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ON A GENERALIZATION OF THE THEOREM OF RIESZ-FROSTMAN-NEVANLINNA

To Professor Yukinari Tôki on the occasion of his 60th birthday

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Introduction

The original form of the theorem of Riesz-Frostman-Nevanlinna is stated as follows: *let* $f(z)$ *be regular and bounded in the unit disc. If* $\lim_{r\to 1} f(re^{i\theta})$ *is equal to zero on a subset of positive measure of* $|z|=1$, *then* $f(z)\equiv 0$. R. Nevanlinna[1) and O. Frostman[2) extended independently this theorem to the case of meromorphic functions of bounded type. However, if we consider arbitrary regular functions this theorem does not hold in general as the example of Lusin-Priwalow shows[3).

Meanwhile, it has been made known by the recent studies of Constantinescu-Cornea[4) that the boundary behavior of analytic maps of Riemann surfaces depends deeply on the harmonic character of maps. In [7], they developed this idea to maps of harmonic spaces satisfying the Brelot's axioms.

In this paper, we shall generalize the theorem of Riesz-Frostman-Nevanlinna for maps of a Green space into a harmonic space. Generalizations are done in some points. One of them is the use of cluster sets along Green lines issuing from a fixed point and of a Green measure instead of radial limits and the Lebesgue measure, respectively. However, an essential point is the validity of the theorem for Fatou maps which include all Lindelöfian maps[5) of hyperbolic Riemann surfaces[6).

In §1, we state the theorem and list up all notations which will be used in the sequel. §2 is devoted to auxiliary lemmas. They are needful to the proof of the theorem. The proof of the theorem is carried out in §3 divided into three

1) Cf. [13], p. 205.
2) Cf. [8], p. 96.
3) Cf. [11] and [14], p. 222.
4) Cf. [4] and [5].
5) Cf. [9].
6) Cf. [5], p. 113 and [4], p. 72.

cases. In the last section, as an application, we shall mention a result which is an improvement of my former one [10].

1. Preliminaries and the theorem

Let Ω be a Green space in the sense of Brelot-Choquet[7]. We consider the Green lines issuing from a fixed point y_0. They are the maximal orthogonal trajectories of

$$\sum^{\lambda} = \{y \in \Omega;\ G_{y_0}(y) = \lambda\}\ ,$$

where G_{y_0} is a Green function of Ω with a pole at y_0 and $0 < \lambda < G_{y_0}(y_0)$. We put

$$D^{\lambda} = \{y \in \Omega;\ G_{y_0}(y) > \lambda\} \qquad \text{for} \quad 0 < \lambda < G_{y_0}(y_0)\,.$$

On the set $\mathcal{L}$ of all Green lines, we can define a topology homeomorphic to the unit sphere and a Radon measure g called the *Green measure*. A Green line l is called *regular*, if $\inf\{G_{y_0}(y);\ y \in l\} = 0$. The set of all regular Green lines will be denoted by $\mathcal{L}'$.

Let X be a harmonic space in the sense of Brelot[8], i.e., X is locally compact, connected and on which it is given a sheaf of continuous functions, called harmonic functions, satisfying the axioms 1, 2 and 3 in [2]. A Green space is a harmonic space in an obvious way. We denote by $\mathcal{P}$ (resp. $\mathcal{H}$) the class of harmonic spaces on which there exists a positive potential (resp. a positive harmonic function). A continuous map φ of a harmonic space X into a second harmonic space X' is called a *harmonic map*, if for any open set $U' \subset X'$ and any harmonic function u' on U', $u' \circ \varphi$ is harmonic on $\varphi^{-1}(U')$.

Let U be an open subset of X, $U \in \mathcal{P}$ and f be a real function defined on U. We denote by $\overline{w}_f^U$ the set of hyperharmonic functions s on U such that

a) s possesses a non-positive subharmonic minorant

and

b) s dominates f outside a compact subset of U.

We denote by $\bar{h}_f^U = \inf\{s;\ s \in \overline{w}_f^U\}$. Also we define

$$\underline{w}_f^U = \{-s;\ s \in \overline{w}_{(-f)}^U\}$$

and

$$\underline{h}_f^U = \sup\{s;\ s \in \underline{w}_f^U\}\ .$$

If $\bar{h}_f^U = \underline{h}_f^U$ and is finite, f is called *harmonizable on* U. A finite continuous function f on X is called a *Wiener function*, if there exists an open set $U \in \mathcal{P}$ with

7) Cf. [3]. For the following facts we refer to [3].

8) Cf. [2]. In [2], it is assumed that X is not compact. In this paper we do not require this.

compact complement such that f is harmonizable on U. A harmonic map φ of X into X' is called a *Fatou map* if for any bounded Wiener function f' on X', $f'\circ\varphi$ is a Wiener function on X. All harmonic maps into $X'\in\mathcal{P}$ are Fatou maps. When $X\in\mathcal{P}$ and $X'\in\mathcal{H}-\mathcal{P}$, in order that a harmonic map φ of X into X' be a Fatou map, it is necessary and sufficient that there exists a closed non-polar set $F'\subset X'$ such that $\hat{R}_1^{\varphi^{-1}(F')}$ is a potential[9].

