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K-GROUPS OF SYMMETRIC SPACES I

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Introduction

In this paper we consider the unitary K-groups of compact homogeneous spaces of Lie groups and in particular, we lay emphasis on the compact symmetric spaces.

For a compact connected Lie group G with $\pi_1(G)$ torsion-free as a symmetric space it is known that the K-group $K^*(G)$ of G is an exterior algebra on the elements of $K^{-1}(G)$ induced by the basic representations of G [2], [3], [7]. Making use of this result Hodgkin constructed the Künneth formula spectral sequence in equivariant K-theory [8], [9]. This spectral sequence is our main tool in the present study. Besides we find some examples of the K-groups of symmetric spaces in [5].

The main theorem of this paper is the following

Theorem A. *Let G be a compact connected simply-connected Lie group together with the involutive automorphism σ and K the subgroup of G consisting of fixed points of σ. When we write M for the homogeneous space G/K, we have*

(i) *There are elements $\rho_1, \cdots, \rho_l$ of $R(G)$ such that*

$$\sigma^*(\rho_k) = \rho_k \ (r+1 \leqq k \leqq l) \qquad \text{for} \quad \text{some } r \text{ and}$$
$$R(G) = Z[\rho_1, \cdots, \rho_r, \sigma^*(\rho_1), \cdots, \sigma^*(\rho_r), \rho_{r+1}, \cdots, \rho_l]\,.$$

(ii) *The natural homomorphism $\alpha\colon Z \underset{R(G)}{\otimes} R(K) \to K^0(M)$ becomes a monomorphism* (*Section* 1) *and if we identify an element of $Z \underset{R(G)}{\otimes} R(K)$ with its image by α, then we can write*

$$K^*(M) = \Lambda(\beta(\rho_1 - \sigma^*(\rho_1)), \cdots, \beta(\rho_r - \sigma^*(\rho_r))) \otimes (Z \underset{R(G)}{\otimes} R(K))\,.$$

where $\beta(\rho_k - \sigma^*(\rho_k))$ is the element of $K^{-1}(M)$ induced by the representations ρ_k and $\sigma^*(\rho_k)$ in (i) *for $k = 1, \cdots, r$* (*Section* 1).

(iii) *$K^*(M)$ is torsion-free.*

The arrangement of this paper is as follows.

In section 1 we describe the definitions of the α- and β- elements of $K^*(M)$ in Theorem A and summarize some of the facts on the Künneth formula spectral sequence.

In sections 2–4 we give a remark (Theorem 2.1) on Snaith's collapsing theorem ([14], Theorem 5.5) for the Künneth formula spectral sequence in equivariant K-theory. Professor V.P. Snaith informed the author that Theorem 2.1 is known to him and the author agreed with him in an outline of a proof. For a proof of Theorem A we have need of Proposition 4.1 obtained as a corollary to the proof of Theorem 2.1.

Sections 5–8 are devoted to the proof of Theorem A.

1. The α and β constructions and the spectral sequence

Let G be a compact Lie group and H a closed subgroup of G. The K-group of the homogeneous space G/H has two kind of elements induced by the unitary representations of G and H.

Over G/H we have the canonical principal H-bundle η. Then, for an H-vector space V, the vector bundle with fibre V associated with η defines an element $\alpha(V)$ of $K^0(G/H)$. Thus η defines a homomorphism of rings $\alpha: R(H)\to K^0(G/H)$ and we see that α is clearly factored through the natural projection $R(H)\to Z\underset{R(G)}{\otimes}R(H)$ where $R(G)$ (resp. $R(H)$) is the complex representation ring of G (resp. H) ([1], [9] §9). We shall denote this factored homomorphism $Z\underset{R(G)}{\otimes}R(H)\to K^0(G/H)$ by the same letter α.

The other elements are defined in the following way. Consider a representation of G viewed as a homomorphism of G to the unitary group $U(n)$. If $\rho_1, \rho_2: G\to U(n)$ are representations of G agreeing on H, then we can define a map $f: G/H \to U(n)$ by $f(gH)=\rho_1(g)\rho_2(g)^{-1}$ for $gH\in G/H$. Then the composition of f and the inclusion of $U(n)$ to the stable unitary group forms an element of $K^{-1}(G/H)$. We denote this element by $\beta(\rho_1-\rho_2)$.

