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HÖLDER CONDITIONS FOR CONTINUOUS GAUSSIAN PROCESSES

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In this paper we sharpen previous results of the author [4] concerning local Hölder conditions for Gaussian processes with stationary increments. Of particular interest is the form of these results which suggests how they may be extended to processes which we have not considered.

Let $X(t)$ be a separable, real valued continuous Gaussian process with stationary increments. For our purposes, it is sufficient to describe these processes by their incremental variance i.e.

$$E\{(X(t)-X(s))^2\} = \sigma^2(|t-s|), \text{ (where } \sigma^2(h) \to 0 \text{ as } h \to 0).$$

We are concerned with finding those processes and the corresponding functions $\varphi(h)$ for which the following inequalities will be satisfied with probability 1:

$$C_0 \leq \limsup_{h \to 0} \frac{|X(t+h)-X(t)|}{\varphi(h)} \leq C_1 \tag{1}$$

where the constants $0 < C_0 \leq C_1 < \infty$ can vary for the different processes. (Also note that the function $\varphi(h)$ depends on $\sigma^2(h)$.)

A major role in this paper is played by the function defined below,

$$f(\log 1/h) = \frac{1}{\sigma^2(h)} \int_0^h \frac{\sigma^2(u)}{u} du. \tag{2}$$

In the various cases considered some or all of the following conditions will also be used:

A.) $f(s)$ is increasing as $s \to \infty$ ($s = \log 1/h$)

B.) $f(2s) \leq 2f(s)$

C.) $\log s \leq f(s) \leq ks$, for some constant k.

The processes for which we can obtain expressions of the form of (1) are given in Theorem 3.

Theorem 3. *Let $X(t)$ be a separable, real valued, mean continuous Gaussian process with stationary increments for which $\sigma^2(h)$ is concave for $h \in [0, \delta]$ for* *some* $\delta>0$. *Let the function* f (*log* $1/h$) *satifsy conditions* A), B) *and* C) *given above. Then with probability* 1

$$C_0 \leq \limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{\left(\int_0^h \frac{\sigma^2(u)}{u} du\right)^{1/2}} \leq C_1 \tag{3}$$

where $0<C_0\leq C_1<\infty$.

We prove this Theorem first by studying lower bounds for the ratio in (1). These results are given in Theorem 1. The function $f(s)$, appearing above, need only satisfy condition A.) and the upper inequality in C.) in this theorem. Next the upper bounds are studied and the results given in Theorem 2. The function ψ is introduced which is assumed to be monotonically increasing and is such that

$$E\{|X(t)-X(s)|^2\} \leq \psi^2(|t-s|);$$

and also a new class of functions

$$p_\alpha(\log 1/h) = \frac{1}{\psi^\alpha(h)} \int_0^h \frac{\psi^\alpha(u)}{u} du\,, \qquad 1\leq\alpha\leq 2,$$

obviously similar to the function defined in (2). Analagous to the discussion following (2) we will occassionally impose conditions A.), B.) and C.) on $p_\alpha(s)$. The results of Theorem 2 are

$$\limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{(\psi^2(h)p_\alpha(\log 1/h))^{1/2}} \leq C \tag{4}$$

where for $\alpha<2$, p_α must satisfy only conditions A.) and the lower inequality in C.). When $\alpha=2$, p_α must satisfy A.), B.) and C.). Note that ψ is not required to be concave. Theorem 3 is obtained by combining Theorems 1 and 2 and including all the conditions necessary for each one.

Theorem 3 extends the results in [4]. In that paper results of this kind were obtained only when the derivative of $\sigma^2(h)$ had certain regularity properties, here the conditions are on $\sigma^2(h)$ and the integral expression. This significantly extends the processes for which expressions like (1) are obtained. This point will be elaborated upon in section 3. The function $\int_0^h \sigma^2(u)/u \cdot du$ is also of interest. It is monotonically increasing for all $\sigma^2(h)$, concave when $\sigma^2(h)$ is concave, and we wonder if it might not provide insight into obtaining Hölder conditions for those processes which we have not been able to handle.

1. Lower bounds

We prove the following Theorem related to the lower bound of the ratio given in (1). For what follows the function f is the one defined in (2).