A compactification X^* of X is a compact space containing X as a dense open subset. A subset A of X^* is called *polar*, if for any domain $U\in\mathcal{P}$ of X there exists a positive superharmonic function s on U such that

$$\lim_{x\to z} s(x) = +\infty \qquad \text{for any} \quad z\in A\cap\bar{U}\,.$$

We shall list up the notations which will be used in the sequel.

Ω: a Green space.

$\{\Omega_n\}$: an exhaustion of Ω, i.e., Ω_n is a relatively compact domain satisfying $\bar{\Omega}_n\subset\Omega_{n+1}$ and $\bigcup_{n=1}^{\infty}\Omega_n=\Omega$.

X: a harmonic space with countable basis, on which there exists a positive superharmonic function, i.e., $X\in\mathcal{P}\cup\mathcal{H}$.

X^*: a compactification of X.

$\{X_n\}$: an exhaustion of X.

Λ: a set of regular Green lines.

$\{\lambda_n\}$: a decreasing sequence of positive numbers tending to zero.

φ: a harmonic map of Ω into X.

$\phi(\ell) = \bigcap_{n=1}^{\infty} \overline{\{\varphi((\lambda_m, \ell));\, m\geq n\}}$, where the closure is taken in the topology of X^*.

A: a polar set in X^*.

In this paper, we shall prove the following theorem, which is a generalization of the theorem of Riesz-Frostman-Nevanlinna.

Theorem. *Let X be a harmonic space. We assume the existence of*

(*) *a countable basis of open sets for X*

and

(**) *a superharmonic function with positive infimum on X. If X is compact, we assume further the existence of*

(***) *a non-polar subset E of X each point of which is polar.*

Let φ be a non-constant Fatou map of a Green space Ω into X and X^* be an arbitrary compactification of X.

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9) Cf. [7], p. 52, th. 6.1.

If there exist a set Λ *of regular Green lines issuing from* $y_0 \in \Omega$, *a decreasing sequence* $\{\lambda_n\}$ *of positive numbers tending to zero and a polar set* A *of* X^* *such that*

$$\phi(l) = \bigcap_{n=1}^{\infty} \overline{\{\varphi((\lambda_m, l)); m \geq n\}} \subset A \qquad \text{for any} \quad l \in \Lambda ,$$

then the outer Green measure of Λ *is zero, i.e.,* $g^*(\Lambda)=0$, *where* (λ_m, l) *denotes the point of* l *on which the value of a Green function* G_{y_0} *is* λ_m, *and the closure is taken in* X^*.

2. Lemmas

2. 1. To prove the theorem stated above, we require some lemmas, which will be given in this section. Throughout this section we shall suppose

$$g^*(\Lambda) > 0 .$$

Lemma 1. *Assume that* X *is non-compact and*

$$\phi(l) \subset A \qquad \text{for any} \quad l \in \Lambda .$$

Then, there exists a sequence $\{D_n\}$ *of relatively compact domains in* X *such that*

$$g^*(\{l \in \Lambda ; \phi(l) \cap \bar{D}_n = \phi\}) > 1/2\, g^*(\Lambda) \qquad (n = 1, 2, \cdots) \tag{2. 1}$$

and

$$(\varphi(\bar{\Omega}_n) \cup \bigcup_{k=1}^{n-1} \bar{D}_k \cup \bar{X}_n) \cap \bar{D}_n = \phi \qquad (n = 1, 2, \cdots) . \tag{2. 2}$$

Proof. Suppose defined $D_1, D_2, \cdots, D_{p-1}$, relatively compact domains and let (2. 1) and (2. 2) hold for $n = 1, 2, \cdots, p-1$. Since X is not compact,

$$X - (\varphi(\bar{\Omega}_p) \cup \bigcup_{k=1}^{p-1} \bar{D}_k \cup \bar{X}_p)$$

is an open non-empty set, so that it is non-polar. Since $A \cap X$ is polar, there exists x such that

$$x \in X - (\varphi(\bar{\Omega}_p) \cup \bigcup_{k=1}^{p-1} \bar{D}_k \cup \bar{X}_p \cup A) . \tag{2. 3}$$

From the second axiom of countability for X[10], we have a sequence $\{E_n\}$ of relatively compact neighbourhoods of x such that

$$\left\{ \begin{array}{l} \bar{E}_{m+1} \subset E_m \qquad (m = 1, 2, \cdots) , \\ \bar{E}_1 \cap [\varphi(\bar{\Omega}_p) \cup \bigcup_{k=1}^{p-1} \bar{D}_k \cup \bar{X}_p] = \phi , \\ \bigcap_{m=1}^{\infty} \bar{E}_m = \{x\} . \end{array} \right. \tag{2. 4}$$

10) It is true that the same conclusion is derived from the first axiom of countability for X. However, in the harmonic space X the two countability axioms are equivalent. (Cf. [6]).

