Nakaoka, M.
Osaka J. Math.
7 (1970), 423-441

GENERALIZATIONS OF BORSUK-ULAM THEOREM

Dedicated to Professor Keizo Asano on his 60th birthday

MINORU NAKAOKA

(Received April 10, 1970)

Introduction

Conner-Floyd proved in their book [1] the following theorem which is a generalization of the classical Borsuk-Ulam theorem: Let $f\colon S^n \to M$ be a continuous map of the n-sphere to a differentiable manifold of dimension m, and T be a fixed point free differentiable involution on S^n. Assume that $m \leqq n$ and $f_*\colon H_n(S^n; \boldsymbol{Z}_2) \to H_n(M; \boldsymbol{Z}_2)$ is trivial. Then the covering dimension of $A(f) = \{y \in S^n \mid f(y) = f(Ty)\}$ is at least $n-m$.

In response to the questions asked in [1, p. 89], Munkholm [4] showed that in the above theorem all differentiability hypotheses can be eliminated if M is assumed to be compact. Furthermore he showed in [5] that S^n can be replaced by a closed manifold which is a mod 2 homology n-sphere if M is the Euclidean space. In the present paper, we shall show the following theorem which is more general.

Main Theorem. *Let N be a closed topological manifold which is a mod 2 homology n-sphere, and T be a fixed point free involution on N. Let $f\colon N \to M$ be a continuous map of N to a compact topological m-manifold M (with or without boundary). Assume that $n \geqq m$ and $f_*\colon H_n(N; \boldsymbol{Z}_2) \to H_n(M; \boldsymbol{Z}_2)$ is trivial. Then the covering dimension of $A(f) = \{y \in N \mid f(y) = f(Ty)\}$ is at least $n-m$.*

Let π denote the cyclic group of order 2 generated by T. Denote by N_π the orbit space of N, and by $N \underset{\pi}{\times} M^2$ the orbit space of $N \times M^2$ on which π acts by $T(y, x, x') = (Ty, x', x)$ $(y \in N,\ x, x' \in M)$. Then $N \underset{\pi}{\times} M^2$ and $N_\pi \times M$ are topological manifolds, and $N_\pi \times M$ is embedded in $N \underset{\pi}{\times} M^2$ by the diagonal map $d\colon M \to M^2$. Assuming M is a closed manifold, let $\theta_0 \in H^m(N \underset{\pi}{\times} M^2; \boldsymbol{Z}_2)$ denote the Poincaré dual of $i^*(w)$, where $w \in H_{n+m}(N \underset{\pi}{\times} M; \boldsymbol{Z}_2)$ is the fundamental class and $i\colon N_\pi \times M \subset N \underset{\pi}{\times} M^2$. Define $s\colon N_\pi \to N \underset{\pi}{\times} M^2$ by $s(y) = (y, f(y), f(Ty))$ $(y \in N)$. Then Conner-Floyd [1] and Munkholm [4] proved their theorems by showing

$s^*(\theta_0) \neq 0$. We also follow this principle (see Lemma 6). However our method of proving $s^*(\theta_0) \neq 0$ is different from theirs, and is purely homological.

Let N be a sufficiently large dimensional sphere, and $T: N \to N$ be the antipodal map. Assume that M is triangulable. Then Haefliger [2] proved a formula giving θ_0 in terms of cohomology classes of M. We shall show that the formula still holds for our N, T and M, and we shall use the formula to prove $s^*(\theta_0) \neq 0$.

The method can be also applied to obtain the Borsuk-Ulam type theorem for a fixed point free homeomorphism of period p on a mod p homology sphere (p: odd prime), and a theorem including the result in [5] will be proved (see Theorem 8 in §9).

1. Generalization of Eilenberg-Zilber theorem

Throughout §1–§3, a principal ideal domain R is fixed, and chain complexes over R are conisdered. Thus, the singular complex of a topological space X with coefficients in R is denoted by simply $S(X) = \{S_q(X)\}$, and the tensor product $\otimes_R$ is denoted by simply $\otimes$.

Let E be a Hausdorff space on which there is given a fixed point free involution T, and such that the reduced homology group $\widetilde{H}_*(E)$ is trivial. Then we have the following generalization of Eilenberg-Zilber theorem.