Theorem 1. *Let $X(t)$ be a separable, real valued, mean continuous Gaussian process with stationary increments for which $\sigma^2(h)$ is concave for $h \in [0, \delta]$ for some $\delta > 0$. Then, with probability* 1,

$$\limsup_{h \to 0} \frac{|X(t+h)-X(t)|}{\left(\int_0^h \frac{\sigma^2(u)}{u} du\right)^{1/2}} \geq C$$

where $C>0$ and $f(s)$ satisfies condition A.) and the upper inequality in C.).

Proof. This proof follows the proof of Theorem 1 in [4]. Some details contained in the earlier paper will not be repeated here.

The sequence t_k is defined as follows:

$$t_k = \{\min t : f(\log 1/t) = \log k\} .$$

Consider the random variables

$$Y_k = \frac{X(t_k)-X(t_{k+1})}{(\sigma^2(t_k)f(\log 1/t_k))^{1/2}} \qquad k \geq k_0$$

where k_0 is chosen so that $t_{k_0} < \delta$. Because of the concavity of $\sigma^2(h)$, $E\{Y_k Y_j\} \leq 0$, $j \neq k$. From this it follows that $P\{Y_k \geq C,\ Y_j \geq C\} \leq P\{Y_k \geq C\} P\{Y_j \geq C\}$, $j \neq k$. Thus by the Chung-Erdös Lemma [1], [4], $P\{Y_k \geqslant C$ infinitely often$\}=1$ if and only if $\Sigma P\{Y_k \geq C\} = \infty$. The same result can be obtained using Slepian's [7] relationship between the elements of the covariance matrix of a Gaussian process and the bounds for the random variables of the joint distributions.

We have

$$P\{Y_k \geq C\} \geq \left[\frac{\text{const. } \sigma^2(t_k)}{\sigma^2(t_k - t_{k+1})} f(\log 1/t_k)\right]^{-1/2} \exp\left\{-C^2/2 \frac{\sigma^2(t_k)}{\sigma^2(t_k - t_{k+1})} f(\log 1/t_k)\right\} .$$

Let us write $\sigma^2(h) = e^{-2g(\log 1/h)}$. Since $\sigma^2(h)$ is concave, $\sigma'(h)$ will be defined everywhere by its right hand derivative. Thus $f'(s)$ is everywhere defined and $f'(s) = 2g'(s)f(s) - 1$. Then

$$\begin{aligned}
\frac{\sigma^2(t_k)}{\sigma^2(t_k - t_{k+1})} &= \exp\left\{\int_{\log 1/t_k}^{\log 1/t_k + \log (t_k/t_k - t_{k+1})} 2g'(s) ds\right\} \\
&= \exp \int_{\log 1/t_k}^{\log 1/t_k + \log (t_k/t_k - t_{k+1})} \left[\frac{f(s')}{f(s)} + \frac{1}{f(s)}\right] ds \\
&= \frac{f\left(\log 1/t_k + \log \frac{t_k}{t_k - t_{k+1}}\right)}{f(\log 1/t_k)} \exp\left\{\int_{\log 1/t_k}^{\log 1/t_k + \log (t_k/t_k - t_{k+1})} \frac{ds}{f(s)}\right\} . \qquad (5)
\end{aligned}$$

The concavity of $\sigma^2(h)$ implies that $2g'(s) < 1$. Hence $f'(s) < f(s)$. By the argument used in [4] this implies that

$$\log \frac{t_k}{t_k - t_{k+1}} \le (1+\varepsilon_1) \log k$$

for any $\varepsilon_1 > 0$ as long as k is sufficiently large. We may as well assume that k_0 is chosen so that $k \ge k_0$ is large enough. Since $f(s)$ is monotonic increasing the exponential term in (5) is bounded above by $e^{1+\varepsilon_1}$. Therefore

$$\begin{aligned} \frac{\sigma^2(t_k)}{\sigma^2(t_k - t_{k+1})} f(\log 1/t_k) &\le e^{1+\varepsilon_1} f\left(\log 1/t_k + \log \frac{t_k}{t_k - t_{k+1}}\right) \\ &\le e^{1+\varepsilon_1} f(\log 1/t_k + (1+\varepsilon_1) \log k) \\ &\le e^{1+\varepsilon_1} f(\log 1/t_k + (1+\varepsilon_1) f(\log 1/t_k)) \end{aligned}$$

and

$$\begin{aligned} \sum_{k \ge k_0} P\{Y_k \ge C\} \le \sum_{k \ge k_0} & \frac{\text{const.}}{[f(\log 1/t_k + (1+\varepsilon_1) f(\log 1/t_k))]^{1/2}} \\ & \times \exp\{-\alpha f(\log 1/t_k + (1+\varepsilon_1) f(\log 1/t_k))\} \end{aligned}$$

where $\alpha = \dfrac{C^2}{2e^{1+\varepsilon_1}}$. The terms of this series are positive and monotonically decreasing thus its divergence is equivalent to the divergence of the following integral in which t_k is replaced by $t(x)$; where $t(x) = \{\min t : f(\log 1/t) = \log x\}$.