Put

$$L_m = \{\ell \in \Lambda;\ \phi(\ell) \cap \bar{E}_m = \phi\}$$

for $m=1, 2, \cdots$. It is easy to see

$$L_1 \subset L_2 \subset \cdots \tag{2. 5}$$

and

$$\bigcup_{m=1}^{\infty} L_m = \Lambda . \tag{2. 6}$$

In fact, if there exists $\ell \in \Lambda$ such that $\ell \notin L_m$ for all m, then

$$\phi(\ell) \cap \bar{E}_m \neq \phi \qquad (m = 1, 2, \cdots) .$$

Since X^* is compact this means

$$\phi(\ell) \cap \bigcap_{m=1}^{\infty} \bar{E}_m \neq \phi ,$$

so that by (2. 4)

$$x \in \phi(\ell) \subset A ,$$

which contradicts (2. 3).

From (2. 5), (2. 6) and the regularity of the outer Green measure[11] we have

$$\lim_{m\to\infty} g^*(L_m) = g^*(\Lambda) .$$

Thus, we have an m_0 such that $g^*(L_{m_0}) > (1/2) g^*(\Lambda)$. $E_{m_0} = D_p$ is the desired one, q.e.d..

Lemma 2. *Suppose that X is non-compact and $X \in \mathcal{H} - \mathcal{P}$, and φ is a Fatou map of Ω into X. Let $\{D_n\}$ be a sequence of relatively compact domains satisfying* (2. 2). *Then, there exists a closed subset F of X such that $\hat{R}_1^{\varphi^{-1}(F)}$*[12] *is a potential, $F_n = F \cap D_n$ is non-polar and compact, and*

$$\lim_{n\to\infty} \hat{R}_1^{\varphi^{-1}(F_n)} = 0 \qquad on \quad \Omega . \tag{2. 7}$$

Proof. Let f_n be a non-negative continuous function on X whose support is in D_n and whose maximum is 1. Since $\{D_n\}$ are mutually disjoint and do not cluster at any point of X, $f = \sum_{n=1}^{\infty} f_n$ is a non-negative bounded continuous function on X, so that f is a Wiener function on X[13]. From the definition of a Fatou map, $\bar{f} = f \circ \varphi$ is a Wiener function on Ω. In virtue of a theorem of Constantinescu-Cornea[14] $\hat{R}_1^{N_\alpha}$ is a potential except for countable values of α,

11) Cf. [15], p. 51.
12) Cf. [2], p. 80, def. 9.
13) Cf. [7], p. 16.

where

$$N_\alpha = \{y \in \Omega; \bar{f}(y) = \alpha\} .$$

We may take α so that $0<\alpha<1$ and $\hat{R}_1^{N_\alpha}$ is a potential. Let us write

$$F = \{x \in X; f(x) = \alpha\} .$$

$\hat{R}_1^{\varphi^{-1}(F)}$ is a potential and $F_n = F \cap D_n$ is compact and non-polar, since

$$F \cap D_n = \{x \in X; f_n(x) = \alpha\}$$

and $X-(F \cap D_n)$ is not connected.

For any $y \in \Omega$ there exists n_0 such that $y \in \Omega_n$ for any $n \geq n_0$. $\varphi^{-1}(F_n) \cap \Omega_n = \phi$ implies that $\hat{R}_1^{\varphi^{-1}(F_n)}$ is bounded and harmonic in Ω_n.
Hence,

$$\hat{R}_1^{\varphi^{-1}(F_n)}(y) = H^{\Omega_n}_{\hat{R}_1^{\varphi^{-1}(F_n)}}(y) \leq H^{\Omega_n}_{\hat{R}_1^{\varphi^{-1}(F)}}(y) .$$

Since $\hat{R}_1^{\varphi^{-1}(F)}$ is a potential, the last term tends to zero as $n \to \infty$, q.e.d..

2. 2.

Let us take $y_1 \in \Omega$ such that $\varphi(y_1) \notin A \cup F$ and we shall fix it. This is possible for $\varphi^{-1}(X \cap A)$ is polar. By the Harnack's inequality we can find $K>1$ satisfying

$$K\, u(y_0) \geq u(y_1) \geq 1/K\, u(y_0) \tag{2. 8}$$

for all non-negative harmonic functions u on Ω.

Let δ be a positive number less than 1/4. By Lemma 2, we have n_0 such that

$$\hat{R}_1^{\varphi^{-1}(F_{n_0})}(y_1) < \delta/(8K)g^*(\Lambda) . \tag{2. 9}$$

Since F_{n_0} is non-polar, each component of $X-F_{n_0} \in \mathcal{P}$. There exists a positive superharmonic function v on $X-F_{n_0}$ such that

$$\begin{cases} \lim\limits_{x \to x'} v(x) = +\infty \qquad \text{for any} \quad x' \in A \cap \overline{(X-F_{n_0})}, \\ v[\varphi(y_1)] < 1/(4K)g^*(\Lambda) . \end{cases} \tag{2. 10}$$

Lemma 3. *Let E be a closed subset of Ω such that $\hat{R}_1^E$ is a potential. Then, for a fixed $y_1 \in \Omega$ we have*

(2. 11) $\hat{R}_1^E(y_1) = \inf \{\hat{R}_1^{\omega \cap \Omega}(y_1);$ *ω is an open subset of Ω_W^* containing $\bar{E}$*$\}$, *where Ω_W^* is a Wiener compactification*[15] *of Ω and the closure is taken in Ω_W^*.*

14) Cf. [7], p. 14, th. 2.6. .
15) Cf. [5], p. 98 and [7], p. 43.