Suppose that G is a compact connected Lie group such that $\pi_1(G)$ is torsion-free and let K_G^* denote the equivariant K-theory associated with G [13]. In [8], [9], Hodgkin constructed a strongly convergent spectral sequence

$$(1.1)\qquad E_2^{*,*}=\mathrm{Tor}_{R(G)}^{*,*}(K_G^*(X),\ K_G^*(Y))\Rightarrow F_G^*(X;\ Y)\,,$$

and showed that there is a natural transformation λ of $F_G^*(X;\ Y)$ to $K_G^*(X\times Y)$ and if either X or Y is a free G-space then λ is an isomorphism ([8], Propositions 6.3 and 7.2).

In particular, when $X=G$ and $Y=G/H$, a homogeneous space, in the spectral sequence (1.1), (1.1) becomes

$$(1.2)\qquad E_2^{*,0}=\mathrm{Tor}_{R(G)}^{*,0}(Z,\ R(H))\Rightarrow K^*(G/H)$$

because of $K_G^*(G/H)=R(H)$.

2. A collapsing theorem for (1.2)

We have the following

Theorem 2.1. *Let G be a compact connected Lie group such that $\pi_1(G)$ is torsion-free and H a closed connected subgroup of G. Then the spectral sequence*

$$E_2^{*,0} = \mathrm{Tor}_{R(G)}^{*,0}(Z, R(H) \Rightarrow K^*(G/H)$$

collapses.

From now we write $E_r^{*,*}(X; Y)_G$ for the r-th term of the spectral sequence (1.1) and also $\{E_r^{*,*}(X; Y)_G\}$ for this spectral sequence.

To prove Theorem 2.1 we reduce this theorem to Theorem 5.5 of [14] (which requires the conditions that $H^*(BG, Z)$ and $H^*(BH, Z)$ are polynomial algebras). For this purpose we prepare two lemmas.

Lemma 2.1. *Let T be a maximal torus of H in Theorem 2.1. If the spectral sequence*

$$E_2^{*,0} = \mathrm{Tor}_{R(G)}^{*,0}(Z, R(T)) \Rightarrow K^*(G/T)$$

collapses, then so does the spectral sequence

$$E_2^{*,0} = \mathrm{Tor}_{R(G)}^{*,0}(Z, R(H)) \Rightarrow K^*(G/H)\,.$$

Proof. The natural projection $G/T \rightarrow G/H$ induces a morphism of the spectral sequences

$$\{\varphi_r\}: \{E_r^{*,*}(G, G/H)_G\} \rightarrow \{E_r^{*,*}(G, G/T)_G\}\,.$$

For a proof of Lemma 2.1 it is sufficient to to prove that φ_2 is injective. However it follows easily from the facts that $\varphi_2=\mathrm{Tor}_{R(G)}^{*,*}(1, i^*)$ where i^* is the restriction of $R(H)$ to $R(T)$ and $R(H)$ is a direct summand of $R(T)$ as an $R(G)$-module (via restriction). q.e.d.

By choosing unitary representations of G suitably, we can embed G into a finite product of unitary groups U such that if we denote this embedding by i: $G \rightarrow U$, then

(2.1) $\qquad i^*: R(U) \rightarrow R(G) \qquad$ is surjective.

Let X be a compact, locally contractible G-space of finite covering dimension. Then we have

Lemma 2.2. *Suppose that the spectral sequence*

$$E_2^{*,*} = \mathrm{Tor}_{R(U)}^{*,*}(Z, K_G^*(X)) \Rightarrow K^*(U \underset{G}{\times} X)$$

collapses, then so does

$$E_2^{*,*} = \mathrm{Tor}_{R(G)}^{*,*}(Z, K_G^*(X)) \Rightarrow K(X)\,.$$

3. Proof of Lemma 2.2

Here we shall give a proof of Lemma 2.2.

Let L be a compact connected Lie group. We define $I(L)$ to be the kernel of the augumentation $\varepsilon: R(L) \to Z$ of $R(L)$ and $J(L)$ the quotient $I(L)/(I(L))^2$. Then we know that if the fundamental group of L is torsion-free then $J(L)$ is a free abelian group of rank l where l is the rank of L ([7], Lemma 4.2).

From (2.1) we see obviously that the homomorphism induced by i^*

(3.1) $\qquad i_1^*: J(U) \to J(G) \qquad$ is surjective.