Theorem 1. *There exist chain maps*

$$\rho : S(E\times X_1\times X_2) \to S(E)\otimes S(X_1)\otimes S(X_2)\,,$$
$$\rho' : S(E)\otimes S(X_1)\otimes S(X_2) \to S(E\times X_1\times X_2)\,,$$

defined for each pair (X_1, X_2) of topological spaces, and satisfying the following conditions:

(i) *ρ and ρ' are functorial, i.e. for any continuous maps $f_1: X_1 \to Y_1$ and $f: X_2 \to Y_2$ we have*

$$(1\otimes f_{1\#}\otimes f_{2\#})\circ\rho = \rho\circ(1\times f_1\times f_2)_\#\,,$$
$$\rho'\circ(1\otimes f_{1\#}\otimes f_{2\#}) = (1\times f_1\times f_2)_\#\circ\rho'\,.$$

(ii) *ρ and ρ' are equivariant in the sense that*

$$(T_\#\otimes T)\circ\rho = \rho\circ(T\times T)_\#$$
$$\rho'\circ(T_\#\otimes T) = (T\times T)_\#\circ\rho\,,$$

where T: $S(X_1)\otimes S(X_2) \to S(X_2)\otimes S(X_1)$ is given by $T(c_1\otimes c_2) = (-1)^{\deg c_1 \deg c_2}$ $c_2\otimes c_1$, and $T\times T$: $E\times X_1\times X_2 \to E\times X_2\times X_1$ by $(T\times T)(e, x_1, x_2) = (Te, x_2, x_1)$ $(e\in E,\ x_1\in X_1,\ x_2\in X_2)$.

(iii) *There exist a chain homotopy* Φ *of* $\rho'\circ\rho$ *to the identity and a chain homotopy* Φ' *of* $\rho\circ\rho'$ *to the identity, which are defined for each pair of topological spaces and which are functorial and equivariant in the same sense as in* (i), (ii).

Proof. The proof is done by the method of acyclic models.

Define a homomorphism ρ_0: $S_0(E\times X_1\times X_2)\to S_0(E)\otimes S_0(X_1)\otimes S_0(X_2)$ by $\rho_0(e, x_1, x_2)=e\otimes x_1\otimes x_2$ ($e\in E$, $x_1\in X_1$, $x_2\in X_2$), and assume inductively that a homomorphism ρ_r: $S_r(E\times X_1\times X_2)\to(S(E)\otimes S(X_1)\otimes S(X_2))_r$ has been defined for $r<n$ so that the conditions

i) $\partial_r\circ\rho_r=\rho_{r-1}\circ\partial_r$,

ii) $\rho_r\circ(1\times f_1\times f_2)_\sharp=(1\otimes f_{1\sharp}\otimes f_{2\sharp})\circ\rho_r$,

iii) $(T_\sharp\otimes T)\circ\rho_r=\rho_r\circ(T\times T)_\sharp$

are satisfied. Take a set $\{e_\lambda^n\}_{\lambda\in\Lambda}$ of singular n-simplexes of E such that $\{e_\lambda^n, T_\sharp e_\lambda^n\}_{\lambda\in\Lambda}$ is a basis of the module $S_n(E)$. For each $\lambda\in\Lambda$, define a singular n-simplex d_λ^n: $\Delta^n\to E\times\Delta^n\times\Delta^n$ by $d_\lambda^n(z)=(e_\lambda^n(z), z, z)$ ($z\in\Delta^n$). It holds that $\partial_{n-1}\rho_{n-1}\partial_n(d_\lambda^n)=0$ ($n>1$) and $\varepsilon\rho_0\partial_1(d_\lambda^n)=0$ ($n=1$) for the augmentation ε. Since the reduced complex of $S(E)\otimes S(\Delta^n)\otimes S(\Delta^n)$ is acyclic, there exists an n-chain $\rho_n(d_\lambda^n)$ of $S(E)\otimes S(\Delta^n)\otimes S(\Delta^n)$ such that $\partial_n\rho_n(d_\lambda^n)=\rho_{n-1}\partial_n(d_\lambda^n)$. The module $S_n(E\times X_1\times X_2)$ is a free module generated by elements of the form $(1\times\sigma_1\times\sigma_2)_\sharp d_\lambda^n$ or $(T\times\sigma_1\times\sigma_2)_\sharp d_\lambda^n$, where σ_i: $\Delta^n\to X_i$ ($i=1, 2$) is any continuous map. Define a homomorphism ρ_n: $S_n(E\times X_1\times X_2)\to S(E)\otimes S(X_1)\otimes S(X_2))_n$ by