$$\begin{aligned} \int^{\infty} & \frac{\text{const.}}{[t(\log 1/t(x) + (1+\varepsilon_1) f(\log 1/t(x))]^{1/2}} \\ & \times \exp\{-\alpha f(\log 1/t(x) + (1+\varepsilon_1) f(\log 1/t(x))\} dx . \end{aligned} \tag{6}$$

Making the substitution $s = e^{f(\log 1/t(x))}$ the integral in (6) becomes

$$\int^{\infty} \frac{\text{const.}}{f(f^{-1}(\log s) + (1+\varepsilon_1) \log s)]^{1/2}} \exp\{-\alpha f(f^{-1}(\log s) + (1+\varepsilon_1) \log s)\} ds . \tag{7}$$

Note that

$$f(f^{-1}(\log s) + (1+\varepsilon_1) \log s) = \log s + \int_{f^{-1}(\log s)}^{f^{-1}(\log s) + (1+\varepsilon_1) \log s} f(u) du . \tag{8}$$

Recall that $f(s) \le ks$. This implies that there exists a sequence $x_n \to \infty$ for which

$$\int_{x_n}^{x_n + (1+\varepsilon_1) \log s} f'(u) du \le (1+\varepsilon_1) k \log s .$$

Set $s_n = e^{f(x_n)}$; from (8) we obtain

$$f(f^{-1}(\log s_n) + (1+\varepsilon_1) \log s_n) \le \log s_n (1 + (1+\varepsilon_1) k) .$$

Returning to equation (7) we see that the integrand is monotonically decreasing and when $s = s_n$ it exceeds

$$\frac{\text{const.}}{(\log s_n)^{1/2}} \frac{1}{s_n^{\alpha(1+(1+\varepsilon_1)k)}}\,.$$

If $\alpha(1+(1+\varepsilon_1)k)<1$ then for sufficiently large N this integral exceeds $1/s_n$ along the sequence s_n, $n\geq N$. Thus the integrand is a monotonic function $h(x)$ for which $\limsup_{x\to\infty} xh(x)\geq 1$. The integral of such an integrand diverges. Thus $P\{Y_k>C\}=\infty$ which is what we set out to prove. The transition from this fact to the statement of the theorem is simple; the details are given in [4]. Hence the theorem is proved.

If we add the condition that $f(2s)\leq 2f(s)$ then the proof is considerably simplified since in this case

$$\frac{\sigma^2(t_k)}{\sigma^2(t_k-t_{k+1})}\leq e^{1+\varepsilon_1}\frac{f(\log 1/t_k+(1+\varepsilon)f(\log 1/t_k))}{f(\log 1/t_k)}\,.$$

For $f(s)\leq ks$, this is bounded and the proof follows easily. The significance of this condition will be discussed in Section 3.

2. Upper bounds

The upper bounds for the Hölder condition in (1) are expressed in terms of a function ψ which dominates the incremental variance of the process, i.e. $E\{(X(t)-X(s))^2\}\leq\psi^2(|t-s|)$ where ψ is considered to be continuous and monotonically increasing. Let $\psi(h)=e^{-g(\log 1/h)}$; the following functions will appear in the sequel:

$$p_\alpha(\log 1/h)=\frac{1}{\psi^\alpha(h)}\int_0^h\frac{\psi^\alpha(u)}{u}du,\qquad 1\leq\alpha\leq 2 \tag{9}$$

where

$$p_\alpha(s)=e^{\alpha g(s)}\int_s^\infty e^{-\alpha g(u)}du \tag{10}$$

and

$$p_\alpha'(s)=\alpha g'(s)p_\alpha(s)-1\,. \tag{11}$$

(Note $p_\alpha'(s)$ is everywhere defined by our previous remark.) Furthermore we assume that the following conditions are satisfied by $p_\alpha(s)$, for s sufficiently large:

2–A) $p_\alpha(s)$ is increasing in s and $p_\alpha(s)\geq\log s$.