Define $J(U, G) = \mathrm{Ker}\, i_1^*$. We can choose a basis $\xi_1, \cdots, \xi_l, \nu_1, \cdots, \nu_s$ for $J(U)$ such that $i_1^*(\xi_1), \cdots, i_1^*(\xi_l)$ form a basis $J(G)$ and $\nu_1, \cdots, \nu_s$ a basis for $J(U, G)$. For brevity we denote the representatives of these elements in $R(U)$ by the same notation and then we may assume that

(3.2) $\qquad i^*(\nu_k) = 0 \qquad$ for $\quad k = 1, \cdots, s\,.$

Then we have the Koszul complex given by

(3.3) $\qquad C^* = \Lambda(x_1, \cdots, x_s, y_1, \cdots, y_l) \otimes R(U)$

where $d(x_i) = \nu_i$ $(1 \leqq i \leqq s)$, $d(y_j) = \xi_j$ $(1 \leqq j \leqq l)$ and d is a derivation.

Proof of Lemma 2.2. The inclusion $X \to U \underset{G}{\times} X$ induces a morphism $\{\phi_r\}$ of $\{E_r^{*,*}(U, U \underset{G}{\times} X)_U\}$ to $\{E_r^{*,*}(U, X)_G\}$.

Using (3.3) we have isomorphisms

$$\begin{aligned} E_2^{*,*}(U, U \underset{G}{\times} X)_U &= \mathrm{Tor}_{R(U)}^{*,*}(Z, K_G^*(X)) \\ &\cong H^*(C^* \underset{R(U)}{\otimes} K_G^*(X)) \\ &\cong \Lambda(x_1, \cdots, x_s) \otimes \mathrm{Tor}_{R(G)}^{*,*}(Z, K_G^*(X)) \qquad \text{by (3.2)}\,. \end{aligned} \tag{3.4}$$

Next we consider $E_2^{*,*}(U, X)_G$. For this we need that

(3.5) $\qquad K^*(U/G) \qquad$ is torsion-free.

Suppose that (3.5) is true for the moment. Then we have an isomorphism

$$\begin{aligned} E_2^{*,*}(U, X)_G &= \mathrm{Tor}_{R(G)}^{*,*}(K^*(U/G), K_G^*(X)) \\ &\cong K^*(U/G) \otimes \mathrm{Tor}_{R(G)}^{*,*}(Z, K_G^*(X)) \end{aligned} \tag{3.6}$$

from (2.1) and (3.5).

From (3.4) and (3.6) we see that ϕ_2 induces an epimorphism from the torsion-part of $E_2^{*,*}(U, U\underset{G}{\times}X)_U$ to that of $E_2^{*,*}(U, X)_G$, and therefore we see by the assumption that $E_2^{*,*}(U, X)_G$ consists of permanent cycles. Moreover when we consider the morphism of the spectral sequences

$$\psi_r\colon \{E_r^{*,*}(U, X)_G\} \to \{E_r^{*,*}(G, X)_G\}$$

induced by the embedding of G to U, it is easy to see that $E_2^{*,*}(G, X)_G$ also consists of permanent cycles.

It remains to prove (3.5). Put $X=G$ in the above. Then, from Lemma 7.3 of [8], it follows that $\{E_r^{*,*}(U, U)_U\}$ collapses, and (3.6) when $X=G$ follows from the facts that $\mathrm{Tor}^{*,*}_{R(G)}(Z, Z)$ is torsion-free ([8], Lemma 7.2). Hence we see that $\{E_r^{*,*}(U, G)_G\}$ collapses by using the above argument and so $K^*(U/G)$ is isomorphic to a subgroup of $K^*(U)$. This shows (3.5). Therefore Lemma 2.2 is proved.

4. Proof of Theorem 2.1 and a corollary

Proof of Theorem 2.1. Putting $X=G/T$ in Lemma 2.2 where T is a maximal torus of H, Lemmas 2.1 and 2.2 imply that $\{E_r^{*,*}(G, G/H)_G\}$ collapses because $\{E_r^{*,*}(U, U/T)_U\}$ does so by Theorem 5.5 of [14]. q.e.d.

Next we describe a result obtained from the proof of Lemma 2.2.

Proposition 4.1 (Cf. [5], Proposition 2.3).