$$\rho_n((1\times\sigma_1\times\sigma_2)_\sharp d_\lambda^n)=(1\otimes\sigma_{1\sharp}\otimes\sigma_{2\sharp})\rho_n(d_\lambda^n)\,,$$
$$\rho_n((T\times\sigma_1\times\sigma_2)_\sharp d_\lambda^n)=(T_\sharp\otimes T)(1\otimes\sigma_{2\sharp}\otimes\sigma_{1\sharp})\rho_n(d_\lambda^n)\,.$$

Then it is easily checked that the conditions i)—iii) are satisfied for $r=n$. Thus there exists a chain map ρ satisfying the conditions (i) and (ii).

Define a homomorphism ρ'_0: $S_0(E)\otimes S_0(X_1)\otimes S(X_2)\to S_0(E\times X_1\times X_2)$ by $\rho'_0(e\otimes x_1\otimes x_2)=(e, x_1, x_2)$ ($e\in E$, $x_1\in X_1$, $x_2\in X_2$), and assume inductively that a homomorphism ρ'_r: $(S(E)\otimes S(X_1)\otimes S(X_2))_r\to S_r(E\times X_1\times X_2)$ has been defined for $r<n$ so that the conditions

i)′ $\partial_r\circ\rho'_r=\rho'_{r-1}\circ\partial_r$,

ii)′ $(1\times f_1\times f_2)_\sharp\circ\rho'_r=\rho'_r\circ(1\otimes f_{1\sharp}\otimes f_{2\sharp})$,

iii)′ $(T\times T)_\sharp\circ\rho'_r=\rho'_r\circ(T_\sharp\otimes T)$

are satisfied. Let $i^k\in S_k(\Delta^k)$ denote the singular simplex given by the identity, and consider $e_\lambda^r\otimes i^s\otimes i^t\in S_r(E)\otimes S_s(\Delta^s)\times S_t(\Delta^t)$ with $r+s+t=n$. It holds that $\partial_{n-1}\rho'_{n-1}\partial_n(e_\lambda^r\otimes i^s\otimes i^t)=0$ ($n>1$) and $\varepsilon\rho'_0\partial_1(e_\lambda^r\otimes i^s\otimes i^t)=0$ ($n=1$). Since the reduced complex of $S(E\times\Delta^s\times\Delta^t)$ is acyclic, there exists an n-chain $\rho'_n(e_\lambda^r\otimes i^s\otimes i^t)\in S_n(E\times\Delta^s\times\Delta^t)$ such that $\partial_n\rho'_n(e_\lambda^n\otimes i^s\otimes i^t)=\rho'_{n-1}\partial_n(e_\lambda^r\otimes i^s\otimes i^t)$. The module $(S(E)\otimes S(X_1)\otimes S(X_2))_n$ is a free module generated by elements of the form $(1\otimes\sigma_{1\#}\otimes\sigma_{2\#})(e_\lambda^r\otimes i^s\otimes i^t)$ and $(T_\#\otimes\sigma_{1\#}\otimes\sigma_{2\#})(e_\lambda^r\otimes i^s\otimes i^t)$, where $\sigma_1\colon \Delta^s\to X$, $\sigma_2\colon \Delta^t\to X_2$ are continuous maps. Define a homomorphism ρ_n': $(S(E)\otimes S(X_1)\otimes S(X_2))_n\to S_n(E\times X_1\times X_2)$ by

$$\begin{aligned}&\rho_n'(1\otimes\sigma_{1\#}\otimes\sigma_{2\#})(e_\lambda^r\otimes i^s\otimes i^t)\\ &=(1\times\sigma_1\times\sigma_2)_\#\rho_n'(e_\lambda^r\otimes i^s\otimes i^t)\,,\\ &\rho_n'(T_\#\otimes\sigma_{1\#}\otimes\sigma_{2\#})(e_\lambda^r\otimes i^s\otimes i^t)\\ &=(-1)^{st}(T\times T)_\#(1\times\sigma_2\times\sigma_1)_\#\rho_n'(e_\lambda^r\otimes i^s\otimes i^t)\,.\end{aligned}$$

Then it is easily checked that the conditions i)′—iii)′ are satisfied for $r=n$. Thus there exists a chain map ρ' satisfying the conditions (i) and (ii).

