in §4). We note that if $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are relatively semi-T-nilpotent, there exists n' for any element x in N_{β_1}, such that $f_{n'+1}g_{n'}f_{n'}\cdots f_2 g_1(x)=0$. If we allow in the above that $\alpha_i=\alpha_{i'}$ $(\beta_j=\beta_{j'})$ for some $i\neq i'$ $(j\neq i')$, we call $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ *relatively T-nilpotent.* It is clear that if either $\{M_\alpha\}_I$ or $\{N_\beta\}_J$ is locally semi-T-nilpotent, then $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are relatively semi-T-nilpotent, however the converse is not ture. If *either I or J is finite,* we assume as a definition that $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are *relatively semi-T-nilpotent,* ($\{M_\alpha\}_I$ or $\{N_\beta\}_J$ is *locally semi-T-nilpotent*).

Let $\{M_\alpha\}_I$ be as before and $\mathfrak{M}_R$ the category of right R-modules. Let $\mathfrak{A}$ be the full sub additive category of $\mathfrak{M}_R$, whose objects consist of all modules which are isomorphic to directsums of some modules in $\{M_\alpha\}_I$. We define the ideal $\mathfrak{J}'$ in $\mathfrak{A}$ as follows: $\mathfrak{J}'\cap[A, B]=\{f\in\mathrm{Hom}_R(A, B)\,|\,p_\beta f i_\alpha$ are non-isomorphisms for all α and $\beta\}$, where $A\approx\sum_{I'}\oplus M_{\alpha'}\in\mathfrak{A}$, $B\approx\sum_{J'}\oplus M_{\beta'}\in\mathfrak{A}$ and $i_{\alpha'}: M_{\alpha'}\to A$ injection, $p_{\beta'}: B\to M_{\beta'}$ projection. By $\mathfrak{A}/\mathfrak{J}'$ we denote the factor category of $\mathfrak{A}$ with respect to $\mathfrak{J}'$. Let A be in $\mathfrak{A}$ and $A=\sum_L\oplus A_\alpha$(A_α are not necessarily in $\mathfrak{A}$). Then there exist submodules A_α' in $\mathfrak{A}$ of A_α for each α such that $\sum_L\oplus A_\alpha'=A$ in $\mathfrak{A}/\mathfrak{J}'$ and those A_α' are unique up to isomorphism. We call those A_α' *dense submodules* of A_α (see [4], [5] and [6] for detail).

2. Main theorem

We recall Krull-Remak-Schmidt-Azumaya theorem. Let $\{M_\alpha\}_I$ be as in §1 and $M=\sum_I\oplus M_\alpha$. We take any other decomposition of M by completely indecomposable modules N_β: $M=\sum_J\oplus N_\beta$. In the theorem above we consider the following two properties for M:

P,1 For a direct summand $\sum_{I'} \oplus M_{\alpha'}$ *of* M $(I' \subseteq I)$, *there exists a one-to-one mapping* φ *of* I' *into* I *such that* $M_{\alpha'} \approx N_{\varphi(\alpha')}$ *for all* $\alpha \in I'$ *and* $M = \sum_{I'} \oplus N_{\varphi(\alpha')} \oplus \sum_{I-I'} \oplus M_{\alpha''}$.

P,2 For a dirsect summand $\sum_{I'} \oplus M_{\alpha'}$ *of* M $(I' \subseteq I)$, *there exists a one-to-one mapping* ψ *of* I' *into* J *such that* $M = \sum_{I'} \oplus M_{\alpha'} \oplus \sum_{J-\psi(I')} \oplus N_{\beta''}$.

Those two properties are general cases of the exchange property in M which is defined in [4] as follows: Let N be a direct summand of M and $M = \sum_{K} \oplus L_{\varepsilon}$ any decomposition of M (L_{ε} are not necessarily indecomposanble). If $M = N \oplus \sum_{K} \oplus L_{\varepsilon}'$ for any decomposition above, where $L_{\varepsilon}' \subseteq L_{\varepsilon}$ for all $\varepsilon \in K$, we say N has *the exchange property in M.* If N has the above property only in case *all* L_{ε} *are indecomposable,* we say N has *the exchange property in M with respect to indecomposable modules* (briefly w. r. t. inde. modules).

It is clear that if N has the exchange property in M, N is in $\mathfrak{A}$ and $(P,2)$ is equivalent to the exchange property in M w. r. t. inde. modules. We have already known that $(P,1)$ is true for any subset I' of I if and only if $\{M_{\alpha}\}_{I'}$ is locally semi-T-nilpotent [8]. Now we fix $\{M_{\alpha}\}_{I}$ and a subset I' of I and consider $(P,1)$ and $(P,2)$ for any decompositions of M. We shall show the following results and give proofs in the next section.

Theorem. *Let* $\{M_{\alpha}\}_{I}$ *be a set of completely indecomposable modules and* $M = \sum_{I} \oplus M_{\alpha}$. *Let* $M = S \oplus T$ *and* $S' = \sum_{I'} \oplus S_{\alpha'}$, $T' = \sum_{I''} \oplus T_{\alpha''}$ *dense submodules of T and S, respectively. Then the following statements are equivalent.*

1) *S has the exchange property in M w. r. t. inde. modules.*
2) *T has the exchange property in M w. r. t. inde. modules.*
3) $\{S_{\alpha'}\}_{I'}$ *and* $\{T_{\alpha''}\}_{I''}$ *are relatively semi-T-nilpotent,*

where $S_{\alpha'}$ *and* $T_{\alpha''}$ *are completely indecomposable modules.*

In those cases S and T are direct sums of completely indecomposable modules.

Corollary 1. *Let M, S and T be as in Theorem. Then S has the exchange property in M if and only if* 3) *in Theorem is satisfied and any direct summand L of M has the following decomposition:* $L = L_1 \oplus L_2$ *and a dense submodule* L_1' *(resp.* L_2'*) of* L_1 *(resp.* L_2*) is isomorphic to a summand of S (resp. T), (cf. Theorem A.2 in* §4).

Corollary 2. *Let* $M = \sum_{i=1}^{n} \oplus M^{(i)}$ *and* $M^{(i)}$ *in* $\mathfrak{A}$ *for all i. Then the following statements are equivalent.*

1) $M^{(1)}$ *has the exchange property in M w. r. t. inde. modules.*
2) $M^{(1)}$ *has the exchange property in* $M^{(1)} \oplus M^{(i)}$ *w. r. t. inde. modules for all i.*

If $M^{(i)}$ and $M^{(j)}$ have the exchange property in M w. r. t. inde. modules, then so does $M^{(i)}\oplus M^{(j)}$. Covnversely, if $M^{(i)}\oplus M^{(j)}$ has the exchange property in M, then $M^{(i)}$ has the exchange property in M if and only if so does $M^{(j)}$. (cf. [2], Lemma 3.11)

Corollary 3. *Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be sets of completely indecomposable modules and $\mathfrak{A}, \mathfrak{B}$ the induced categories from $\{M_\alpha\}_I$ and $\{N_\beta\}_J$, respectively. Then $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are relatively T-nilpotent if any onlf if for any modules M and N in $\mathfrak{A}, \mathfrak{B}$ respectively, M (resp. N) has the exchange property in $M\oplus N$ w. r. t. inde. modules.*

Corollary 4. *Let $\{S_{\alpha'}\}_{I'}$ (resp. $\{T_{\alpha''}\}_{I''}$) be a set of noetherian (resp. injective) and completely indecomposable modules. Then $\sum_{I'}\oplus S_{\alpha'}$ (and $\sum_{I''}\oplus T_{\alpha''}$) have the exchange property in $\sum_{I'}\oplus S_{\alpha'}\oplus\sum_{I''}\oplus T_{\alpha''}$ w. r. t. inde. modules.*

Corollary 5. *Let S, T and M be as in Theorem. If $Hom_R(S, T)=0$ or $Hom_R(T, S)=0$, then S and T have the exchange propety in M w. r. t. inde. modules.*

3. Proof of Theorem

Let $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ be as in §2. We shall rearrange them as follwows: $\{M_\alpha\}_I=\{M_{kj}\}_{k\in K, j\in I_k}$ and $\{N_\beta\}_J=\{N_{kj}\}_{k\in K, j\in J_k}$, where $M_{kj}\approx M_{kj'}\approx N_{kj}\approx N_{kj'}$ and $M_{kj}\not\approx M_{k'j'}$, $N_{kj}\not\approx N_{k'j'}$ if $k\neq k'$.