2–B) $p_\alpha(2s)\leq 2p_\alpha(s)$.

In proving Theorem 2 we use a version of Fernique's lemma due to the author [6]. This is

Lemma 1. *Let $X(t)$ be a real valued, separable Gaussian process on* [0, 1].

Suppose $E((X(t)-X(s))^2)<\psi^2(|t-s|)$ *and* $\int^\infty \psi(e^{-x^2})dx<\infty$. *Let* $n(p)$ *denote* n^{2^p}, *n an integer greater than* 1. *Then*

$$P\left\{\|X\|_\infty \geq C\sqrt{\log n}\left[\Gamma^{1/2}+\sqrt{2}\sum_{p=1}^{\infty} 2^{p/2}\psi\left(\frac{1}{n(p)}\right)\right]\right\}$$
$$\leq 4n^2\int_{C\sqrt{\log n}}^{\infty} e^{-u^2/2}du \tag{12}$$

where $\|X\|_\infty$ *is the maximum of* $|X(t)|$ *on* $[0,1]$ *and* Γ *is the maximum of* $E\{X(t)X(s)\}$, $t, s\in[0, 1]$.

We proceed to Theorem 2.

Theorem 2. *Let* $X(t)$ *be a real valued separable Gaussian Process. Let* ψ *and* p_α *be defined as above. Assume* $\psi(t)>t^{1+\eta}$ *for some* $\eta>0$ *and* $t\in[0, \delta]$ *for some* $\delta>0$. *Then*

$$P\left\{\limsup_{h\to 0}\frac{|X(t+h)-X(t)|}{(\psi^2(h)p_\alpha(\log 1/h))^{1/2}}\leq C_1\right\}=1$$

(*where* C_1 *is a positive constant*) *with the following additional restrictions*:

1.) *When* $1\leq\alpha<2$ *condition* 2–*A*) *holds.*
2.) *When* $\alpha=2$ *condtions* 2–*A*) *and* 2–*B*) *hold.*

The constant C_1 *can depend on* ψ.

Proof. Lemma 1 relates to the maximum of the process on the unit interval. In order to change the scale we define

$$Y_k(t)=X(tt_k+t_0)-X(t_0) \qquad 0\leq t\leq 1.$$

Then

$$E\{(Y_k(t)-Y_k(s))^2\}\leq\psi^2(|t-s|t_k)$$
$$E\{Y(t)Y(s)\}\leq\psi(tt_k)\psi(st_k)\leq\psi^2(t_k) \qquad 0\leq s,\ t\leq 1.$$

From (12) we obtain

$$P\left\{\|Y_k\|_\infty\geq C\psi(t_k)\sqrt{\log n}\left(1+\frac{\sqrt{2}}{\psi(t_k)}\sum_{p=1}^{\infty}\psi\left(\frac{t_k}{n(p)}\right)2^{p/2}\right)\right\}$$
$$\leq 4n^2\int_{C\sqrt{\log n}}^{\infty}e^{-u^2/2}du\,. \tag{13}$$

Starting with some positive, small t_0 we define the sequence t_k by the equation $\psi(t_k)=\theta\psi(t_{k-1})$, where $\theta<1$. For a fixed value of t_k we choose the integer n as follows:

$$n=[\exp\{p_\alpha(\log 1/t_k)\}]+1$$

(where $[x]$ denotes the integral part of x). Obviously n also depends on α. Let us assume that it is fixed; we can do the problem separately for each α. We have

$$\sqrt{p_\alpha(\log 1/t_k)} \leq \sqrt{\log n} \leq \sqrt{p_\alpha(\log 1/t_k)} + 1 \tag{14}$$

Since $t_k^{1+\eta} < \psi(t_k) < \theta^k \psi(t_0)$ and

$$\log n \sim p_\alpha(\log 1/t_k) \geq \log_2 1/t_k \geq (1-\varepsilon) \log k$$

for some $\varepsilon > 0$ and k sufficiently large, we see that for an appropriate constant C the right side of (13) is a term of a convergent series. If we show that

$$\frac{1}{\psi(t_k)} \sum_{p=1}^{\infty} \psi\left(\frac{t_k}{n(p)}\right) 2^{p/2} < M \tag{15}$$