By the similar method we can construct chain homotopies Φ and Φ' in (iii). This completes the proof of Theorem 1.

The following is obvious from the proof above.

Corollary. *Let E' be a subspace of E which is invariant under T and such that $\tilde{H}_q(E')=0$ for $q<n$. Then ρ and ρ' can be taken in such a way that $\rho_q(S(E'\times X_1\times X_2))\subset S(E')\otimes S(X_1)\otimes S(X_2)$ and $\rho_q'(S(E')\otimes S(X_1)\otimes S(X_2))\subset S(E'\times X_1\times X_2)$ for $q\leqq n$.*

2. Algebraic lemmas

Given a chain complex C, the module $Z(C)$ of cycles of C and the homology module $H(C)$ of C are regarded as chain complexes with trivial boundary operator. Then the inclusion $\xi\colon Z(C)\to C$ and the projection $\eta\colon Z(C)\to H(C)$ are chain maps.

Let π be a cyclic group of order 2, and T its generator. Let W be a π-free acyclic complex, and define an action of π on the chain complex $C^2=C\otimes C$ by $T(c_1\otimes c_2)=(-1)^{\deg c_1\deg c_2}c_2\otimes c_1$ $c_1, c_2\in C)$. Consider the diagonal action of π on $W\otimes C^2$, and let $W\underset{\pi}{\otimes}C^2$ denote the quotient complex.

For the homomorphisms

$$\begin{aligned}\xi_*&\colon H(W\underset{\pi}{\otimes}Z(C)^2)\to H(W\underset{\pi}{\otimes}C^2)\,,\\ \eta_*&\colon H(W\underset{\pi}{\otimes}Z(C)^2)\to H(W\underset{\pi}{\otimes}H(C)^2)\end{aligned}$$

induced by ξ and η, we have

Lemma 1. *If C is a free chain complex such that $H(C)$ is free, then $\xi_*\circ\eta_*^{-1}$: $H(W\underset{\pi}{\otimes}H(C)^2)\to H(W\underset{\pi}{\otimes}C^2)$ is well defined and is an isomorphism.*

Proof. There exist chain maps $\eta'\colon H(C)\to Z(C)$ and $\zeta'\colon C\to H(C)$ such that $\eta\circ\eta'=1$, $\zeta'\circ\xi=\eta$. Put $\zeta=\xi\circ\eta'\colon H(C)\to C$. Then $\zeta_*\colon H(C)\to H(C)$ is the identity, and hence ζ is a chain equivalence. Therefore, by a lemma due to

Steenrod (see [8], p. 125), $1\otimes\zeta^2\colon W\underset{\pi}{\otimes}H(C)^2\to W\underset{\pi}{\otimes}C^2$ is a chain equivalence, and we have

$$\zeta_*\colon H(W\underset{\pi}{\otimes}H(C)^2)\cong H(W\underset{\pi}{\otimes}C^2)\,.$$

Since ζ'_* is the inverse of ζ_*, it follows from $\zeta'_*\circ\xi_*=\eta_*$ that $\xi_*=\zeta_*\circ\eta_*$. Since $\eta_*\circ\eta'_*=1$, η_* is surjective. Thus we have $\zeta_*=\xi_*\circ\eta_*^{-1}$ which completes the proof.

Denote by C^* the cochain complex dual to a chain complex C. We regard the module $Z(C^*)$ of cocycles of C^* and the cohomology module $H(C^*)$ as cochain complexes with trivial coboundary operator.