Lemma 1. *Let $\{M_{kj}\}_{k\in K, j\in I_k}$ and $\{N_{kj}\}_{k\in K, j\in J_k}$ be as above and K' a subset of K. We assume $\{M_{kj}\}$ and $\{N_{kj}\}$ are relatively semi-T-nilpotent. Then 1) if I_k and J_k are infinite for all $k\in K'$, then $\{M_{kj}\}_{K', I_k}$ is locally T-nilpotent. 2) If either $|I_k|\leqslant|J_k|$ or I_k is finite and $J_k\neq\phi$ for all $k\in K'$, then $\{M_{kj}\}_{K', I_k}$ is locally semi-T-nilpotent, where $|I|$ means the cardinal number of a set I.*

Proof. 1) and the first part of 2) are clear. We assume I_k is finite and $J_k\neq\phi$. Let $\{M_i\}_1^\infty$ be a countable subset of $\{M_{kj}\}_{K_1', I_k}$ and $\{f_i\colon M_i\to M_{i+1}\}$ a set of non-isomorphisms. We assume $M_1=M_{k_1j_1}$. Since I_{k_1} is finite, there exists n_2 such that $M_{n'}\not\approx M_{k_1j_1}$ for all $n'\geqslant n_2$. Next, we assume $M_{n_2}=M_{k_2j_2}$ $(k_1\neq k_2)$. Again there exists $n_3\geqslant n_2$ such that $M_{n'}\not\approx M_{k_2j_2}$ for all $n'\geqslant n_3$. Repeating those arguments, we obtain a subset $\{M_{n_i}\}$ of $\{M_i\}$, $(n_1=1)$ such that $M_{n_i}\not\approx M_{n_j}$ for all $i\neq i$ and a set $\{g_i=f_{n_{i+1}-1}f_{n_{i+1}-2}\cdots f_{n_i}\colon M_{n_i}\to M_{n_{i+1}}\}$. It is clear that no one of g_i is isomorphic by [3], Lemma 4. Furthermore, $M_{n_i}\approx N_{kj}$ for some k. Hence, $\{g_i\}$ is locally T-nilpotent from the assumption. Therefore, $\{f_i\}$ is locally T-nilpotent.

Lemma 2. *If $\{M_\alpha\}_I$ and $\{N_\beta\}_J$ are locally semi-T-nilpotent, then so is $\{M_\alpha, N_\beta\}_{I\cup J}$.*

Proof. Let $\{T_i\}_1^\infty$ be a countable subset of $\{M_\alpha, N_\beta\}_{I\cup J}$ and $\{f_i\colon T_i\to T_{i+1}\}$ a set of non-isomorphisms. We assume $\{T_i\}_1^\infty$ contains an infinite number of modules in $\{M_\alpha\}_I$; say $\{T_i\}\supseteq\{M_{\alpha_i}\}_1^\infty$. Then $\{g_i=f_{\alpha_{i+1}-1}f_{\alpha_{i+1}-2}\cdots f_{\alpha_i}\}$ is locally T-nilpotent and hence so is $\{f_i\}$. If $\{T_i\}$ contains only a finite number of modules in $\{M_\alpha\}_I$, then there exists n such that $\{T_i\}_{i\geqslant n}\subseteq\{N_\beta\}_J$. Therefore, $\{f_i\}$ is also locally T-nilpotent.

Lemma 3 ([5], Theorem 2). *Let M be as before and N_1 a direct summand of M: $M=N_1\oplus N_2$. We assume $N_1=\sum_{I'}\oplus N_\alpha'$ and $\{N_\alpha'\}_{I'}$, a set of completely indecomposable modules, is locally semi-T-nilpotent. Then N_i has the exchange property in M for $i=1,2$.*[1)]

Lemma 4. *Let $\{M_\alpha\}_I$ and M be as before. Let $M=T\oplus S$ and N a direct summand of M such that $N=\sum_J\oplus N_\beta$ and every N_β is completely indecomposable and not isomorphic to any summand of S. If $\{N_\beta\}_J$ is locally semi-T-nilpotent, $M=N\oplus T'\oplus S$, where $T'\subseteq T$.*

Proof. We have $M=N\oplus T'\oplus S'$, $T=T'\oplus T''$ and $S=S'\oplus S''$ by the assumption and Lemma 3. Since $T''\oplus S''\approx N$, $S''=(0)$.

Lemma 5. *Let $M=M_1\oplus M_2\oplus N_1\oplus N_2$ and $M_i=\sum_{K_i\ni\alpha_i}\sum_{I\alpha_i}\oplus M_{\alpha_ij}$, $N_i=\sum_{L_i\ni\beta_i}\sum_{J_{\beta_i}}\oplus N_{\beta_ik}$; the M_{α_ij}, N_{β_ik} are completely indecomposable. We assume $M_{\alpha_1j}\not\approx M_{\alpha_2j'}$ and $M_{\alpha_1j}\not\approx N_{\alpha_2j''}$, $N_{\beta_1j}\not\approx N_{\beta_2j'}$ and $N_{\beta_1j}\not\approx M_{\beta_2j''}$ for any j, j' and j''. Let $M=T\oplus S$ and T', S' dense submodules of T and S, respectively. If $\{M_{\alpha_1}\}_{K_1, I\alpha_1}$ is locally semi-T-nilpotent and $T'\approx M_1\oplus M_2$, $S'\approx N_1\oplus N_2$, then we have the following decompositions:*

$$M=(M_1\oplus T_1'')\oplus M_2\oplus N_1'\oplus N_2=(M_1\oplus T_1'')\oplus T''\oplus S'',$$

where 1) $T=T_1'\oplus T''$, $T_1'=T_1''\oplus T_1'''$ $(T=T_1''\oplus T''\oplus T_1''')$, 2) $S=S''\oplus S'''$, 3) $N_1=N_1'\oplus N_1''$, 4) T_1'', T_1''', S''' *and* N_1' *are in* $\mathfrak{A}$ *and* 5) T'' (*resp.* S'') *contains a dense submodule which is isomorphic to* M_2 (*resp.* $N_1'\oplus N_2'$; N_2' *is in* $\mathfrak{A}$ *and is a summand of* N_2).

Proof. Since $T'\approx M_1\oplus M_2$ and $\{M_{\alpha_1}\}_{K_1, I\alpha_1}$ is semi-T-nilpotent, $T'=T_1'\oplus$

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1) Correction to the proof of [5], Theorem 2.
Replace the following words on the left side by ones on the right.
"p" on 4, 5 and 6 th lines (from the bottom)$\Rightarrow p_\beta$. "$\sum\oplus T_\beta'''$" on 5th line$\Rightarrow T_\beta''$. "$\mathrm{Ker}\,\bar{p}=\sum\oplus T_\beta'$" on 4th line$\Rightarrow\mathrm{Ker}\,\bar{p}_\beta=\bar{T}_\beta'\oplus\sum_{\gamma\neq\beta}\oplus\bar{T}_\gamma$. "$L=\sum\oplus T_\beta^*$ and $\bar{L}=\mathrm{Ker}\,\bar{p}$" on 3rd line$\Rightarrow\bar{T}_\beta^*\subseteq\mathrm{Ker}\,\bar{p}_\beta\cap\bar{T}_\beta=\bar{T}_\beta'$. "Therefore" on 2nd line$\Rightarrow$Finally, let p be the projection of M onto $\sum_K\oplus T_\beta''$ with $\mathrm{Ker}\,p=\sum_K\oplus T_\beta^*$. Then

T_2'; $T_i' \approx M_i$ and $T = T_1' \oplus T''$ by [4], Proposition 2. Now, we consider all modules in $\bar{\mathfrak{A}} = \mathfrak{A}/\mathfrak{J}'$. By $\bar{A}$ we shall denote the residue class of A in $\bar{\mathfrak{A}}$. If B is a direct summand of A, $\bar{B}$ means the image of $\bar{e}$ in $\bar{\mathfrak{A}}$, where e is the projection of A to B. Then $\bar{T} = \bar{T}_1' \oplus \bar{T}'' = \bar{T}' = \bar{T}_1' \oplus \bar{T}_2'$ and hence $\bar{T}'' \approx \bar{T}_2' \approx \bar{M}_2$ in $\bar{\mathfrak{A}}$. Accordingly T'' contains a dense submodule isomorphic to M_2. If we apply Lemma 4 to the decomposition $M = T_1' \oplus T'' \oplus S$, then $M = M_1 \oplus T_1'' \oplus S''$, where $T_1'' \subseteq T_1'$ and $S'' \subseteq S$. We put $T_1' = T_1'' \oplus T_1'''$, $S = S'' \oplus S'''$, and $\bar{\bar{M}} = M/M_1$. Then $S''' \oplus T_1' \approx M/(S'' \oplus T'') \approx M_1 \oplus T_1'' \in \mathfrak{A}$ and hence, $S''' \in \mathfrak{A}$ by Lemma 3. Further

$$\bar{\bar{M}} \approx T_1'' \oplus T'' \oplus S'' \approx M_2 \oplus N_1 \oplus N_2 \qquad \cdots\cdots\cdots(*).$$