Let G and H be as in Theorem 2.1. Suppose that $\pi_1(H)$ is torsion-free and the restriction $i^*\colon R(G)\to R(H)$ is surjective. Then we have

(i) *There exist elements $\nu_1, \cdots, \nu_s$ of $R(G)$ such that $i^*(\nu_k)=0$ for $k=1, \cdots, s$ and $\pi(\nu_1), \cdots, \pi(\nu_s)$ form a basis for the free abelian group* Ker $(J(G)\to J(H))$ *where π is the composition of the natural projections $R(G)\to I(G)\to J(G)$.*

(ii) *$K^*(G/H)$ is an exterior algebra on $\beta(\nu_1), \cdots, \beta(\nu_s)$.*

Proof. By Theorem 2.1 the spectral sequence

$$E_2^{*,0} = \mathrm{Tor}^{*,0}_{R(G)}(Z, R(H)) \Rightarrow K^*(G/H)$$

collapses. Here we consider the E_2-term of this spectral sequence. In section 3 we can substitute the pair (G, H) for the pair (U, G) by the assumption and put $X=$a point. Then we have an isomorphism

$$E_2^{*,0}(G, G/H)_G \cong \Lambda(x_1, \cdots, x_s)$$

by using the notation of (3.4) and we see that the edge homomorphism of this spectral sequence sends x_j to $\beta(v_j)$ for $j=1, \cdots, s$ (See [9], §10). This completes the proof.

5. The classification of symmetric spaces

Let G, K, σ and M be as in Theorem A and let $j\colon K\to G$ be the inclusion of K throughout the remainder of this paper.

We know that K is connected ([11, I], Theorem 3.4 in Chapter IV). Now if K is of maximal rank, then we can easily check Theorem A as follows: Since the restriction $i^*\colon R(G)\to R(K)$ is injective, all elements of $R(G)$ are fixed by σ^*, and since $R(K)$ is a stably-free module as an $R(G)$-module [12], we see that

$$\alpha\colon Z \underset{R(G)}{\otimes} R(K) \to K^*(G/K) \qquad \text{is an isomorphism} \tag{5.1}$$

from the spectral sequence (1.2) ([9], §9) and $Z \underset{R(G)}{\otimes} R(K)$ is torsion-free.

Therefore it suffices to prove Theorem A when rank $G>$ rank K. When M is a simply-connected Lie group as a symmetric space, we refer the reader to [2] and [3] or [7]. M is simply-connected and so it is a direct product of irreducible symmetric spaces. Hence we consider only the irreducible symmetric spaces such that rank $G>$ rank K. According to the classification of irreducible symmetric spaces [6], such irreducible symmetric spaces are the following six types:

$$\begin{array}{lll} AI & SU(n)/SO(n) & \\ AII & SU(2n)/Sp(n) & \\ BDI(a) & Spin(p+q)/Spin(p)\underset{Z_2}{\times} Spin(q) & (\textit{where } p \textit{ and } q \textit{ are odd and} \\ & Z_2=\{(1,1), (-1,-1)\}\} & \\ BDII(a) & Spin(n)/Spin(n-1) & (\textit{where } n \textit{ is even}) \\ EI & E_6/PSp(4) & \\ EIV & E_6/F_4 & \end{array} \tag{5.2}$$

6. Proofs for *AII*, *BDII(a)* and *EIV*

The symmetric spaces of types AII, $BDII(a)$ and EIV have the properties such that $\pi_1(K)$ is torsion-free and $i^*\colon R(G)\to R(K)$ is surjective. Hence we can apply Proposition 4.1 to this case.

Here we describe Proposition 4.1 for the above three symmetric spaces explicitly.

Type AII ($M=SU(2n)/Sp(n)$). Let I_n denote the $n\times n$ unit matrix and put $J=\begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix}$. Then σ is given by

$$\sigma(g) = J\bar{g}J^{-1} \qquad \text{for} \quad g\in SU(2n)$$

where $\bar{g}$ is the complex conjugate of g.