(i.e. that the series is uniformly bounded for all t_k) we can use the Borel-Cantelli lemma to show that

$$P\left\{\limsup_{k\to\infty} \frac{|X(tt_k+t_0)-X(t_0)|}{(\psi^2(t_k)p_\alpha(\log 1/t_k))^{1/2}} \leq \text{const.}\right\} = 1\,.$$

It is a simple step to pass from here to the desired result (see [4]). Thus our problem is reduced to showing that (15) is uniformly bounded in the t_k's. We write (15) as follows:

$$\begin{aligned}
&\sum_{p=1}^{\infty} \exp\left\{-\int_{\log 1/t_k}^{\log 1/t_k + 2^p \log n} g'(s)\,ds\right\} 2^{p/2} \\
&= \sum_{p=1}^{\infty} \left[\frac{p_\alpha(\log 1/t_k)}{p_\alpha(\log 1/t_k + 2^p \log n)}\right]^{1/\alpha} 2^{p/2} \exp\left\{-\frac{1}{\alpha}\int_{\log 1/t_k}^{\log 1/t_k+2^p\log n} \frac{du}{p_\alpha(u)}\right\}
\end{aligned} \tag{16}$$

where we have made use of (11). Since $p_\alpha(s)$ is increasing this sum is dominated by

$$\begin{aligned}
&\sum_{p=1}^{\infty} \left[\frac{p_\alpha(\log 1/t_k)}{p_\alpha(\log 1/t_k + 2^p \log n)}\right]^{1/\alpha} 2^{p/2} \exp\left\{-\frac{1}{\alpha}\, \frac{2^p \log n}{p_\alpha(2^p \log n + \log 1/t_k)}\right\} \\
&\leq \sum_{p=1}^{\infty} \left[\frac{p_\alpha(\log 1/t_k)}{p_\alpha(\log 1/t_k + 2^p \log n)}\right]^{1/\alpha} 2^{p/2} \exp\left\{-\frac{1}{\alpha}\, \frac{2^p p_\alpha(\log 1/t_k)}{p_\alpha(2^p \log n + \log 1/t_k)}\right\}
\end{aligned} \tag{17}$$

(using (14)). The sum in (17) is of the form $\sum_{p=1}^{\infty} 2^{p/2} z^{1/\alpha} \exp\{-2^p z/\alpha\}$. The maximum value of each summand occurs for $z = 1/2^p$; substituting into (16) we obtain

$$\sum_{p=1}^{\infty} \frac{1}{2^p(1/\alpha - 1/2)} \exp\{-1/\alpha\} \tag{18}$$

which converges for $\alpha < 2$. When $\alpha = 2$ the situation is not so simple.

Returning to equation (16) we wish to show the uniform boundedness of this sum when $\alpha=2$. For simplicity we denote $p_2(\log 1/t_k)$ by $p(\log 1/t_k)$. Replacing $\log n$ by $p(\log 1/t_k)$ increases the sum in (16). Also we replace $\log 1/t_k$ by s. We show that with the additional condition 2–B)

$$\sum_{p=1}^{\infty}\left[\frac{2^p p(s)}{p(2^p p(s)+s)}\right]^{1/2}\exp\left\{-1/2\int_s^{2^p p(s)+s}\frac{du}{p(u)}\right\} \tag{19}$$

converges uniformly in s. Equation (19) is bounded above by

$$\sqrt{2}\sum_{p=1}^{\infty}\left[\frac{2^p p(s)}{p(2^p p(s)+s)}\right]^{1/2}\exp\left[-\frac{2^p p(s)}{2p(2^p p(s)+s)}\right]$$
$$\exp\left\{-1/2\sum_{j=1}^{p-1}\int_{2^{j-1}p(s)+s}^{2^j p(s)+s}\frac{du}{p(u)}\right\}. \tag{20}$$

For fixed s, let p_0 denote the integer for which

$$2^{p_0}p(s)\leq s\leq 2^{p_0+1}p(s)$$

(If there is no p_0 for which this is true ignore the following step.) We need only be concerned with s large. By 2–B) $p(2s)\leq 2p(s)$; therefore

$$\frac{2^p p(s)}{2p(2^p p(s)+s)}\geq\frac{2^p p(s)}{2p(2s)}\geq 2^{p-2}\qquad\text{for } p\leq p_0\,. \tag{21}$$