We may assume those modules are equal to each other through isomorphisms. Since T_1'' is isomorphic to summand of M_1, $\bar{\bar{M}} = T_1'' \oplus M_2 \oplus N_1' \oplus N_2$ form (*) and Lemma 4, where $N_1 = N_1' \oplus N_1''$. $\bar{\bar{M}}/T_1'' \approx M_2 \oplus N_1' \oplus N_2 \approx T'' \oplus S''$. On the other hand, $T_1'' \approx \bar{\bar{M}}/(M_2 \oplus N_1' \oplus N_2) \approx N_1''$ and hence, N_1' is in $\mathfrak{A}$ by Lemma 3. Since T'' contains a dense submodule isomorphic to M_2 and S'' is a direct summand of S, S'' contains a dense submodule isomorphic to $N_1' \oplus N_2'$, where N_2' is in $\mathfrak{A}$ and a direct summand of N_2. Therefore, $M = M_1 \oplus T_1'' \oplus M_2 \oplus N_1' \oplus N_2 = M_1 \oplus T_1'' \oplus S'' \oplus T''$ are the desired decompositions.

Lemma 6. *Let $\{M_\alpha\}_I$ and M be as in the theorem. We assume $M = N_1 \oplus N_2$, $N_1 = \sum_{I'} \oplus M_{\alpha'}$, $N_2 = \sum_{I''} \oplus M_{\alpha''}$ and $M_{\alpha'} \not\approx M_{\alpha''}$ for any pair $(\alpha', \alpha'') \in I' \times I''$, $(I = I' \cup I'')$. We assume further that $M = T \oplus S$, $T = T_1 \oplus T_2$ and T (resp. S) contains a dense submodule which is isomorphic to N_1 (resp. N_2). Then for any element t in T_1 there exists a direct summand $\sum_1^n \oplus T_{i'}$ of T_1 such that $t \in \sum_1^n \oplus T_{i'}$ and $T_{i'} \approx M_{\alpha_i'}$ for som $\alpha_i' \in I'$.*

Proof. Let $t \in (\sum_1^{n_1} \oplus M_{\alpha_i'} \oplus \sum_1^{n_2} \oplus M_{\alpha_i''}) = M'$. Then $M = M' \oplus T_1' \oplus T_2' \oplus S'$ by Lemma 3, where $T_i = T_i' \oplus T_i''$, $S = S' \oplus S''$ and $T_1'' = T_1 \cap (M' \oplus T_2' \oplus S') \ni t$. Since $T_1'' \oplus T_2'' \oplus S'' \approx M'$, $T_1'' = \sum_1^n \oplus T_{i'}$, $T_{i'} \approx M_{\alpha_i'}$ by the assumption and Lemma 3.

Lemma 7. *Let $M = N_1 \oplus N_2$ be as in Lemma 6. Then $M = T \oplus N_2 = N_1 \oplus S$ for any decomposition $M = T \oplus S$ as in Lemma 6 if and only if $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I''}$ are relatively semi-T-nilpotent. In this case $T \approx N_1$ and $S \approx N_2$*, (cf. Corollary 2 to Theorem A.1 in §4).

Proof. We assume $M = N_1 \oplus S = T \oplus N_2$ as in the lemma. We may assume that I' and I'' are infinite. Let $\{M_{2i-1}\}_1^\infty$ and $\{M_{2i}\}_1^\infty$ be countable subsets of

$\{M_\alpha'\}_{I'}$ and $\{M_\alpha''\}_{I''}$, respectively and let $\{f_n: M_n \to M_{n+1}\}$ be a set of non-isomorphisms. We shall make use of the same argument as the proof in [3], Lemma 9 and [5], Lemma 2. Put $M_n' = \{x+f_n(x) \mid x \in M_n\} \subseteq M_n \oplus M_{n+1} \subseteq M$. Then $M = \sum_1^\infty \oplus M_{2i-1}' \oplus N_1^* \oplus N_2 = N_1 \oplus \sum_1^\infty \oplus M_{2i}' \oplus N_2^*$, where $N_1^* = \sum_{I'-\aleph_0'} \oplus M_\alpha'$, $\aleph_0' = \{1, 3, \cdots, 2i-1, \cdots\}$ and $N_2^* = \sum_{I''-\aleph_0''} \oplus M_\alpha''$, $\aleph_0'' = \{2, 4, \cdots, 2i, \cdots\}$. It is clear that $\sum_1^\infty \oplus M_{2i-1} \overset{\varphi}{\approx} \sum_1^\infty \oplus M_{2i-1}'$. We define an automorphism Φ of M by setting $\Phi = \varphi + I_{(\sum_{I'-\aleph_0'} \oplus M_\alpha \oplus N_2)}$. Then $M = \Phi^{-1}(M) = \Phi^{-1}(N_1) \oplus \sum \oplus \Phi^{-1}(M_{2i}') \oplus \Phi^{-1}(N_2^*)$. Hence we have $M = N_1 \oplus \sum \oplus \Phi^{-1}(M_{2i}') \oplus \Phi^{-1}(N_2^*)$ from the assumption. Therefore, $M = \Phi(M) = \Phi(N_1) \oplus \sum \oplus M_{2i}' \oplus N_2^* = \sum_1^\infty \oplus M_{2i}' \oplus \sum_1^\infty \oplus M_{2i-1}' \oplus N_1^* \oplus N_2^*$. We can easily show from this decomposition that $\{f_i\}$ is locally T-nilpotent, (cf. [3], Lemma 9). Hence, $\{M_\alpha'\}_{I'}$ and $\{M_\alpha''\}_{I''}$ are relatively semi-T-nilpotent. We shall show the converse. Put $M^* = N_1 \cup S$. Let $s \in N_1 \cap S$, then there exists a direct summand S_1 of S (and hence of M) such that $s \in S_1$ and $S_1 \approx \sum_1^n \oplus M_{\alpha_i}''$ by Lemma 6. On the other hand, $M = S_1 \oplus N_1 \oplus N_2'$ from Lemma 4. Hence, $N_1 \cap S = (0)$. If either I' or I'' is finite, $M = N_1 \oplus S' \oplus T'$ by Lemma 3. where $S' \subseteq S$ and $T' \subseteq T$. Since $N_2 \approx M/N_1 \approx S' \oplus T'$, $T' = (0)$ and so $M = N_1 \oplus S' \subseteq M^* \subseteq M$. We assume I' are infinite and $M \neq M^*$. Since $M^* \supseteq N_1$, there exists $\alpha_1'' \in I''$ such that $M_{\alpha_1}'' \not\subseteq M^*$. Let $m_{\alpha_1}'' \in M_{\alpha_1}'' - M^*$. From the decomposition

$$M = T \oplus S \qquad \cdots\cdots\cdots(1)$$

$m_{\alpha_1}'' = t + s$; $t \in T$, $s \in S$. Since $s \in S \subseteq M^*$, $t \in T - M^*$. We obtain, from Lemma 6, a direct summand $\sum_1^n \oplus T_i$ of T (and hence of M) such that $T_i \approx M_{\alpha_i}'$ and $t \in \sum \oplus T_i$. Let $p_i^{(1)}$ be the projection of M to T_i in that decomposition. Then there exists $i\,(=\alpha_2')$ such that $p_{\alpha_2}'^{(1)}(m_{\alpha_1}') = t_{\alpha_2}' \in T_{\alpha_2}' - M^*$. From Lemma 3 and the assumption, S contains a summand $S_{\alpha''(1)}$ isomorphic to M_{α_1}''. We obtain again from Lemma 3

$$M = N_1 \oplus \sum_{I''-\varepsilon_1} \oplus M_\alpha'' \oplus S_{\alpha''(1)} \qquad \cdots\cdots\cdots\cdots(2),$$

where, ε_1 is some index.
Let $p_{\alpha_i}''^{(2)}$ be the projection of M to M_{α_i}'' in (2). Since $N_1 \oplus S_{\alpha''(1)} \subseteq M^*$, there exists $\alpha_3'' \in I'' - \varepsilon_1$ such that $p_{\alpha_3}''^{(2)}(t_{\alpha_2}') = m_{\alpha_3}'' \in M_{\alpha_3}'' - M^*$, (it is possible $\alpha_1'' = \alpha_3''$. In this case (2) implies M contains at least two summands isomorphic to M_{α_1}''). We may assume $T_{\alpha_2}' \approx M_{\alpha'(2)}$ for some $\alpha'(2) \in I'$. Then again from Lemma 3 and (1) we have