Proof. Denoting by α_1 the right-hand side of (2. 11), we have clearly

$$\alpha_0 = \hat{R}_1^E(y_1) \leq \alpha_1 .$$

Suppose $\alpha_0 < \alpha_1$. In the first place, we shall show that there exists an open subset G_1 of Ω_W^* such that

$$\bar{E} \cap \Delta_W{}^{15)} \subset G_1$$

and

$$R_1^{G_1 \cap \Omega}(y_1) < (\alpha_1 - \alpha_0)/4 .$$

In fact, since $\hat{R}_1^E$ is a potential $\bar{E} \cap \Delta_W \cap \Gamma_W = \phi$[16], we have an open subset G of Ω_W^* such that

$$\bar{E} \cap \Delta_W \subset G \quad \text{and} \quad \bar{G} \cap \Gamma_W{}^{15)} = \phi .$$

$\hat{R}_1^{G \cap \Omega}$ is a potential. Hence we have an Ω_n such that

$$(\alpha_1 - \alpha_0)/4 > H^{\Omega_{n-1}}_{\hat{R}_1^{G \cap \Omega}}(y_1) \geq H^{\Omega_{n-1}}_{\hat{R}_1^{(G - \bar{\Omega}_n) \cap \Omega}}(y_1) = \hat{R}_1^{(G - \bar{\Omega}_n) \cap \Omega}(y_1) .$$

We have $G_1 = G - \bar{\Omega}_n$.

On the other hand, since $R_1^E(y_1)$ defines a capacity in the sense of Choquet[17], there exists an open subset ω_0 of Ω such that

$$E \subset \omega_0 \quad \text{and} \quad R_1^{\omega_0}(y_1) \leq (\alpha_0 + \alpha_1)/2 .$$

$\omega = \omega_0 \cup G_1$ is open in Ω_W^*, $\bar{E} \subset \omega$ and

$$\begin{aligned} R_1^{\omega \cap \Omega}(y_1) &\leq R_1^{\omega_0}(y_1) + R_1^{G_1 \cap \Omega}(y_1) \leq (\alpha_0 + \alpha_1)/2 + (\alpha_1 - \alpha_0)/4 \\ &< \alpha_1 , \end{aligned}$$

which contradicts the definition of α_1. Hence $\alpha_0 = \alpha_1$, q.e.d..

Lemma 4. *Let E be a closed subset of Ω. If $\hat{R}_1^E$ is a potential and $\hat{R}_1^E(y_1) < \alpha$, then there exists a closed non-polar subset Q of Ω such that $E \subset Q$, $\hat{R}_1^Q$ is a continuous potential and*

$$R_1^Q(y_1) < \alpha . \tag{2. 12}$$

Proof. The proof is obvious if E is empty. We assume E is not empty. By Lemma 3, we have an open subset ω of Ω_W^* such that

$$\bar{E} \subset \omega \quad \text{and} \quad \hat{R}_1^{\omega \cap \Omega}(y_1) < \alpha .$$

We have also an open subset ω_1 of Ω_W^* such that

16) Cf. [7], p. 45, th. 5.6. .

17) Cf. [2], p. 122. .

$$\bar{E} \cap \Delta_W \subset \omega_1, \quad \omega_1 \subset \omega \quad \text{and} \quad \bar{\omega}_1 \cap \Gamma_W = \phi .$$

$E-\omega_1$ is compact. Denoting by ω_2 a relatively compact neighbourhood of $E-\omega_1$ whose closure is contained in ω and putting

$$G = (\omega_1 \cap \Omega) \cup \omega_2$$

we have

$$\hat{R}_1^{G \cap \Omega} \leq \hat{R}_1^{\omega_1 \cap \Omega} + \hat{R}_1^{\omega_2} .$$

Since $\bar{G} \cap \Delta_W \subset (\bar{\omega}_1 \cap \Delta_W) \cup (\bar{\omega}_2 \cap \Delta_W) = \bar{\omega}_1 \cap \Delta_W \subset \Delta_W - \Gamma_W$, $\hat{R}_1^{G \cap \Omega}$ is a potential and $\hat{R}_1^{G \cap \Omega}(y_1) \leq \hat{R}_1^{\omega \cap \Omega}(y_1) < \alpha$.

Next, for each point y of $\bar{\Omega}_2 \cap E$ we assign a regular neighbourhood[18] of y contained in $G \cap \Omega_3$. A finite number of them, say V_i $(1 \leq i \leq m_1)$, covers $\bar{\Omega}_2 \cap E$. In general, for each point y of $(\bar{\Omega}_{n+1} - \Omega_{n-1}) \cap E$ we assign a regular neighbourhood of y contained in $G \cap (\Omega_{n+2} - \bar{\Omega}_{n-2})$ and cover $(\bar{\Omega}_{n+1} - \Omega_{n-1}) \cap E$ by a finite number of them, say $V_i (m_{n-1} + 1 \leq i \leq m_n)$. Put $Q = \bigcup_{i=1}^{\infty} V_i$. Q is closed since $\{V_i\}$ is locally finite. It is clear $E \subset Q \subset G$, therefore $\hat{R}_1^Q$ is a potential and $\hat{R}_1^Q(y_1) < \alpha$. Since Q is not thin at every boundary point y of $\Omega - Q$, y is regular for $\Omega - Q$ with respect to the Dirichlet problem. Thus $\hat{R}_1^Q$ is continuous, q.e.d..