Using the notation in [10] we have

$$R(SU(2n)) = Z[\lambda_1, \cdots, \lambda_{2n-1}] \quad \text{and} \quad R(Sp(n)) = Z[\lambda_1, \cdots, \lambda_n] \tag{6.1}$$

([10], Theorems 3.1 and 6.1 in §13). Then it is clear that

$$i^*(\lambda_k) = i^*(\lambda_{2n-k}) = \lambda_k \quad \text{and} \quad \sigma^*(\lambda_k) = \lambda_{2n-k} \qquad \text{for} \quad k = 1, \cdots, n\,. \tag{6.2}$$

From (6.2) we see easily that i^* is surjective and $\pi(\lambda_k - \lambda_{2n-k})$ $(1 \leqq k \leqq n-1)$ form a basis for the free abelian group $\mathrm{Ker}(J(SU(2n)) \to J(Sp(n)))$. Therefore we get from Proposition 4.1 that

Proposition 6.1. *The notation being as in* (6.1)),

$$K^*(SU(2n)/Sp(n)) = \Lambda(\beta(\lambda_1 - \lambda_{2n-1}), \cdots, \beta(\lambda_{n-1} - \lambda_{n+1}))\,.$$

Type *BDII(a)* ($M = Spin(2n)/Spin(2n-1)$). σ is given by

$$\sigma(g) = -e_{2n} g e_{2n} \qquad \text{for any} \quad g \in Spin(2n)$$

where e_{2n} is the generator of the Clifford algebra C_{2n} in the $2n$-th position ([10], §11).

From Theorem 10.3 in §13 of [10],

$$\begin{aligned} &R(Spin(2n)) = Z[\lambda^1(\rho_{2n}), \cdots, \lambda^{n-2}(\rho_{2n}), \Delta_{2n}^+, \Delta_{2n}^-]\,, \\ &R(Spin(2n-1)) = Z[\lambda^1(\rho_{2n-1}), \cdots, \lambda^{n-2}(\rho_{2n-1}), \Delta_{2n-1}] \end{aligned} \tag{6.3}$$

using the same notation. Then we can easily verify that

$$\begin{aligned} &i^*(\lambda^k(\rho_{2n})) = \lambda^k(\rho_{2n-1}) + \lambda^{k-1}(\rho_{2n-1}) \qquad (1 \leqq k \leqq n-2)\,, \\ &i^*(\Delta_{2n}^{\pm}) = \Delta_{2n-1}\,, \\ &\sigma^*(\lambda^k(\rho_{2n})) = \lambda^k(\rho_{2n}) \qquad (1 \leqq k \leqq n-2) \quad \text{and} \\ &\sigma^*(\Delta_{2n}^{\pm}) = \Delta_{2n}^{\pm}\,. \end{aligned} \tag{6.4}$$

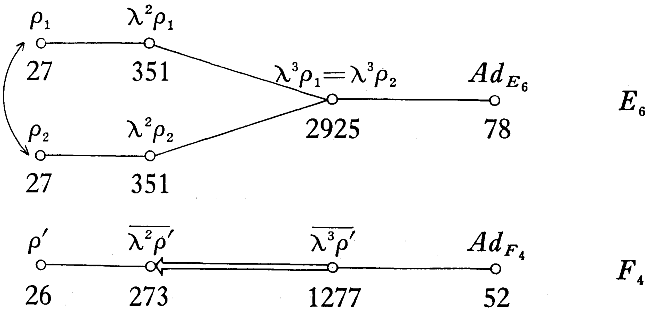
From (6.4) we see that i^* is surjective and the element $\Delta_{2n}^+ - \Lambda_{2n}^-$ holds the conditions required in (i) of Proposition 4.1 and therefore we have by Proposition 4.1

Propositiom 6.2. *The notation being as in* (6.3),

$$K^*(Spin(2n)/Spin(2n-1)) = \Lambda(\beta(\Delta_{2n}^+ - \Delta_{2n}^-))\,.$$

Type *EIV* ($M = E_6/F_4$). We look at the Dynkin diagrams of E_6 and F_4 with the irreducible representations corresponding to the vertexes and their dimensions written next to the vertexes:

(6.5)



where $\overline{\lambda^k\rho'}$ ($k=1$ or 2) is the greatest component of $\lambda^k\rho'$ (See Supplement and Table 30 of [4]).