$$M=M_{\alpha'(2)}\oplus T'\oplus S \qquad \cdots\cdots\cdots(3),$$

where $T'\subseteq T$. Then $m_{\alpha_3}{''}=x+t$; $x\in M^*$, $t\in T'-M^*$. Similarly to the argument after the decomposition (1), we obtain $\alpha_4{'}$ and a homomorphism $p_{\alpha_4}{'}^{(3)}$ of M to $T_{\alpha_4}{'}$ such that $p_{\alpha_4}{'}^{(3)}(m_{\alpha_3}{''})=t_{\alpha_4}{'}\in T_{\alpha_4}{'}-M^*$ and $T_{\alpha_4}{'}$ is a summand of T' (it is possible $T_{\alpha_2}{'}\approx T_{\alpha_4}{'}$. In this case (3) implies M contains at least two summands isomorphic to $M_{\alpha'(2)}$). From the remark after (2) and Lemma 3, there exists $\alpha''(3)$ such that

$$M=N_1\oplus\sum_{I''-\{\varepsilon_1',\varepsilon_2'\}}\oplus M_{\alpha''}\oplus S'_{\alpha''(1)}\oplus S_{\alpha''(3)} \qquad \cdots\cdots\cdots(4),$$

where $S\supseteq S_{\alpha''(3)}\approx M_{\alpha_3{''}}$ and $S'_{\alpha''(1)}\approx S_{\alpha''(1)}$. Hence, we obtain $\alpha_5{''}\in I''-\{\varepsilon_1{'},\varepsilon_2{'}\}$ such that $p_{\alpha_5}{''}^{(4)}(t_{\alpha_4}{'})=m_{\alpha_5}{''}\in M_{\alpha_5}{''}-M^*$. Similarly we have

$$M=M'_{\alpha'(2)}\oplus M'_{\alpha'(4)}\oplus T''\oplus S \qquad \cdots\cdots\cdots(5),$$

where $M'_{\alpha'(4)}\approx T_{\alpha_4}{'}$, $M'_{\alpha(2)}\approx M_{\alpha'(2)}$ and $T''\subseteq T$. Repeating those arguments, we have a series of indecomposable modules; $M_{\alpha_1}{''}$, $T_{\alpha_2}{'}$, $M_{\alpha_3}{''}$, $T_{\alpha_4}{'},\cdots$ and a series of homomorphisms $p_1=p_{\alpha_2}{'}^{(1)}\,|\,M_{\alpha_1}{''}$, $p_2=p_{\alpha_3}{''}^{(2)}\,|\,T_{\alpha_2}{'}$, $\cdots$ (it is possible $\alpha_i{''}=\alpha_j{''}$ (resp. $T_{\alpha_k}{'}\approx T_{\alpha_l}{'}$) for $i\neq j$ (resp. $k\neq l$)). Put $M^{(n)}=\sum_1^n\oplus M_{\alpha_{2i+1}}{''}$, $T^{(n)}=\sum_1^n\oplus T_{\alpha_{2i}}{'}$ (external directsum). Then $M^{(n)}$ and $T^{(n)}$ are isomorphic to direct summands of M for all n from the decompositions (n). We now concentrate to find a contradiction to the assumption of relative semi-T-nilpotency and hence, after replacing $M_{\alpha_i}{''}$ (resp. $T_{\alpha_k}{'}$) by another isomorphic summands when $\alpha_i{''}=\alpha_j{''}$ (resp. $\alpha'(k)=\alpha'(l)$) for $i\neq i$ (resp. $k\neq l$), we may assume $\alpha_i{''}\neq\alpha_j{''}$ (resp. $\alpha'(k)\neq\alpha'(l)$). It is clear that any p_i are non-isomorphic and $p_np_{n-1}\cdots p_1(m_{\alpha_1}{''})\neq 0$ for all n. This is a contradiction to the relative semi-T-nilpotency. Therefore, $M=M^*$. Similarly, we have $M=T\oplus N_2$.

Lemma 8. *Let M and $\{M_\alpha\}_I$ be as in the theorem and I' a subset of I. Put $M=(\sum_I\oplus M_\alpha=)N_1\oplus N_2$, where $N_1=\sum_{I'}\oplus M_\alpha$ and $N_2=\sum_{I-I'}\oplus M_{\alpha''}$. Then the following statements are equivaletnt.*

1) *N_1 satisfies $(P,2)$, (equivalently N_2 satisfies $(P,1)$).*
2) *N_2 satisfies $(P,2)$, (equivalently N_1 satisfies $(P,1)$).*
3) *$\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent.*

Proof. Since the condition in 3) is symmetric, we may show 1) is equivalent to 3). We know already from [5], Lemma 2 that 1) implies 3) (see Corollary 2 to Theorem A.1 in §4). Now we assume 3) and $\{M_\alpha\}_I=\{M_{kj}\}_{k\in K,j\in I_k}$ as in the beginning and $N_1=\sum_K\sum_{I_k}\oplus M_{kj}$, $N_2=\sum_K\sum_{J_k}\oplus N_{kj}$. We consider a partition of K as follows:

$K_1=\{k\in K \mid I_k$ and J_k are infinite$\}$,
$K_2=\{k\in K \mid I_k\neq\phi$ and J_k is finite$\}$,
$K_3=\{k\in K \mid I_k$ is finite and J_k is infinite$\}$,
$K_4=\{k\in K \mid I_k=\phi\}$ and
$K_5=\{k\in K \mid J_k=\phi\}$.

We put $M(i)=\sum_{K_i}\sum_{I_k}\oplus M_{kj}$ and $N(i)=\sum_{K_i}\sum_{J_k}\oplus N_{kj}$. $\{M_{kj}\}_{K_1,I_k}$, $\{N_{kj}\}_{K_1,J_k}$ are locally T-nilpotent and $\{M_{kj}\}_{K_3,I_k}$, $\{N_{kj}\}_{K_2,J_k}$ are locally semi-T-nilpotent by the assumption and Lemma 1. Hence, $\{M_{kj}\}_{K_1\cup K_3,I_k}$ is locally semi-T-nilpotent from Lemma 2. Let $M=\sum_I\oplus L_\varepsilon$ be any decomposition with L_ε indecomposable. Then $M=M(1)\oplus M(3)\oplus\sum_{I'}\oplus L_{\varepsilon'}$ for some $I'\subseteq I$ by Lemma 3. We put $\bar{M}=M/(M(1)\oplus M(3))$ then

$$\bar{M}\approx M(2)\oplus M(5)\oplus\sum_{i=1}^{4}\oplus N(i)\approx\sum_{I'}\oplus L_{\varepsilon'}\qquad\cdots\cdots\cdots(**).$$

Since $\{N_{kj}\}_{K_2,J_k}$ is locally semi-T-nilpotent, we obtain from (**) and Lemma 3 $\bar{M}\approx M(2)\oplus M(5)\oplus\sum_{i\neq 2}\oplus N(i)\oplus\sum_{I''}\oplus L_{\varepsilon''}$ for some $I''\subseteq I'$. Hence, $\bar{M}/(\sum_{I''}\oplus L_{\varepsilon''})\approx M(2)\oplus M(5)\oplus\sum_{i\neq 2}\oplus N(i)\approx\sum_{I'''}\oplus L_{\varepsilon'''}$, where $I'''=I'-I''$. We consider a partition of I''' as follows: $I_1'''=\{\varepsilon'''\in I'''\mid L_{\varepsilon'''}\approx M_{kj}$ for some $k\in K_2\cup K_5\}$ and $I_2'''=I'''-I_1'''$. Then $\sum_{I_1'''}\oplus L_{\varepsilon'''}$ is a dense submodule of itself, which satisfies the assumption in Lemma 7. Hence, $\bar{M}/(\sum_{I'''}\oplus L_{\varepsilon'''})\approx M(2)\oplus M(5)\oplus\sum_{I_2'''}\oplus L_{\varepsilon'''}$. Therefore, $M=N_1\oplus\sum_{I''}\oplus L_{\varepsilon''}\oplus\sum_{I_2'''}\oplus L_{\varepsilon'''}$.