2. 3.

By Lemma 4 we may construct a continuous potential $p = \hat{R}_1^Q$ for $E = \varphi^{-1}(F_{n_0})$ and $\alpha = \delta/(8K) g^*(\Lambda)$ (see (2. 9)). Put

$$V_0 = \{y \in \Omega;\, p(y) > 1 - \delta\} ,$$
$$V_1 = \{y \in \Omega;\, p(y) > 1 - 2\delta\}$$

and

$$p_1 = \min(v \circ \varphi, 1) \qquad \text{(see (2. 10))}$$

$\partial V_0 = \{x \in \Omega;\, p(y) = 1 - \delta\}$ and each point of ∂V_0 is regular for $\Omega - \bar{V}_0$ with respect to the Dirichlet problem. p_1 is superharmonic on each component of $\Omega - \varphi^{-1}(F_{n_0})$ and $0 < p_1 \leq 1$.

Lemma 5.

$$s = \begin{cases} (1-\delta)/\delta & on \quad \bar{V}_0 \\ (1-\delta)/\delta \cdot \hat{R}_1^{V_0} + (\widehat{R_{p_1}^{\Omega - V_1}})_{\Omega - \bar{V}_0}{}^{19)} & on \quad \Omega - \bar{V}_0 \end{cases}$$

18) We use this terminology in the following sense: a neighbourhood V is regular if it is compact and its local image is a sphere. If V is regular, then both V and $\Omega - V$ are not thin at each point of ∂V.

19) Cf. [2], p. 82, def. 10.

is a superharmonic function on Ω. *Especially, on* $\Omega-\bar{V}_1$ *we have*

$$s \geq \min(v\circ\varphi, 1).$$

Proof. $(1-\delta)/\delta-p/\delta$ is a positive superharmonic function on $\Omega-\bar{V}_0$ and ≥ 1 on $\Omega-V_1$. Therefore we have

$$(1-\delta)/\delta-p/\delta \geq (\widehat{R_{p_1}^{\Omega-V_1}})_{\Omega-\bar{V}_0} \quad on \quad \Omega-\bar{V}_0.$$

On $\Omega-\bar{V}_0$, we have further

$$\begin{aligned}&(1-\delta)/\delta-(1-\delta)/\delta\cdot\hat{R}_1^{\bar{V}_0}-(\widehat{R_{p_1}^{\Omega-V_1}})_{\Omega-\bar{V}_0}\\ &\geq(1-\delta)/\delta-(1-\delta)/\delta\cdot\hat{R}_1^{\bar{V}_0}-[(1-\delta)/\delta-p/\delta]\\ &\geq(1-\delta)/\delta-(1-\delta)/\delta\cdot p/(1-\delta)-[(1-\delta)/\delta-p/\delta]=0,\end{aligned}$$

since $p\geq 1-\delta$ on $\bar{V}_0$. This means

$$(1-\delta)/\delta \geq (1-\delta)/\delta\ \hat{R}_1^{\bar{V}_0}+(\widehat{R_{p_1}^{\Omega-V_1}})_{\Omega-\bar{V}_0} \quad on \quad \Omega-\bar{V}_0. \tag{2. 13}$$

From the regularity of each point $y\in\partial V_0$ for $\Omega-\bar{V}_0$ we have

$$\lim_{z\to y}[(1-\delta)/\delta\ \hat{R}_1^{\bar{V}_0}(z)+(\widehat{R_{p_1}^{\Omega-V_1}})_{\Omega-\bar{V}_0}(z)]=(1-\delta)/\delta,$$

so that s is continuous on ∂V_0. Combining this with (2. 13), s is superharmonic. On $\Omega-\bar{V}_1$, we have

$$s\geq(\widehat{R_{p_1}^{\Omega-V_1}})_{\Omega-\bar{V}_0}=p_1=\min(v\circ\varphi, 1), \quad \text{q.e.d.}.$$

2. 4.

We shall put

$$A_n=\{l\in\mathcal{L}'; (\lambda_n, l)\in\Omega-\bar{V}_1\}$$

and

$$\tilde{A}_n=\{(\lambda_n, l); l\in A_n\}.$$

$\tilde{A}_n$ is the set of points on which a Green line of A_n intersects $\sum^{\lambda_n}$. Correspondingly, we put

$$B_n=\{l\in\mathcal{L}'; (\lambda_n, l)\in\bar{V}_1\}$$

and

$$\tilde{B}_n=\{(\lambda_n, l); l\in B_n\}.$$

It is known that A_n (resp. B_n) differs from an analytic set only in a set of dg-measure zero. They are dg-measurable. The difference between $\tilde{B}_n$ and $\bar{V}_1\cap\sum^{\lambda_n}$ is within a set of $d\,\omega_{y_0}^{\lambda_n}$-measure[20] zero.