Thus the sum of (20) from $p=1$ to $p=p_0$ is dominated by

$$\sqrt{2}\sum_{p=1}^{p_0}(2^{p-2})^{1/2}\exp\{-2^{p-2}\}<1\,.$$

For simplicity assume now that $p_0>4$. The sum of (20) from p_0+1 to ∞ is bounded above by

$$\sum_{p=p_0+1}^{\infty}\exp\left\{-1/2\sum_{j=1}^{p-1}\int_{2^{j-1}p(s)+s}^{2^j p(s)+s}\frac{du}{p(u)}\right\}$$
$$\leq\sum_{p=p_0+1}^{\infty}\exp\left\{-\sum_{j=1}^{p_0}2^{j-3}-\sum_{j=p_0+1}^{p-1}\frac{2^{j-1}p(s)}{2p(2^{j-1}p(s)+s)}\right\} \tag{22}$$

where we have again used (21). Since for $j>p_0$

$$\frac{2^{j-1}p(s)}{2p(2^{j-1}p(s)+s)}\geq\frac{2^{j-1}p(s)}{2p(2^{j-p_0}s)}\geq\frac{2^{j-1}p(s)}{2^{j-p_0+1}p(s)}=2^{p_0-2}\,,$$

equation (22) is dominated by

$$\sum_{p=p_0+1}^{\infty}\exp\{-\sum_{j=1}^{p_0}2^{j-3}-(p-p_0-2)2^{p_0-2}\}$$
$$\leq\sum_{p=p_0+1}^{\infty}\exp\{-(p-p_0-1)2^{p_0-2}\}\leq 1\,.$$

Thus the series is uniformly bounded in s; this completes the proof of the theorem.

It is of interest to note that this theorem could also have been proved in the $\alpha=2$ case under the weaker condition that $f(2s)\le kf(s)$ for any k as long as $f(s) \le \text{Const. } s$.

3. Discussion of results

Theorem 3, which is stated in the Introduction follows immediately from Theorems 1 and 2 since it includes as hypotheses the conditions necessary for each of them. We shall not bother to restate it here.

There are a number of points relating to the results in Theorems 1 and 2 that are worth mentioning. Our goal in studying these results is to hopefully gain insight into the form that the function $\varphi(h)$ in equation (1) should take in order to provide a solution to our problem for all separable, mean continuous, Gaussian processes with stationary increments. One might conjecture that when

$$\int_0^h \frac{\sigma^2(u)}{u}\,du \ge \sigma^2(h)\log_2 1/h\,,$$

$\varphi(h)=\left(\int_0^h \frac{\sigma^2(u)}{u}\,du\right)^{1/2}$. This is probably incorrect since it appears that $\varphi(h)$ will have to be smoother than this function. We shall proceed to attempt to justify this opinion and at the same time examine the relationship between the different bounds that appear in the theorems and in [4].

Note that the functions $p_\alpha(\log 1/h)$ $1\le\alpha<2$ can be written in terms of $p_2(\log 1/h)$, since

$$\begin{aligned}\frac{1}{\psi^\alpha(h)}\int_0^h \frac{\psi^\alpha(u)}{u}\,du &= \frac{1}{\psi^\alpha(h)}\int_0^h \frac{1}{\psi^{2-\alpha}(u)}\,d\int_0^u \frac{\psi^2(v)}{v}\,dv \\ &= p_2(\log 1/h)+\left(\frac{2-\alpha}{\alpha}\right)\int_0^h p_2(\log 1/u)\,d\frac{\psi^\alpha(u)}{\psi^\alpha(h)}\,. \end{aligned} \tag{23}$$

This integration is possible as long as $\int_0^h \frac{\psi^\alpha(u)}{u}\,du$ exists.

We can obtain from (23) that it is possible for $\limsup_{h\to0} p_\alpha/p_2=\infty$, $\alpha<2$. Let us now assume that $\psi^2(|t-s|)=E\{(X(t)-X(s))^2\}=\sigma^2(|t-s|)$. Note that only for $\alpha=2$ does $p_\alpha(\log 1/h)$ exist whenever $\int^\infty \sigma(e^{-x^2})\,dx<\infty$, i.e. only for $\alpha=2$ do the integrals $\int_0^h \frac{\sigma^2(u)}{u}\,du$ converge for all those processes that Fernique has shown to be continuous.