Lemma 9. *Let M and $\{M_\alpha\}_I$ be as in Theorem. We assume $M=T\oplus S$, and T', S' are dense submodules of T and S, respectively*: $T'=\sum_{I'}\oplus T_{\alpha'}$ $S'=\sum_{I''}\oplus S_{\alpha''}$. *If $\{T_{\alpha'}\}_{I'}$ and $\{S_{\alpha''}\}_{I''}$ are relatively semi-T-nilpotent, then T and S are in $\mathfrak{A}$, where $T_{\alpha'}$ and $S_{\alpha''}$ are completely indecomposable.*

Proof. Let $M=T\oplus S$ and T', S' dense submodules of T and S, respectively. Then $M\overset{\varphi}{\approx}T'\oplus S'$. We shall use the same notations as in the proof of Lemma 8 and put $N_1=\varphi^{-1}(T')$, $N_2=\varphi^{-1}(S')$. Since $\{M_{kj}\}_{K_1\cup K_3}$ is locally semi-T-nilpotent,

$$\begin{aligned}M&=M(1)\oplus M(3)\oplus T_1''\oplus M(2)\oplus M(5)\oplus(N(1)\oplus N(3))'\oplus N(2)\oplus N(4)\\&=M(1)\oplus M(3)\oplus T_1''\oplus T''\oplus S'',\ T=T_1'\oplus T''\oplus T_1''\text{ and }S=S''\oplus S'''\end{aligned}$$

by Lemma 5, where $(N(1)\oplus N(3))'\subseteq N(1)\oplus N(3)$ and T'' (resp. S'') contains a dense submodule isomorphic to $M(2)\oplus M(5)$ (resp. $(N(1)\oplus N(3))'\oplus N(2)'\oplus N(4)$), where $N(2)'$ is a summand of $N(2)$; $N(2)'\approx\sum_{K_2}\sum_{J_{\beta_2}'}\oplus N_{\beta_2 j}$, $(N(4)'=N(4)$

in this case). Then a dense submodule of T'' is isomorphic also to $M(2)\oplus M(5)\oplus \sum_{K_2}\sum_{J_{\beta_2}-J_{\beta_2}'}\oplus N_{\beta_2 j}$. Put $N(2)''=\sum_{K_2}\sum_{J_{\beta_2}-J_{\beta_2}'}\oplus N_{\beta_2 j}$, and $\bar{M}=M/(M(1)\oplus M(3)\oplus T_1'')(\approx M(2)\oplus M(5)\oplus(N(1)\oplus N(3))'\oplus N(2)\oplus N(4)\approx T''\oplus S'')$. $\{N_{kj}\}_{K_2, J_k}$ is locally semi-T-nilpotent. We apply Lemma 5 to a decomposition $(N(2)')\oplus((N(1)\oplus N(3))'\oplus N(4))\oplus(M(2)\oplus N(2)'')\oplus(M(5))=S''\oplus T''$, then $\bar{M}\approx N(2)'\oplus S_1^{(4)}\oplus(N(1)\oplus N(3))'\oplus N(4)\oplus(M(2)\oplus N(2)'')'\oplus M(5)=N(2)'\oplus S_1^{(4)}\oplus S^{(4)}\oplus T^{(4)}$, where $S^{(4)}$ contains a dense submodule isomorphic to $(N(1)\oplus N(3))'\oplus N(4)$ and $T^{(4)}$ does a dense submodule isomorphic to $(M(2)\oplus N(2)'')'\oplus M(5)$ from the structure of $M(i)$ and $N(j)$, and $S''=S_1^{(4)}\oplus S^{(4)}\oplus S_1^{(5)}$, $T''=T^{(4)}\oplus T^{(5)}$. Accordingly $(N(1)\oplus N(3))'\oplus N(4)\oplus(M(2)\oplus N(2)'')'\oplus M(5)\approx S^{(4)}\oplus T^{(4)}$. Since J_k is finite for $k\in K_2$ and $\{N_{kj}\}_{K_2, J_k'}$ is locally semi-T-nilpotent, $(N(1)\oplus N(3))'\oplus N(4)$ and $(M(2)\oplus N(2)'')'\oplus M(5)$ satisfy the assumption in Lemma 7 (cf. the proof of Lemma 2). Hence, $S^{(4)}\approx(N(1)\oplus N(3))'\oplus N(4)$ and $T^{(4)}\approx(M(2)\oplus N(2)'')'\oplus M(5)$ by Lemma 7. Since $T=T_1''\oplus T''\oplus T_1'''$ and $T''=T^{(4)}\oplus T^{(5)}$, T is in $\mathfrak{A}$ from Lemma 5. Similarly, $S=S''\oplus S'''$ and $S''=S_1^{(4)}\oplus S^{(4)}\oplus S^{(5)}$.

Proof of Theorem. Since the condition 3) is symmetric, we may show that 1) is equivalent to 3). We assume 1). Then T and S are in $\mathfrak{A}$ and hence, we obtain 3) from Lemma 8. We assume 3). Then T and S are again in $\mathfrak{A}$ from Lemma 9. Hence, we have 1) from Lemma 8.

Proofs of Corollaries.

1: We assume S has the exchange property in M. Let $M=L\oplus L'$. Then $M=S\oplus L_1\oplus L_1'$ and $L=L_1\oplus L_2$, $L'=L_1'\oplus L_2'$. Since $L_1\oplus L_1'\approx T$ and $L_2'\oplus L_2\approx S$, L_1 (resp. L_2) contains a dense submodule isomorphic to a direct summand of T (resp. S). Conversely, we assume the above fact and 3) in Theorem. Then $S\approx\sum_{I'}\oplus M_{\alpha'}$ and $T\approx\sum_{I''}\oplus N_{\alpha''}$ from Lemma 9. We use the same notations as in the proof of Lemma 8, and put $S=N_1$ and $T=N_2$. Let $M=\sum_K\oplus L_\varepsilon$ be any decomposition of M. Then $\bar{M}\approx M(2)\oplus M(5)\oplus\sum_{i=1}^{4}\oplus N(i)\approx\sum_K L_\varepsilon'$ by Lemma 3, where $L_\varepsilon=L_\varepsilon'\oplus L_\varepsilon''$. Again from Lemmas 2 and 3 we have $\bar{M}\approx M(2)\oplus M(5)\oplus\sum_{i=3,4}\oplus N(i)\oplus\sum_K\oplus L_\varepsilon'''$, where $L_\varepsilon'=L_\varepsilon'''\oplus L_\varepsilon^{(4)}$. Hence,

$$\bar{\bar{M}}=\bar{M}/(\sum_K\oplus L_\varepsilon''')\approx M(2)\oplus M(5)\oplus\sum_{i=3,4}\oplus N(i)\approx\sum_K L_\varepsilon^{(4)}\cdots\cdots\cdots(***).$$

Now we shall apply the assumption to $L_\varepsilon^{(4)}$. Put $L_\varepsilon^{(4)}=L$. Then $L=L_1\oplus L_2$ and L_i contains a dense submodule L_i' which is isomorphic to a direct summand of N_i. Hence, $L_1'=M^*(2)\oplus M^*(5)\oplus N^*(3)$ and $L_2'=M^*(2)'\oplus N^*(3)'\oplus N^*(4)'$ from (***), where $M^*(\ )$ and $M^*(\ )'$ (resp. $N^*(\ )$ and $N^*(\ )'$) are isomorphic to direct summands of $M(\ )$ (resp. $N(\ )$). On the other hand, $N^*(3)$ is isomorphic to a direct summand of N_1 and hence of $M^*(3)$. Therefore, $N^*(3)$ has the exchange property in M by Lemma 3. Similarly, $M^*(2)'$ has the same property. Therefore, $N^*(3)$ and $M^*(2)'$ are direct summands of M (and hence of L) by [4], Proposition 2: $L_1=N^*(3)\oplus L_1''$, $L_2=M^*(2)'\oplus L_2''$ and L_1'' (resp. L_2'') contains a dense submodule isomorphic to $M^*(2)\oplus M^*(5)$ (resp. $N^*(3)'\oplus N^*(4)'$), (see the proof of Lemma 5). Accordingly $M(2)\oplus M(5)\oplus N(3)\oplus N(4)=\sum_K L_\varepsilon{}^{(4)}{}_1 \oplus L_\varepsilon{}^{(4)}{}_2=\sum_K (L_\varepsilon{}^{(4)}{}_1{}''\oplus N^*(3)_\varepsilon)\oplus\sum_K (L_\varepsilon{}^{(4)}{}_2{}''\oplus M^*(2)_\varepsilon{}')=\sum_K (M^*(2)_\varepsilon{}'\oplus L_\varepsilon{}^{(4)}{}_1{}'')\oplus \sum_K (N^*(3)_\varepsilon\oplus L_\varepsilon{}^{(4)}{}_2{}'')$. Therefore, we obtain from Lemma 7 that $\overline{\overline{M}}\approx M(2)\oplus M(5)\oplus\sum_K (N^*(3)_\varepsilon\oplus L_\varepsilon{}^{(4)}{}_2{}'')$. Thus, $M=N_1\oplus\sum_K (L_\varepsilon{}'''\oplus N^*(3)\oplus L_\varepsilon{}^{(4)}{}_2{}'')$ and $L_\varepsilon{}'''\oplus N^*(3)_\varepsilon\oplus L_\varepsilon{}^{(4)}{}_2{}''$ is a direct summand of L_ε.