20) $\omega_{y_0}^{\lambda_n}$ denotes a harmonic measure on $\sum^{\lambda_n}$ with respect to D^{λ_n} and y_0.

Lemma 6.

$$\lim_{n\to\infty} g(B_n) = 0 .$$

Proof.
$$\begin{aligned} g(B_n) &= \int_{B_n} dg = \int_{\tilde{B}_n} d\omega_{y_0}^{\lambda_n} = \int_{\Sigma^{\lambda_n}\cap\bar{V}_1} d\omega_{y_0}^{\lambda_n} \\ &\leq \int_{\Sigma^{\lambda_n}\cap\bar{V}_1} p/(1-2\delta)\, d\omega_{y_0}^{\lambda_n} \\ &\leq \int_{\Sigma^{\lambda_n}} p/(1-2\delta)\, d\omega_{y_0}^{\lambda_n} \\ &\leq 1/(1-2\delta) H_p^{D^{\lambda_n}}(y_0) . \end{aligned}$$

Since p is a potential, we have $\lim_{n\to\infty} H_p^{D^{\lambda_n}}(y_0)=0$, q.e.d..

Put

$$U = \{x \in X - F_{n_0};\, v(x) > 1\}$$

and

$$\Lambda' = \{\ell \in \Lambda;\, \phi(\ell) \subset A - \bar{D}_{n_0}\} ,$$

where D_{n_0} is the domain defined in Lemma 1. $A-\bar{D}_{n_0}$ is a polar subset of X^* and by Lemma 1 we have

$$g^*(\Lambda') > 1/2 g^*(\Lambda) . \tag{2. 14}$$

Lemma 7. *If we put*

$$C_n = \{\ell \in \Lambda';\, \varphi((\lambda_m, \ell)) \in U \quad \textit{for any} \quad m \geq n\} ,$$

then we have

$$C_1 \subset C_2 \subset \cdots \quad \textit{and} \quad \bigcup_{n=1}^{\infty} C_n = \Lambda' \tag{2. 15}$$

and

$$\lim_{n\to\infty} g^*(A_n \cap C_n) = g^*(\Lambda') . \tag{2. 16}$$

Proof. First, we shall prove (2. 15). The first part is obvious from the definition of C_n. Suppose we have $\ell \in \Lambda'$ such that $\ell \notin C_n$ for any n. Then, there should exist numbers $\{\nu_n\}$ satisfying $\nu_n \geq n$ and $\varphi((\lambda_{\nu_n}, \ell)) \notin U$. From

$$\lim_{x\to x'} v(x) = +\infty \qquad \text{for any} \quad x' \in \phi(\ell)$$

we have an open neighbourhood W of $\phi(\ell)$ in X^* such that

$$v > 2 \qquad \text{on} \quad W \cap X .$$

Then, we have an n such that $W \supset \overline{\{\varphi((\lambda_m, \ell));\, m \geq n\}}$, so that

$$v[\varphi((\lambda_m, \ell))] \geq 2 \qquad \text{for any} \quad m \geq n\,.$$

On the other hand, $v[\varphi((\lambda_m, \ell))] \leq 1$ for infinitely many m, which leads to a contradiction.

To prove (2. 16) we shall remark

$$\lim_{n\to\infty} g^*(C_n) = g^*(\Lambda')\,. \tag{2. 17}$$

This is an immediate consequence of (2. 15) and the regularity of g^*. Since A_n is dg-measurable we have

$$g^*(C_n) = g^*(C_n \cap A_n) + g^*(C_n - A_n)\,.$$

$\mathcal{L} - A_n$ and B_n differ in a set of dg-measure zero each other, so that we have

$$\begin{aligned} g^*(C_n) &= g^*(C_n \cap A_n) + g^*(C_n \cap B_n) \\ &\leq g^*(C_n \cap A_n) + g^*(B_n)\,. \end{aligned} \tag{2. 18}$$

Letting $n \to \infty$ in (2. 18) and in view of Lemma 6 and (2. 17) we have

$$\begin{aligned} g^*(\Lambda') &\leq \varliminf_{n\to\infty} g^*(C_n \cap A_n) \leq \varlimsup_{n\to\infty} g^*(C_n \cap A_n) \\ &\leq \lim_{n\to\infty} g^*(C_n) = g^*(\Lambda'), \text{ q.e.d..} \end{aligned}$$

3. The proof of the theorem

3. 1. In this section, we shall give the proof of the theorem stated in § 1. We consider three cases: (1) $X \in \mathcal{H} - \mathcal{P}$ and non-compact, (2) $X \in \mathcal{H} - \mathcal{P}$ and compact, and (3) $X \in \mathcal{P}$.