Define an action of π on the cochain complex $C^{*2}=C^*\otimes C^*$ by $T(u_1\otimes u_2)=(-1)^{\deg u_1 \deg u_2}u_2\otimes u_1(u_1,\ u_2\in C^*)$, and consider the cochain complex $\mathrm{Hom}_\pi(W,\ C^{*2})$ consisting of equivariant homomorphisms of W to C^{*2}. The inclusion $\xi\colon Z(C^*)\to C^*$ and the projection $\eta\colon Z(C^*)\to H(C^*)$ induces homomorphisms

$$\xi_*\colon H(\mathrm{Hom}_\pi(W,\ Z(C^*)^2))\to H(\mathrm{Hom}_\pi(W,\ C^{*2}))\,,$$
$$\eta_*\colon H(\mathrm{Hom}_\pi(W,\ Z(C^*)^2))\to H(\mathrm{Hom}_\pi(W,\ H(C^*)^2))\,.$$

Let $\mu\colon C^{*2}\to C^{2*}$ denote the canonical cochain map defined by

$$\langle\mu(u_1\otimes u_2),\ c_1\otimes c_2\rangle=u(c_1)u_2(c_2)\qquad(u_1,\ u_2\in C^*,\ c_1,\ c_2\in C)\,.$$

The cochain map dual to $T\colon C^2\to C^2$ defines an action of π on C^{2*}. Then μ is equivariant, and so it induces a homomorphism

$$\mu_*\colon H(\mathrm{Hom}_\pi(W,\ C^{*2}))\to H(\mathrm{Hom}_\pi(W,\ C^{2*}))\,.$$

Lemma 2. *Let C be a free non-negative chain complex such that $H(C)$ is of finite type and is free. Then $\xi_*\circ\eta_*^{-1}\colon H(\mathrm{Hom}_\pi(W,\ H(C^*)^2))\to H(\mathrm{Hom}_\pi(W,\ C^{*2}))$ is well defined, and both μ^* snd $\xi_*\circ\eta_*^{-1}$ are isomorphisms.*

Proof. There is a free non-negative chain complex C' of finite type such that C and C' are chain equivalent (see [7], p. 246). Let $\varphi\colon C\to C'$ be a chain equivalence, and consider the following commutative diagram

$$\begin{tikzcd}
H(\mathrm{Hom}_\pi(W,\ H(C'^*)^2)) \arrow[r, "\varphi_*"] & H(\mathrm{Hom}_\pi(W,\ H(C^*)^2)) \\
H(\mathrm{Hom}_\pi(W,\ Z(C'^*)^2)) \arrow[r, "\varphi_*"] \arrow[u, "\eta_*"] \arrow[d, "\xi_*"] & H(\mathrm{Hom}_\pi(W,\ Z(C^*)^2)) \arrow[u, "\eta_*"] \arrow[d, "\xi_*"] \\
H(\mathrm{Hom}_\pi(W,\ C'^{*2})) \arrow[r, "\varphi_*"] \arrow[d, "\mu_*"] & H(\mathrm{Hom}_\pi(W,\ C^{*2})) \arrow[d, "\mu_*"] \\
H(\mathrm{Hom}_\pi(W,\ C'^{2*})) \arrow[r, "\varphi_*"] & H((\mathrm{Hom}_\pi(W,\ C^{2*}))
\end{tikzcd}$$

Since C'^* and $H(C^{*\prime})$ are free, the argument similar to the proof of Lemma 1 shows that $\xi_* \circ \eta_*^{-1}$ in the left side is well defined and is an isomorphism. Since C' is of finite type and is free, μ: $C'^{*2} \to C'^{2*}$ is an isomorphism, and so is μ_* in the left side. Since φ is a chain equivalence, it follows that φ_* in the 3-rd and the 4-th rows are isomorphisms. Obviously φ_* in the 1-st row is also an isomorphism. Thus we obtain the desired result.

By the definitions, $H(W \underset{\pi}{\otimes} H(C)^2)$ is the homology group $H(\pi; H(C)^2)$ of the group π with coefficients in the module $H(C)^2$ on which π acts by $T(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a$ $(a, b \in H(C))$, and $H(\mathrm{Hom}_\pi(W, H(C^*)^2))$ is the cohomology group $H(\pi; H(C^*)^2)$ of the group π with coefficients in the module $H(C^*)^2$ on which π acts by $T(\alpha \otimes \beta) = (-1)^{\deg \alpha \deg \beta} \beta \otimes \alpha$ $(\alpha, \beta \in H(C^*))$.