2 and 3: They are clear from Lemma 8.

4: Let $\{S_{2i-1}\}$ and $\{N_{2i}\}$ be countable subsets of $\{S_{\alpha'}\}_{I'}$ and $\{N_{\alpha''}\}_{I''}$, respectively and $\{f_{2i-1}: S_{2i-1}\to N_{2i}\}$, $\{g_{2i}: N_{2i}\to S_{2i+1}\}$ sets of non-isomorphisms. Since N_{2i} is injective, Ker $g_{2i}\neq 0$ is essential in N_{2i}. Hence, Ker $f_{2i-1}g_{2i-2}\cdots f_1 \subsetneqq$ Ker $g_{2i}f_{2i-1}\cdots f_1$ and so $\{f_{2i-1}, g_{2i}\}$ is T-nilpotent.

5: Let T' and S' be dense submodules of T and S, respectively. We take indecomposable summands T_1 and S_1 of T' and S'. Then $T=T_1\oplus T''$ and $S=S_1\oplus S''$ by [4], Proposition 2. Hence, $\mathrm{Hom}_R(S_1, T_1)=0$ or $\mathrm{Hom}_R(T_1, S_1)=0$.

EXAMPLES. 1. Let Z be the ring of integers and p, q primes. Then $\{Z/p^i\}_1^\infty$ and $\{Z/q^j\}_1^\infty$ are relatively T-nilpotent, but $\{Z/q^i\}_1^\infty$ is not T-nilpotent. Put $N_1=\sum_1^\infty \oplus Z/p^{2i-1}$ and $N_2=\sum_1^\infty \oplus Z/p^{2i}$. Then all Z/p^n have finite composition series, but N_i does not have the exchange property in $\sum_1^\infty \oplus Z/p^i$.

2. Let K be a field and R the ring of lower tri-angular and column summable matrices over K with degree. $\aleph_0$ Let $\{e_{ij}\}$ be a set of matrix units in R. We put $N_1=\sum_1^\infty \oplus e_{2i-1\ 2i-1}R$ and $N_2=\sum_1^\infty \oplus e_{2i\ 2i}R$. Then all $e_{ii}R$ are projective and noetherian (artinian), but N_i does not have the exchange property in $N_1\oplus N_2$.

4. Appendix (The finite exchange property)

In §3 we have used Lemma 2 in [5]. However, I gave, in [5], only an idea of the proof of this lemma. In this section we shall give its proof as a more general form for the sake of completeness. Making use of a remark by K. Yamagata [12], [13], and [14], we shall deal with a relation between the finite exchange property and the exchange property and give generalizations of [6], Lemma 5 and [14], Theorem.

Let M be an R-module. In §2 we have defined the exchange property in M for a direct summand N. If we consider only decompositions $M=\sum_K \oplus L_\varepsilon$ with $|K|\leqslant m$ in that definition, we say N has *the m-exchange property in M*. In

[2] we have several properties on modules with m-exchange property (not necessarily in M), however they are note valid in our restricted case. Hence, we shall give proofs for some results in [2], if we are necessary to change some parts of proofs.

The following lemma is substantially due to K. Yamagata [9].

Lemma A.1. *Let T be an R-module and $T=A_1\oplus A_2=M\oplus N$. We assume $M=\sum_K \oplus M_\alpha$ and every M_α has the finite exchange property (in the usual sense) and $A_1\cap M\neq(0)$. Then there exists a finite subset $\{1, 2, \cdots, m\}$ in K such that $T=\sum_1^m \oplus M_i^*\oplus A_1^*\oplus A_2$, where $M_i^*\subseteq M_i$ ($M_j^*\neq(0)$ for some j) and $A_1^*\subseteq A_1$.*

Proof. There exists a finite subset $\{1, 2, \cdots, m\}$ in K such that $A_1\cap(\sum_1^m \oplus M_i)\neq(0)$. We put $M^\triangle=\sum_1^m \oplus M_i$, then $M^\triangle$ has the finite exchange property by [2], Lemma 3.10. Hence, $T=M^\triangle\oplus A_1'\oplus A_2'$, where $A_i=A_i'\oplus A_i''$. Since $M^\triangle\cap A_1\neq(0)$, $A_1'\neq A_1$ and so $A_1''\neq(0)$. Put $\bar{T}=T/(A_1'\oplus A_2')=\bar{A}_1''\oplus \bar{A}_2''=\bar{M}^\triangle$. By [2], Lemma 3.10 $\bar{A}_1''$ has the finite exchange property and hence $\bar{T}=\bar{A}_1''\oplus \sum_1^m \oplus \bar{M}_i'$, where $M_i=M_i'\oplus M_i''$.[2)] Then $\bar{\bar{T}}=\bar{T}/\sum \oplus \bar{M}_i'=\bar{\bar{A}}_1''=\sum_1^m \oplus \bar{\bar{M}}_i''$. We may assume $\bar{\bar{M}}_1''\neq(0)$. Then $A_1''=A_1'''\oplus A_1^{\mathrm{iv}}$ and $\bar{\bar{A}}_1'''=\bar{\bar{M}}_1''\neq(0)$, $\bar{\bar{A}}^{\mathrm{iv}}=\sum_{i\geqslant 2} \oplus \bar{\bar{M}}_i''$. Accordingly $\bar{T}=\sum_1^m \oplus \bar{M}_i'\oplus \bar{A}_1^{\mathrm{iv}}\oplus \bar{M}_1''$. On the other hand, $\bar{A}_2''$ has also the finite exchange property. Hence, we have $\bar{T}=\bar{A}_2''\oplus \bar{A}_1^{\mathrm{iv}*}\oplus \sum_1^m \oplus \bar{M}_i^*$, where $A_1^{\mathrm{iv}*}\subseteq A_1^{\mathrm{iv}}$ and $M_i^*\subseteq M_i$. Since $A_1^{\mathrm{iv}*}\subseteq A_1^{\mathrm{iv}}\subsetneqq A_1''$, $\sum_1^m \oplus M_i^*\neq(0)$. Hence, $T=\sum_1^m \oplus M_i^*\oplus(A_1^{\mathrm{iv}*}\oplus A_1')\oplus A_2$ is a desired decomposition.

Lemma A.2 ([5], Lemma 1). *Let T be an R-module and $T=N_1\oplus N_2$. We assume that N_1 has the m-exchange property in M and $T=N_1'\oplus N_2'$; $N_i'\approx N_i$, $i=1, 2$. Then N_1' has the m-exchange property in M.*

It is clear (cf. the proof of Lemma 7).

Lemma A.3 ([2], Lemma 3.10). *Let $T=B_1\oplus B_2\oplus B_3$ be R-modules. We assume B_1 has the m-exchange property in T and B_2 has the m-exchange property in $B_2\oplus B_3$. Then $B_1\oplus B_2$ has the m-exchange property in T.*

It is clear.

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2) *added in proof*: Use $\bar{A}_2''$ instead of $\bar{A}_1''$ and we obtain $\bar{T}=\bar{A}_2''\oplus \sum_1^m \oplus \bar{M}_i'$. Hence, $T=A_1'\oplus \sum_1^m \oplus M_i'\oplus A_2$.

Lemma A.4 ([2], Lemma 3.11). *Let T be an R-module and N a direct summand of T. If N has the 2-exchange property in T, then N has the finite exchange property in T.*

Proof. It is sufficient to show that N has the 3-exchange property in T. Let $T=N\oplus N_1=\sum_1^3 \oplus A_i$. Then

$$T = N\oplus A_1'\oplus(A_2\oplus A_3)',$$

where $A_1=A_1'\oplus A_1''$, $A_2\oplus A_3=(A_2\oplus A_3)'\oplus(A_2\oplus A_3)''$ and $(A_2\oplus A_3)''=(A_2\oplus A_3)\cap(N\oplus A_1')$. On the other hand, $N\approx A_1''\oplus(A_2\oplus A_3)''$ and $N_1\approx A_1'\oplus(A_2\oplus A_3)'$. Hence, $A_1''\oplus(A_2\oplus A_3)''$ has the 2-exchange property in T by Lemma A.2. Accordingly $T=(A_1\oplus A_2)\oplus A_3=A_1''\oplus(A_2\oplus A_3)''\oplus(A_1\oplus A_2)'\oplus A_3'$, where $(A_1\oplus A_2)'\subseteq(A_1\oplus A_2)$ and $A_3'\subseteq A_3$. Hence, since $(A_2\oplus A_3)''\oplus A_3'\subseteq A_2\oplus A_3$,

$$A_2\oplus A_3 = (A_2\oplus A_3)''\oplus A_3'\oplus D,$$

where $D=(A_2\oplus A_3)\cap(A_1''\oplus(A_1\oplus A_2)')\subseteq(A_2\oplus A_3)\cap(A_1\oplus A_2)=A_2$, namely D is a direct summand of A_2. Put $N\oplus A_1'=(A_2\oplus A_3)''\oplus K$. Then $T=N\oplus A_1'\oplus(A_2\oplus A_3)'=(A_2\oplus A_3)''\oplus K\oplus(A_2\oplus A_3)'=(A_2\oplus A_3)\oplus K=(A_2\oplus A_3)''\oplus A_3'\oplus D\oplus K=N\oplus A_1'\oplus D\oplus A_3'$.