The proof of the case (1). Suppose, on the contrary, $g^*(\Lambda) > 0$. Denoting by s the function defined in Lemma 5,

$$s(y) \geq H_s^{D^\lambda}(y) \qquad \text{for all} \quad \lambda\,.$$

Therefore

$$s(y) \geq \lim_{n\to\infty} H_s^{D^{\lambda_n}}(y) = u(y)\,,$$

where $u(y)$ is the best harmonic minorant[21] of s in Ω. It is clear that u is non-negative. We assert u is positive and $u(y_0) \geq g^*(\Lambda')$ (for the definition of Λ', see 2. 4, § 2). In fact,

$$\begin{aligned} u(y_0) &= \lim_{n\to\infty} H_s^{D^{\lambda_n}}(y_0) = \lim_{n\to\infty} \int_{\Sigma^{\lambda_n}} s\, d\omega_{y_0}^{\lambda_n} \\ &\geq \varliminf_{n\to\infty} \int_{\Sigma^{\lambda_n} \cap (\Omega - \overline{V}_1)} \min(v \circ \varphi, 1)\, d\omega_{y_0}^{\lambda_n} \qquad \text{(by Lemma 5)} \end{aligned}$$

21) Cf. [1], p. 434.

$$= \varliminf_{n\to\infty} \int_{\tilde{A}_n} \min(v\circ\varphi,\ 1)\, d\omega_{y_0}^{\lambda_n}$$

$$\geq \varlimsup_{n\to\infty} \int_{\widetilde{A_n\cap C_n}} \min(v\circ\varphi,\ 1)\, d\omega_{y_0}^{\lambda_n},$$

where $\widetilde{A_n\cap C_n} = \{(\lambda_n, \ell);\ \ell\in A_n\cap C_n\}$. Since $(\lambda_n, \ell)\in\widetilde{A_n\cap C_n}$ implies $\varphi((\lambda_n, \ell))\in U$, so that $v[\varphi((\lambda_n, \ell))]>1$, the last term is equal to

$$\varlimsup_{n\to\infty} \int_{\widetilde{A_n\cup C_n}} d\omega_{y_0}^{\lambda_n}$$

$$= \varlimsup_{n\to\infty} \int_{A_n\cap C_n} dg = \lim_{n\to\infty} g^*(A_n\cap C_n) = g^*(\Lambda') \qquad \text{(by Lemma 7)}.$$

Thus, we have $u(y_0)\geq g^*(\Lambda')$.

From (2. 8) and (2. 14)

$$\begin{aligned} s(y_1) &\geq u(y_1)\geq 1/K\, u(y_0) \\ &\geq 1/K\, g^*(\Lambda')\geq 1/(2K) g^*(\Lambda)\,. \end{aligned} \tag{3. 1}$$

On the other hand, in virtue of (2. 12)

$$\begin{aligned} p(y_1) = \hat{R}_1^{Q}(y_1) &< \delta/(8K) g^*(\Lambda) \\ &\leq \delta/(8K) < 1-2\delta\,, \end{aligned}$$

which means $y_1\in\Omega-\bar{V}_1$. Therefore

$$\begin{aligned} s(y_1) &= (1-\delta)/\delta\cdot \hat{R}_1^{\bar{V}_0}(y_1) + (\widehat{\hat{R}_{p_1}^{\Omega-V_1}})_{\Omega-\bar{V}_0}(y_1) \\ &\leq (1-\delta)/\delta\cdot p(y_1)/(1-\delta) + p_1(y_1) \\ &= 1/\delta\, p(y_1) + \min(v[\varphi(y_1)],\ 1) \\ &\leq 1/\delta\cdot\delta/(8K) g^*(\Lambda) + 1/(4K) g^*(\Lambda) \qquad \text{(by (2. 10))} \\ &\leq 3/(8K) g^*(\Lambda)\,. \end{aligned}$$

This contradicts (3. 1). Hence we conclude $g^*(\Lambda)=0$.

3. 2. *The proof of the case* (2)

Next, we proceed to the case (2). From our assumption (***), we have a non-polar set E each point of which is polar.

$$E-(A\cup\{\varphi(y_0)\})\neq\phi\,,$$

for if $E\subset A\cup\{\varphi(y_0)\}$, then $A\cup\{\varphi(y_0)\}$ is non-polar. Since A is polar, this implies $\varphi(y_0)\notin E$, so that $E\subset A$, which is absurd. Let us take $x_0\in E-(A\cup\{\varphi(y_0)\})$. $\varphi^{-1}(\{x_0\})$ is a polar subset of Ω. Let us write

$$\Omega_0 = \Omega-\varphi^{-1}(\{x_0\}),\ X_0 = X-\{x_0\}$$

and let φ_0 denote the restriction of φ on Ω_0. φ_0 is a Fatou map of Ω_0 into X_0. In fact, since φ is a Fatou map, we have a closed non-polar subset F of X such that $\hat{R}_1^{\varphi^{-1}(F)}$ is a potential. $F_0=F\cap X_0$ is closed and non-polar in X_0. Our assertion is derived at once from the facts $\hat{R}_1^{\varphi^{-1}(F_0)}$ is a potential and $X_0\in\mathcal{H}-\mathcal{P}$. The Green function of Ω_0 is the restriction on Ω_0 of the Green function of Ω. Denoting by Λ_0 the set of Green lines issuing from y_0 and passing no points of $\varphi^{-1}(\{x_0\})$, the condition $\phi(l)\subset A$ is reduced to

$$\phi_0(l) = \bigcap_{n=1}^{\infty} \overline{\{\varphi_0((\lambda_m, l));\ m\geqq n\}}\subset A$$

for any $l\in\Lambda_0$ Thus, we can reduce the case (2) to the previous one, since a set of Green lines passing the points of $\varphi^{-1}(\{x_0\})$ is of dg-measure zero.