The involutive automorphism σ of E_6 for EIV is the normal extension of a symmetry of the Dynkin diagram E_6 indicated by the arrow in the diagram (6.5) (See [11, II], *p*. 130). Hence it follows immediately that

$$\sigma^*(\lambda^k\rho_1) = \lambda^k\rho_2 \ (1\leqq k\leqq 3) \quad \text{and} \quad \sigma^*(Ad_{E_6}) = Ad_{E_6}\,. \tag{6.6}$$

Consider the highest weights of ρ_1 and ρ' and their dimensions, then we get

$$i^*(\rho_1) = \rho'+1 \tag{6.7}$$

and moreover we obtain

$$i^*(Ad_{E_6}) = Ad_{F_4}+\rho' \tag{6.8}$$

by enumerating the all weights of the adjoint representations Ad_{E_6} and Ad_{F_4}.

From (6.7) and (6.8) we see that $i^*\colon R(E_6)\to R(F_4)$ is surjective because of $R(F_4)=Z[\rho', \lambda^2\rho', \lambda^3\rho', Ad_{F_4}]$ [15], and therefore from (6.6) and Proposition 4.1 follows

Proposition 6.3 (Cf. [5]). *The notation being as in the diagram* (6.5),

$$K^*(E_6/F_4) = \Lambda(\beta(\rho_1-\rho_2),\ \beta(\lambda^2\rho_1-\lambda^2\rho_2))\,.$$

7. Proofs for *BDI(a)* and *EI*

Let L be a compact connected Lie group, H be a closed connected subgroup of maximal rank of L and $j\colon H\to L$ the inclusion of H. Then,

Proposition 7.1. *For a compact L-space X, there is a natural homomorphism of $K_L^*(X)$-modules $j_*\colon K_H^*(X)\to K_L^*(X)$ such that $j_*(1)=1$, and therefore j_*j^* is an identity isomorphism where j^* is the restriction $K_L^*(X)\to K_H^*(X)$.*

Proof. The proof is immediate from Proposition (3.8) of [13].

Type $BDI(a)$ ($M = Spin(2m+2n+2)/Spin(2m+1)\underset{Z_2}{\times} Spin(2n+1)$). σ is given by

$$\sigma(g) = -(e_1 \cdots, e_{2m+1})g(e_{2m+1}\cdots e_1)$$

for any $g \in Spin(2m+2n+2)$ where e_k is the generator of the Clifford algebra C_{2m+1} in the k-th position for $k=1, \cdots, 2m+1$ ([10], §11). Then we have

$$(7.1)\qquad \begin{aligned} &\sigma^*(\lambda^k(\rho_{2m+2n+2})) = \lambda^k(\rho_{2m+2n+2}) \qquad (1 \leqq k \leqq m+n-1), \\ &\sigma^*(\Delta^{\pm}_{2m+2n+2}) = \Delta^{\pm}_{2m+2n+2} \end{aligned}$$

using the notation in (6.3).

Put $G=Spin(2m+2n+2)$, $K=Spin(2m+1)\underset{Z_2}{\times} Spin(2n+1)$, $G_1=Spin(2m+2n+1)$ and $K_1=Spin(2m+1)\underset{Z_2}{\times} Spin(2n)$, then we have an isomorphism induced by the external product homomrphism

$$(7.2)\qquad K^*(G/G_1) \underset{R(G_1)}{\otimes} R(K_1) \cong K^*(G/K_1)$$

by use of the Künneth formula spectral sequence in $K^*_{G_1}$ ([8], [9] and [12]). Furthermore, since the restriction $R(G) \to R(G_1)$ is surjective, we have isomorphisms

$$(7.3)\qquad \begin{aligned} K^*(G/K_1) &\cong K^*(G/G_1) \otimes (Z \underset{R(G_1)}{\otimes} R(K_1)) && \text{by (7.2)} \\ &\cong \Lambda(\beta(\Delta^+_{2m+2n+2} - \Delta^-_{2m+2n+2})) \otimes (Z \underset{R(G_1)}{\otimes} R(K_1)) && \text{by Prop. 6.2} \\ &\cong \Lambda(\beta(\Delta^+_{2m+2n+2} - \Delta^-_{2m+2n+2})) \otimes (Z \underset{R(G)}{\otimes} R(K_1)) \end{aligned}$$

Let $j^*: K^*(G/K) \to K^*(G/K_1)$ be the homomorphism induced by the projection $G/K_1 \to G/K$ and $j_*: K^*(G/K_1) \to K^*(G/K)$ the homomorphism of $K^*(G/K)$-modules mentioned in Proposition 7.1. $K^*(G/K_1)$ is torsion-free by (7.3) and j^* is injective by the property of j_*. Therefore,

(7.4) $\qquad K^*(G/K)$ is torsion-free.