In [4] the lower bound for equation (1) was given as $1/g'(\log 1/h)$, (recall that $\sigma^2(h)\equiv e^{-2g(\log 1/h)}$ and we are considering concave $\sigma^2(h)$). The functions $p_\alpha(\log 1/h)$ are smoothings of this function since

$$\frac{1}{\sigma^\alpha(h)}\int_0^h \frac{\sigma^\alpha(u)}{u}du = 1/\alpha\int_0^h \frac{\sigma(u)}{u\sigma'(u)}d\frac{\sigma^\alpha(u)}{\sigma^\alpha(h)}$$

and $1/g'(\log 1/u)=\dfrac{\sigma(u)}{u\sigma'(u)}$. The best result in [4] relating to the problem considered here is that $\varphi(h)=(\sigma^2(h)\,1/g'(\log 1/h))^{1/2}$ where $1/g'(s)$ is a regularly varying function with exponent less than 1. When this is the case $\lim_{s\to\infty} g'(s)p_\alpha(s) =1/\alpha$, so we can replace for $1/g'(s)$ any of the functions $p_\alpha(s)$.

We see from (11) and the above remarks that $1/g'(s)$ regularly varying implies that $p_\alpha'(s)\to 0$. However $p_2(s)$ can be regularly varying and monotonic without $p_2'(s)$ converging to 0 and Theorem 3 holds for regularly varying $p_2(s)$ (exponent less than 1). In fact the conditions $f(2s)\le 2f(s)$ and $f(s)$ monotone is equivalent to $f(s)$ being of dominated variation [2], a more relaxed condition than regular variation. In comparing this paper to [4] we see that both the lower bound and the upper bound have been smoothed and this is the direction that improvements should take. Returning to the question of whether $\int_0^h \frac{\sigma^2(u)}{u}du$ is the appropriate Hölder condition in general (assuming that it is greater than $\sigma^2(h)\log_2 1/h$). My guess is that it is still not smooth enough.

In Theorem 3 it is required that $p_2(s)\le ks$ for some constant k. However, it is possible for $p_2(s)$ to take on larger values; (the fact that $\int^s \frac{du}{p_2(u)}$ must diverge shows what values $p_2(s)$ can take.) Note that no upper bounds are placed on $p_\alpha(\log 1/h)$, $1\le\alpha<2$; therefore we can take for $\varphi(h)$ in (1) the function $(\sigma^2(h)p_\alpha(\log 1/h))^{1/2}$, $\alpha<2$. However, when $p_\alpha(s)>\log s$ a better choice for $\varphi(h)$ is given in [4] namely

$$\int_0^h \frac{\sigma(u)}{u\sqrt{\log 1/u}}du\,. \tag{24}$$

Nevertheless, $\left(\int_0^h \frac{\sigma^2(u)}{u}du\right)^{1/2}$ is less than (24); in fact it converges for more processes than does $\int^\infty \sigma(e^{-x^2})dx$. (Equation (24) is $\int_{\sqrt{\log 1/h}}^\infty \sigma(e^{-x^2})dx$ under a change of variables.)

As for the smoothness of $\int_0^h \frac{\sigma^2(u)}{u}du$ itself, the hypothesis that $p_2(s)\ge\log s$ implies that this function is slowly varying and the hypothesis that $\sigma^2(u)$ is concave $0\le u\le h$ implies that this function is concave. Thus for those processes for which (1) is actually obtained, the Hölder condition is both concave and slowly varying.

Our major concern in this paper has been processes for which $\log s\le f_2(s) \le ks$. In those cases studied in [4] for which $f_2(s)\le\log s$ we obtained

$$C_0 \leq \limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{(\sigma^2(h)\log_2 1/h)^{1/2}} \leq C_1$$

and in some of these cases $C_0 = C_1 = \sqrt{2}$.

Another interesting question is the following: Suppose $h/\sigma^2(h)\to 0$, is it true that if $\sigma_1(h)\sim\sigma_2(h)$ then the two processes have the same Hölder condition? From Theorem 2 we see that the upper bounds are the same in this case, the problem for the lower bounds is not clear. The question is answered affirmatively for the processes considered in Theorem 6 in [4] and one would also expect it to be true when the uniform Hölder condition is being considered.

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