3. Homology and cohomology of $E \underset{\pi}{\times} X^2$

Given a topological space Y on which π acts, we denote by Y_π the orbit space. For a topological space X, consider the space $E \times X^2$ on which π acts by $T(e, x, x') = (Te, x', x)$ $(e \in E,\ x, x' \in X)$. We write $(E \times X^2)_\pi = E \underset{\pi}{\times} X^2$.

For the singular homology group and the singular cohomology group of $E \underset{\pi}{\times} X^2$, we have

Theorem 2. (i) *There exists a functorial isomorphism*

$$\kappa:\ H_*(\pi; H_*(X)^2) \cong H_*(E \underset{\pi}{\times} X^2),$$

defined for each topological space X such that $H_*(X)$ is free.

(ii) *There exists a functorial isomorphism*

$$\kappa:\ H^*(\pi; H^*(X)^2) \cong H^*(E \underset{\pi}{\times} X^2),$$

defined for each topological space X such that $H_*(X)$ is of finite type and is free.

Proof. (i) The action of π on $E \times X^2$ makes the singular complex $S(E \times X^2)$ a π-complex. Let $S(E \times X^2)/\pi$ denote the quotient complex. Since the projection p: $E \times X^2 \to E \underset{\pi}{\times} X^2$ is a fibering with discrete fiber, it follows that $p_\sharp$: $S(E \times X^2) \to S(E \underset{\pi}{\times} X^2)$ induces an isomorphism

$$S(E \times X^2)/\pi \cong S(E \underset{\pi}{\times} X^2).$$

Define an action of π on $S(E) \otimes S(X)^2$ by $T(c \otimes c_1 \otimes c_2) = (-1)^{\deg c_1 \deg c_2} T_\sharp(c) \otimes c_2 \otimes c_1$ $(c \in S(E),\ c_1, c_2 \in S(X))$. Then it follows that ρ' in Theorem 1 induces a chain equivalence

$$S(E) \underset{\pi}{\otimes} S(X)^2 \to S(E \times X^2)/\pi.$$

Therefore an isomorphism

$$\chi\colon H(S(E)\underset{\pi}{\otimes}S(X)^2)\cong H_*(E\underset{\pi}{\times}X^2)$$

is induced by the chain map $p_\sharp\circ\rho'$. Since $p_\sharp$ and ρ' are functorial, so is χ.

Since $S(E)$ is a π-free acyclic complex, by Lemma 1 we have

$$\xi_*\circ\eta_*^{-1}\colon H(S(E)\underset{\pi}{\otimes}H(X)^2)\cong H(S(E)\underset{\pi}{\otimes}S(X)^2)\,.$$

Obviously ξ_* and η_* are functorial. Therefore the desired isomorphism κ is given by $\kappa=\chi\circ\xi_*\circ\eta_*^{-1}$.

(ii) The cochain complex $\mathrm{Hom}_\pi(S(E),\,S(X)^{2*})$ is canonically isomorphic with the cochain complex $(S(E)\underset{\pi}{\otimes}S(X)^2)^*$, the above proof of (i) shows that an isomorphism

$$\chi'\colon H(\mathrm{Hom}_\pi(S(E),\,S(X)^{2*}))\cong H^*(E\underset{\pi}{\times}X^2)$$

is induced by the chain map $p_\sharp\circ\rho'$. On the other hand, by Lemma 2 we have

$$\mu_*\circ\xi_*\circ\eta_*^{-1}\colon H(\mathrm{Hom}_\pi(S(E),\,H^*(X)^2))\cong H(\mathrm{Hom}_\pi(S(E),\,S(X)^{2*}))\,.$$

Therefore the desired isomorphism κ is given by $\kappa=\chi\circ\xi_*\circ\eta_*^{-1}$ with $\chi=\chi'\circ\mu_*$. This completes the proof of Theorem 2.