Here we have a natural homomorphism of rings

$$\varphi: \Lambda(\beta(\Delta^+_{2m+2n+2} - \Delta^-_{2m+2n+2})) \otimes (Z \underset{R(G)}{\otimes} R(K)) \to K^*(G/K)$$

which is well-defined by (7.4). Then φ is injective because $j^*\varphi$ is so, and also it is easy to see that φ is surjective by the fact that $j_*j^* = \text{identity}$. Hence we conclude that

Proposition 7.2. *The notation being as in* (6.3), $K^*(Spin(2m+2n+2)/Spin(2m+1)\underset{Z_2}{\times} Spin(2n+1))$ *is torsion-free and equals the ring*

$$\Lambda(\beta(\Delta^{+}_{2m+2n+2}-\Delta^{-}_{2m+2n+2}))\otimes(Z\underset{R(\mathrm{Spin}(2m+2n+2))}{\otimes}R(Spin(2m+1)\underset{Z_2}{\times}Spin(2n+1)))\,.$$

Type EI ($M=E_6/PS(4)$). σ is the composition of the involutive automorphism of E_6 for EIV and the inner automorphism (See [11, II], p. 131). So we have from (6.6)

$$\sigma^*(\lambda^k\rho_1)=\lambda^k\rho_2\ (1\leqq k\leqq 3)\quad\text{and}\quad \sigma^*(\mathrm{A}d_{E_6})=Ad_{E_6} \tag{7.5}$$

using the notation the diagram (6.5).

From the argument in p. 131 of [11, II] we know that E_6 has $Sp(3)\underset{Z_2}{\times}SU(2)$ as a subgroup which is contained in $PSp(4)$ and F_4 where Z_2 is the subgroup of $Sp(3)\times SU(2)$ consisting of (1,1) and $(-1,-1)$.

Setting $G=E_6$, $K=PSp(4)$, $G_1=F_4$ and $K_1=Sp(3)\underset{Z_2}{\times}SU(2)$ the similar argument to $BDI(a)$ shows that

Proposition 7.3. *The notation being as in the diagram* (6.5), $K^*(E_6/PSp(4))$ *is torsion-free and equals the ring*

$$\Lambda(\beta(\rho_1-\rho_2),\ \beta(\lambda^2\rho_1-\lambda^2\rho_2))\otimes(Z\underset{R(E_6)}{\otimes}R(PSp(4)))\,.$$

8. Proof for AI

In the case of AI-type, σ is given by $\sigma(g)=\bar{g}$ for any $g\in SU(n)$ where $\bar{g}$ is the complex conjugate of g.

From Theorems 3.1 and 10.3 in §13 of [10], we have

$$\begin{aligned} &R(SU(n))=Z[\lambda_1,\cdots,\lambda_{n-1}]\,,\\ &R(SO(2m+1))=Z[\lambda_1,\cdots,\lambda_m]\quad\text{where }\lambda_k=\lambda^k(\rho_{2m+1})\ (1\leqq k\leqq m)\,,\\ &R(SO(2m))=Z[\lambda_1,\cdots,\lambda_{m-1},\lambda_m^+,\lambda_m^-]/\sim\quad\text{where }\lambda_k=\lambda^k(\rho_{2m})\\ &(1\leqq k\leqq m-1)\quad\text{and}\quad \lambda_m^{\pm}=\lambda_{\pm}^m(\rho_{2m}) \end{aligned} \tag{8.1}$$

using the same notation.

First we consider the case when n is odd. Put $n=2m+1$. Then,

$$i^*(\lambda_k)=i^*(\lambda_{2m+1-k})=\lambda_k\quad\text{and}\quad \sigma^*(\lambda_k)=\lambda_{2m+1-k}\ \text{ for }\ k=1,\cdots,m \tag{8.2}$$

clearly.