The following theorem is a generalization of [6], Lemma 5.

Theorem A.1. *Let $\{P_\alpha\}_I$ be an infinite set of R-modules which have the finite exchange property and $P=\sum_I \oplus P_\alpha$. Let I' be an infinite subset of I with infinite complement I-I'. We assume $P_{I'}=\sum_{I'}\oplus P_{\alpha'}$ has the 2 (finite)-exchange property in P. Then if we take any countable subsets $\{P_{2i-1}\}_1^\infty$ and $\{P_{2i}\}_1^\infty$ of $\{P_\alpha\}_{I'}$ and $\{P_\alpha\}_{I-I'}$, respectively and any sets of homomorphisms $f_i\colon P_i\to P_{i+1}$ such that for any direct summands X in P_{2i-1} (or Y in P_{2i}) $f_{2i-1}(X)$ (or $f_{2i-1}^{-1}(Y)$) is not a direct summand, provided $f_{2i-1}(X)\neq(0)$ (or $f_{2i-1}^{-1}(Y)\neq P_{2i-1}$) for all i (e.g. $Im\, f_{2i-1}$ is small in P_{2i} or $Ker\, f_{2i-1}$ is large in P_{2i-1}), then there exists n, depending on x in P_1 such that $f_nf_{n-1}\cdots f_1(x)=0$.*

Proof. We can prove the theorem similarly to [6], Lemma 5 and so we shall give a sketch of the proof. We shall use the same notations as in the proof of Lemma 7, changing M_α by P_α. Put $P_i'=\{p_i+f_i(p_i)\,|\,p_i\in P_i\}\subseteq P_i\oplus P_{i+1}$. Then $P=\sum_1^\infty \oplus P_{2i-1}'\oplus P^{(1)}\oplus \sum_1^\infty \oplus P_{2i}\oplus P^{(2)}=\sum_1^\infty \oplus P_{2i-1}\oplus P^{(1)}\oplus \sum_1^\infty \oplus P_{2i}'\oplus P^{(2)}$. Since $\sum_1^\infty \oplus P_{2i-2}'\oplus P^{(1)}$ has the finite exchange property in P from the assumption and Lemma A.2, we obtain from the decomposition above

$$P=\sum_1^\infty \oplus P_{2i-1}'\oplus P^{(1)}\oplus X\oplus Y\oplus Z \qquad \cdots\cdots\cdots(1),$$

where $X\subseteq \sum_1^\infty \oplus P_{2i-1}$, $Y\subseteq \sum_1^\infty \oplus P_{2i}'$ and $Z\subseteq P^{(2)}$. We shall show $X=(0)$. We assume contrary $X\neq(0)$. Then we have from Lemma A.1

$$P=\sum_1^t \oplus P_{2i-1}{}^*\oplus \sum_1^\infty \oplus P_{2i-1}{}'\oplus P^{(1)}\oplus X'\oplus Y\oplus Z \qquad \cdots\cdots\cdots(2),$$

where $P_{2i-1}{}^*\subseteq P_{2i-1}$ ($P_{2j-1}{}^*\neq(0)$ for some j) and $X'\subseteq X$, We consider the following modules and a decomposition of P:

$$P_1{}^*\oplus \sum_1^\infty \oplus P_{2i-1}{}'\oplus P^{(1)} \quad \text{and} \quad P=P_1\oplus(\sum_{i\geqslant 2}\oplus P_{2i-1})\oplus P^{(1)}$$
$$\oplus P_2{}'\oplus(\sum_{i\geqslant 2}\oplus P_{2i}{}')\oplus P^{(2)} \qquad \cdots\cdots\cdots(3).$$

Since the former module has the finite exchange property in P by Lemma A.3 and [2], Lemma 3.10, we obtain from (3)

$$P=(P_1{}^*\oplus \sum_1^\infty \oplus P_{2i-1}{}'\oplus P^{(1)})\oplus P_1{}^{**}\oplus A\oplus P_2{}'^*\oplus B\oplus C\,,$$

where $P_1{}^{**}\subseteq P_1$, $A\subseteq \sum_{i\geqslant 2}\oplus P_{2i-1}$, $P_2{}'^*\subseteq P_2{}'$, $B\subseteq \sum_{i\geqslant 2}\oplus P_{2i}{}'$ and $C\subseteq P^{(2)}$. Hence

$$P=(P_1{}^{\S}\oplus P_1{}')\oplus(P_2{}'^*\oplus D) \qquad \cdots\cdots\cdots(4),$$

where $P_1{}^{\S}=P_1{}^*\oplus P_1{}^{**}$ and $D=\sum_{i\geqslant 2}\oplus P_{2i-1}{}'\oplus P^{(1)}\oplus A\oplus B\oplus C\subseteq \sum_{i\geqslant 3}\oplus P_i$. Using only a fact $D\subseteq \sum_{i\geqslant 3}\oplus P_i$ in (4), we shall show that $P_1{}^{\S}=(0)$. Let x be in $P_1{}^{\S}$. If $f_1(x)\in(P_2{}'^*\oplus D)$, $x=0$ from (4). Hence, $f_i\,|\,P_1{}^{\S}$ is monomorphic and

$$P_2=f_1(P_1{}^{\S})\oplus N \qquad \cdots\cdots\cdots(5),$$

where $N=\{x\in P_2\,|\,f_2(x)\in D\}$, (see [6], Lemma 5). Furthermore

$$P_1=P_1{}^{\S}\oplus f_1{}^{-1}(N) \qquad \cdots\cdots\cdots(6),$$

since $f_1\,|\,P_1{}^{\S}$ is monomorphic. Hence, $P_1{}^{\S}=(0)$ from (5), (6) and the assumptions. Therefore, $P_1{}^*=(0)$ in (2). Next, we consider similarly to (3)

$$P_3{}^*\oplus \sum_1^\infty \oplus P_{2i-1}{}'\oplus P^{(1)} \quad \text{and} \quad P=P_1\oplus P_3\oplus(\sum_{i\geqslant 3}\oplus P_{2i-1})$$
$$\oplus P^{(1)}\oplus P_2{}'\oplus P_4{}'\oplus(\sum_{i\geqslant 3}\oplus P_{2i}{}')\oplus P^{(2)} \qquad \cdots\cdots\cdots(3').$$

Then $P=(P_3{}^*\oplus \sum_1^\infty \oplus P_{2i-1}{}'\oplus P^{(1)})\oplus P_1{}^{\triangle}\oplus P_3{}^{\triangle}\oplus A'\oplus P_2{}'^*\oplus P_4{}'^*\oplus B'\oplus C'$, where $P_{2i}{}'^*\subseteq P_{2i}{}'$, $P_{2i-1}{}^{\triangle}\subseteq P_{2i-1}$, $A'\subseteq \sum_{i\geqslant 3}\oplus P_{2i-1}$, $B'\subseteq \sum_{i\geqslant 3}\oplus P_{2i}{}'$ and $C'\subseteq P^{(2)}$.

From the argument after (4) we know $P_1{}^{\triangle}=(0)$. Thus, we have

$$P=(P_1{}'\oplus P_2{}'^*)\oplus\{(P_3{}^{\S}\oplus P_3{}')+(P_4{}'^*\oplus D')\} \qquad \cdots\cdots\cdots(4'),$$

where $P_3{}^{\S}=P_3{}^*\oplus P_3{}^{\triangle}$ and $D'=\sum_{i\geqslant 3}\oplus P_{2i-1}{}'\oplus P^{(1)}\oplus A'\oplus B'\oplus C'\subseteq \sum_{i\geqslant 5}\oplus P_i$.