3. 3. *The proof of the case* (3).

It remains to be proved the case (3), i.e., $X\in\mathcal{P}$. In this case, the situation is rather simple and we can prove without resorting many lemmas.

There exists a positive superharmonic function v defined on the whole X such that

$$\lim_{x\to x'} v(x) = +\infty \qquad \text{for any} \quad x'\in A\,.$$

$s=v\circ\varphi$ defines a positive superharmonic function on Ω and as before

$$u = \lim_{n\to\infty} H_s^{D^{\lambda_n}}$$

is non-negative and harmonic on Ω.

Assuming, as in 3. 1, $g^*(\Lambda)>0$, let us take $M>0$ such that

$$M/2\, g^*(\Lambda)>u(y_0)\,. \tag{3.2}$$

We define

$$U = \{x\in X;\ v(x)>M\}$$

and

$$C_n = \{l\in\Lambda;\ \varphi((\lambda_m, l))\in U \qquad \text{for} \quad m\geq n\} \quad (n=1, 2, \cdots)\,.$$

Quite in the same way, we can prove

$$\lim_{n\to\infty} g^*(C_n) = g^*(\Lambda)\,.$$

$$u(y_0) = \lim_{n\to\infty} H_s^{D^{\lambda_n}}(y_0) = \lim_{n\to\infty} \overline{\int_{\Sigma^n}} s\,d\,\omega_{y_0}^{\lambda_n}$$

$$\geq \lim_{n\to\infty} \overline{\int_{\tilde{C}_n}} s\,d\,\omega_{y_0}^{\lambda_n} \geq \lim_{n\to\infty} \overline{\int_{\tilde{C}_n}} M\,d\,\omega_{y_0}^{\lambda_n}\,,$$

since $\varphi((\lambda_n, l))\in U$, so that $s=v\circ\varphi>M$ on C_n. The last term is equal to

$$\overline{\lim_{n\to\infty}}\int_{C_n} M\,dg = M\lim_{n\to\infty} g^*(C_n) = Mg^*(\Lambda)\,.$$

Combining this with (3. 2)

$$M/2\, g^*(\Lambda) \geq u(y_0) \geq Mg^*(\Lambda)\,,$$

which is a contradiction. Thus, the proof is completed.

4. Consequences

In this section, we shall consider the case where Ω is a hyperbolic Riemann surface and φ is an analytic map of Ω into a Riemann surface X. A Fatou map in our definition is the same as in [5][22]. We have then

Corollary 1. *Let R be a hyperbolic Riemann surface and φ be a Fatou map of R into a Riemann surface R'. If*

$$\bigcap_{n=1}^{\infty} \overline{\{\varphi((\lambda_m, l));\ m \geqq n\}}$$

is polar in R'^* for every $l \in \Lambda$, where R'^* is an arbitrary compactification of R', $\{\lambda_n\}$ is a decreasing sequence of positive numbers tending to zero and the outer Green measure of Λ is posive, then φ is a constant map.

Since an AD function on R (a holomorphic function with finite Dirichlet integral) is a Dirichlet map of R into a Riemann sphere[23] and a Dirichlet map is a Fatou map, this is an extension of a result of M. Nakai[24].

A meromorphic function defined on $|z|<1$ is a Lindelöfian map of $R=\{|z|<1\}$ into a Riemann sphere if and only if it is of bounded type. The theorem of Riesz-Frostman-Nevanlinna for these functions is classical. Our theorem is a generalization even in the classical case, since we have known an example of a Fatou map which is not a Lindelöfian map[25].

In [10], we have investigated the boundary behavior of harmonic functions on a Green space along Green lines. As an application, we have given there a theorem of Riesz type for holomorphic functions f in the Smirnov class (i.e., $\log^+|f|$ has a quasi-bounded harmonic majorant) on a hyperbolic Riemann surface. For functions in the class AL (i.e., $\log^+|f|$ has a harmonic majorant, or equivalently, f is lindelöfian) we have proved under some assumption. Now, we can remove the restriction:

Corollary 2. *Let Ω be a hyperbolic Riemann surface. Let $f \in AL(\Omega)$, that*

22) Cf. [5], p. 110 and [7], p. 52.
23) Cf. [5], p. 115, Folgesatz 10.3.
24) Cf. [12], p. 19 and [16], p. 206.
25) Cf. [4], p. 72.

is, f is holomorphic on Ω and $\log^+|f|$ *has a harmonic majorant. If there exists a sequence* $\{\lambda_n\}$ *of positive numbers tending to zero such that*

$$\lim_{n\to\infty} f((\lambda_n, l)) = 0 \qquad \text{for all} \quad l \in \alpha\,,$$

where α is a set of Green lines of positive outer Green measure, then

$$f \equiv 0\,.$$

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