Using the Koszul complex

$$C^*=\Lambda(x_1,\cdots,x_{2m})\otimes R(SU(2m+1))$$

where $d(x_k)=\tilde{\lambda}_{2m+1-k}(=\lambda_{2m+1-k}-\varepsilon(\lambda_{2m+1-k}))$ $(1\leqq k\leqq 2m)$ and d is a derivation, we shall show

$$\mathrm{Tor}^{*,0}_{R(SU(2m+1))}(Z,\ SO(2m+1))=\Lambda(x_1-x_{2m},\cdots,x_m-x_{m+1})\,. \tag{8.3}$$

For a proof we define E_l $(1 \leqq l \leqq 2m)$ to be the subcomplex $\Lambda_{R(SO(2m+1))}(x_1, \cdots, x_l)$ of $C^* \underset{R(SU(2m+1))}{\otimes} R(SO(2m+1))$. Then there exist a natural short exact sequence of complexes

$$0 \rightarrow E_l \rightarrow E_{l+1} \rightarrow E_{l+1}/E_l \rightarrow 0$$

and an isomorphism of complexes

$$E_l \cong E_{l+1}/E_l$$

defined by the correspondence $z \rightarrow zx_{l+1}$, $z \in E_l$ for $=1, \cdots, 2m-1$. This permits us to apply the induction on l and then we obtain

$$H^*(E_l) = R(SO(2m+1))/(\tilde{\lambda}_1, \cdots, \tilde{\lambda}_l)\,,$$
$$H^*(E_{m+l}) = \Lambda(x_{m-l+1} - x_{m+l}, \cdots, x_m - x_{m+1})$$

for $l=1, \cdots, m$. Thus (8.3) is proved.

From [9], §10 it follows that the element $x_k - x_{2m+1-k}$ converges to $\beta(\lambda_k - \lambda_{2m+1-k})$ in the spectral sequence (1.2) for $k=1, \cdots, m$. Hence we have

Proposition 8.1. *The notation being as in* (8.1),

$$K^*(SU(2m+1)/SO(2\mathrm{m}+1)) = \Lambda(\beta(\lambda_1 - \lambda_{2m}), \cdots, \beta(\lambda_m - \lambda_{m+1}))\,.$$

In a similar way when n is even, we can prove the following

Proposition 8.2. *The notation being as in* (8.1),

$$K^*(SU(2m)/SO(2m)) = \Lambda(\beta(\lambda_1 - \lambda_{2m-1}), \cdots, \beta(\lambda_{m-1} - \lambda_{m+1})) \otimes \Lambda(\tilde{\lambda}_m^+)$$

where $\tilde{\lambda}_m^+ = \lambda_m^+ - \frac{1}{2}\binom{2m}{m}$.

This completes the proof of Theorem A.

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References

[1] M.F. Atiyah and F. Hirzebruch: *Vector bundles and homogeneous spaces*, Proc. of Symposia in Pure Math., Differential Geometry, Amer. Math. Soc., 1961, 7–38.

[2] M.F. Atiyah: *On the K-theory of compact Lie groups*, Topology **4** (1965), 95–99.

[3] S. Araki: *Hopf structures attached to K-theory: Hodgkin's theorem*, Ann. of Math. **85** (1967), 508–525.

[4] E.B. Dynkin: *Maximal subgroups of the classical groups*, Trudy Moskov. Mat. Obšč. **1** (1952), 39–166; Amer. Math. Soc. Transl. **(2) 6** (1957), 245–378.

[5] B. Harris: *The K-theory of a class of homogeneous spaces,* Trans. Amer. Math. Soc. **131** (1968) 323–332.
[6] S. Helgason: Differential Geometry and Symmetric Spaces, Academic Press, 1962.
[7] L. Hodgkin: *On the K-theory of Lie groups,* Topology **6** (1967), 1–36.
[8] ————: *An equivariant Künneth formula in K-theory,* University of Warwick preprint, 1968
[9] ————: *Künneth formula spectral sequence,* University of London King's College preprint, 1972.
[10] D. Husemoller: Fibre Bundles, McGraw-Hill, Inc., 1966.
[11] O. Loos: Symmetric Spaces I, II, W.A. Benjamin, Inc., 1969.
[12] H.V. Pittie: *Homogeneous vector bundles on homogeneous spaces,* Topology **11** (1972), 199–203.
[13] G. Segal: *Equivariant K-theory,* Inst. Hautes Etudes Sci. Publ. Math. (Paris) **34** (1968), 129–151.
[14] V.P. Snaith: *Massey products in K-theory* II, Proc. Camb. Phil. Soc. **69** (1971), 259–289.
[15] I. Yokota: *Exceptional Lie group F_4 and its representation rings,* J. Fac. Sci. Shinshu Univ. **3** (1968), 35–60.