Applying the same arguments on $P_3^{\S}\oplus P_3'$ and P_4 in (4′) as ones after (4), we obtain $P_3^{\S}=P_3^*=(0)$. Continuing those arguments, we have a contradiction to the assumption $X\neq(0)$ in (1). Therefore, we have from (1)

$$P=\sum_1^\infty\oplus P_{2i-1}'\oplus P^{(1)}\oplus Y\oplus P^{(2)} \quad \text{and} \quad Y\subseteq\sum_1^\infty\oplus P_{2i}' \cdots\cdots\cdots(7).$$

Hence, $\{f_i\}$ is locally T-nilpotent.

From Theorem A.1 and [10], [11] we have

Corollary 1. *Let E be an injective module. If $\sum_1^\infty\oplus E_{2i}$ has the finite exchange property in $\sum_1^\infty\oplus E_i$; $E_i\approx E$ (e.g. E is $\sum$-injective), then the radical of $End_R(E)$ is locally T-nilpotent.*

Corollary 1′. *Let P be a projective module with finite exchange property. If $\sum_1^\infty\oplus P_{2i}$ has the finite exchange property in $\sum_1^\infty\oplus P_i$; $P_i\approx P$, then the radical of $End_R(P)$ is locally T-nilpotent.*

Corollary 2 ([5], Lemma 2). *Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M=\sum_I\oplus M_\alpha$. Put $N_i=\sum_{I_i}\oplus M_{\alpha'}$, where $I=I_1\cup I_2$ and $I_1\cap I_2=\phi$. If N_1 has the 2-exchange property in M, then $\{M_{\alpha'}\}_{I_1}$ and $\{M_{\alpha''}\}_{I_2}$ are relatively semi-T-nilpotent.*

Proof. We may assume that I_i are infinite. Let $\{M_{2i-1}\}_1^\infty$ and $\{M_{2i}\}_1^\infty$ be any countable subsets of $\{M_{\alpha'}\}_{I_1}$ and $\{M_{\alpha''}\}_{I_2}$, respectively and $\{f_r\colon M_n\to M_{n+1}\}$ a set of non-isomorphisms. We shall show that f_{2i-1} satisfies the assumptions in Theorem A.1. Since M_i is completely indecomposable, M_i has the (finite) exchange property by [9], Proposition 1. If $\operatorname{Ker} f_{2i-1}$ is a direct summand of M_{2i-1}, $\operatorname{Ker} f_{2i-1}=M_{2i-1}$ or $\operatorname{Ker} f_{2i-1}=(0)$. The former case implies $f_{2i-1}=0$. We assume $\operatorname{Ker} f_{2i-1}=(0)$. If $\operatorname{Im} f_{2i-1}$ is a direct summand of M_{2i}, then f_{2i-1} is isomorphic. Hence, $\operatorname{Im} f_{2i-1}$ is not direct summand of M_{2i}. Therefore, f_{2i-1} satisfies the assumptions in Theorem A.1.

Corollary 3. *Let $\{M_\alpha\}_I$ and M be as in Corollary 2. For any subset I' of I we put $M_{I'}=\sum_{I}\oplus M_{\alpha'}$. Then the following statements are equivalent.*

1) *$\{M_\alpha\}_I$ is locally semi-T-nilpotent.*
2) *$M_{I'}$ has the 2-exchange property in M for any $I'\subseteq I$.*
3) *$M_{I'}$ has the finite exchange property in M for any $I'\subseteq I$.*
4) *$M_{I'}$ has the exchange property in M for any $I'\subseteq I$,* (cf. [14]).

Proof. It is clear from Lemma A.5 and [8], Theorem.

Theorem A.2. *Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules and $M=\sum_I \oplus M_\alpha$. We put $M_{I'}=\sum_{I'} \oplus M_{\alpha'}$ for some $I'\subseteq I$. Then the following statements are equivalent.*

1) *$M_{I'}$ has the 2-exchange property in M.*
2) *$M_{I'}$ has the finite exchange property in M.*
3) *$M_{I'}$ has the exchange property in M.*
4) *$M_{I-I'}$ has the exchange property in M.*

Proof. 3)$\rightarrow$2)$\rightarrow$1) are clear. We assume 1). Then $\{M_{\alpha'}\}_I$ and $\{M_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent by Corollary 2 to Theorem A.1. Hence, $M_{I'}$ (resp. $M_{I-I'}$) satisfies conditions in Corollary 1 of Theorem (cf. its proof) and so $M_{I'}$ and $M_{I-I'}$ have the exchange property in M.

Corollary 1. *Let M be as above and $M=T\oplus S$. Then T has the exchange property in M if and only if so does S.*

Corollary 2. *Let $M=\sum_I\oplus M_\alpha$ be as above. We assume $M=S\oplus T$ and any indecomposable direct summands of S are not isomorphic to direct summands of T. Then S has the 2-exchange property in M if and only if S has the exchange property in M.*

Proof. Let S' and T' be dense submodules of S and T, respectively. Since $S'\oplus T'\approx M$, $M=\sum_{I'}\oplus M'_{\alpha'}\oplus\sum_{I''}\oplus M'_{\alpha''}$; $\sum_{I'}\oplus M'_{\alpha'}\approx S'$ and $\sum_{I''}\oplus M'_{\alpha''}\approx T'$. We assume S has the 2-exchange property in M. Then $M=S\oplus\sum_{I''}\oplus M'_{\alpha''}$ from the assumption, (cf. the proof of Lemma 6). Hence, $S\approx\sum_{I'}\oplus M'_{\alpha'}$ and $T\approx\sum_{I''}\oplus M'_{\alpha''}$. Therefore, S has the exchange property in M by Theorem A.2.

Corollary 3. *Let $M=\sum\oplus M_\alpha$. We assume $M_\alpha\not\approx M_{\alpha'}$ if $\alpha\neq\alpha'$. Then a direct summand S of M has the 2-exchange property in M if and only if S has the exchange property in M.*

Corollary 4. *Let $M=\sum_I M_\alpha=S\oplus T$. We assume S has the exchange property in M. If $M=S_1\oplus T_1$ and a dense submodule of S_1 (resp. T_1) is isomorphic to S (resp. T), then S_1 has the exchange property in M.*

Proof. We may assume $S=M_{I'}$ and $T=M_{I''}$. Then $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I''}$ are relatively semi-T-nilpotent. Hence, S_1 and T_1 are in $\mathfrak{A}$ by the assumption and Theorem. Therefore, $S_1\approx S'\approx S$ ($T_1\approx T'\approx T$).

REMARKS. 1. If every direct summand of M is in $\mathfrak{A}$ (e.g. all M_α are countably generated), then Theorem $A.2$ shows that 2-exchange property in M of a direct summand is equal to the exchange property in M. Furthermore, it is eqivalent to a fact that $\{M_{\alpha'}\}_{I'}$ and $\{M_{\alpha''}\}_{I-I'}$ are relatively semi-T-nilpotent.

2. Let $M=\sum_I \oplus M_\alpha = S\oplus T$ be as before. We assume that S has the 2-exchange property in M. Then the proof of Theorem A.1 shows that for any direct summands $\sum_K \oplus M'_{\alpha'}$ and $\sum_{K'} \oplus M'_{\alpha''}$ of S and T, respectively $\{M'_{\alpha'}\}_K$ and $\{M'_{\alpha''}\}_{K'}$ are relatively semi-T-nilpotent.

3. In the definition of relative semi-T-nilpotency in §1, we took a set of non-isomorphisms $\{f_i, g_i\}$. However, this definition is equivalent to a stronger one in which we assume only $\{f_i\}$ or $\{g_i\}$ is a set of non-isomorphisms, (cf. Theorem A.1).

4. Let $\{M_\alpha\}_I$ be a set of completely indecomposable modules such that $\{M_\alpha\}_I$ is locally semi-T-nilpotent. We assume $M=\sum_I \oplus M_\alpha$ and $T=M\oplus N=\sum_1^\infty \oplus A_i$. Then we obtain, from [6], Lemma 8, decompositions $A_i=A_i'\oplus A_i''$ such that $(\sum_1^\infty \oplus A_i')\cap M=(0)$ and $\sum_1^n \oplus A_i''$ is isomorphic to a a direct summand of M. We further assume that N does not contain any direct summands isomorphic to some M_α in $\{M_\alpha\}_I$. Then if we make use of the same argument in the proof of Lemma 7, we can prove $T=M\oplus \sum \oplus A_i'$, namely M has the $\aleph_0$-exchange property in T, because if $T\neq M\oplus \sum A_i'$, there exist a subset $\{M_{\alpha_i}\}$ of $\{M_\alpha\}_I$, an element $x\in M_{\alpha_1}$ and a set of homomorphisms $f_{2i-1}\colon M_{\alpha_{2i-1}}\to N$ and $f_{2i}\colon N\to M_{\alpha_{2i}}$ such that $f_{2i}f_{2i-1}\cdots f_1(x)\neq 0$ for all i.

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