Define a pairing of $H^*(X)^2$ and $H^*(X)^2$ to $H^*(X)^2$ by

$$(\alpha\otimes\beta)\cdot(\gamma\otimes\delta)=(-1)^{\deg\beta\deg\gamma}(\alpha\smile\gamma)\otimes(\beta\smile\delta)$$
$$(\alpha,\,\beta,\,\gamma,\,\delta\in H^*(X))\,.$$

Since this pairing is equivariant with respect to the action on $H^*(X)^2$, it gives rise to a cup product

$$\smile\colon H^*(\pi;\,H^*(X)^2)\otimes H^*(\pi;\,H^*(X)^2)\to H^*(\pi;\,H^*(X)^2)\,.$$

Similarly, an equivariant pairing of $H^*(X)^2$ and $H_*(X)^2$ to $H_*(X)^2$ defined by

$$(\alpha\otimes\beta)\cdot(a\otimes b)=(-1)^{\deg\alpha(\deg b-\deg\beta)}(\alpha\frown a)\otimes(\beta\frown b)$$
$$(\alpha,\,\beta\in H^*(X),\,a,\,b\in H_*(X))$$

gives rise to a cap product

$$\frown\colon H^*(\pi;\,H^*(X)^2)\otimes H_*(\pi;\,H_*(X)^2)\to H_*(\pi;\,H_*(X)^2)\,.$$

Theorem 3. *The isomorphisms κ in Theorem 2 preserve the cup products and the cap products, i.e. the following diagrams are commutative.*

$$\begin{array}{ccc} H^*(\pi;\,H^*(X)^2)\otimes H^*(\pi;\,H^*(X)^2) & \xrightarrow{\smile} & H^*(\pi;\,H^*(X)^2)) \\ \big\downarrow{\scriptstyle\kappa\otimes\kappa} & & \big\downarrow{\scriptstyle\kappa} \\ H^*(E\underset{\pi}{\times}X^2)\otimes H^*(E\underset{\pi}{\times}X^2) & \xrightarrow{\smile} & H^*(E\underset{\pi}{\times}X^2)\,, \end{array}$$

$$\begin{array}{ccc} H^*(\pi;\, H^*(X)^2)\otimes H_*(\pi;\, H_*(X)^2) & \xrightarrow{\frown} & H_*(\pi;\, H_*(X)^2) \\ \downarrow{\scriptstyle \kappa\otimes\kappa} & & \downarrow{\scriptstyle \kappa} \\ H^*(E\underset{\pi}{\times}X^2)\otimes H_*(E\underset{\pi}{\times}X^2) & \xrightarrow{\frown} & H_*(E\underset{\pi}{\times}X^2)\,. \end{array}$$

Proof. For $C=S(X)$ and $Z(S(X))$, a cup product

$$\smile:\ \mathrm{Hom}_\pi(S(E),\, C^{*2})\otimes\mathrm{Hom}_\pi(S(E),\, C^{*2})\to\mathrm{Hom}_\pi(S(E),\, C^{*2})$$

and a cap product

$$\frown:\ \mathrm{Hom}_\pi(S(E),\, C^{2*})\otimes(S(E)\underset{\pi}{\otimes}C^2)\to S(E)\underset{\pi}{\otimes}C^2$$

are defined similarly to the above, by using of the cup product and the cap product for cochains and chains. Then it is obvious that the homomorphisms ξ_* and η_* in the proof of Theorem 2 preserve the cup products and the cap products. Therefore it suffices to prove that the homomorphisms χ in the proof of Theorem 2 preserve the cup products and the cap products.

For any topological space Y, let Δ: $S(Y)\to S(Y)^2$ denote the diagonal approximation (see [7], p. 250). Consider a diagram

$$\begin{array}{ccccc} S(E)\otimes S(X)^2 & \xrightarrow{\Delta\otimes\Delta^2} & S(E)^2\otimes(S(X)^2)^2 & \xrightarrow{\tau} & (S(E)\otimes S(X)^2)^2 \\ \downarrow{\scriptstyle \rho'} & & & & \downarrow{\scriptstyle \rho'^2} \\ S(E\times X^2) & & \xrightarrow{\Delta} & & S(E\times X^2)^2 \\ \downarrow{\scriptstyle p_\sharp} & & & & \downarrow{\scriptstyle p_\sharp^2} \\ S(E\underset{\pi}{\times}X^2) & & \xrightarrow{\Delta} & & S(E\underset{\pi}{\times}X^2)^2\,, \end{array}$$

where τ is the appropriate chain map shuffling factors. Since Δ is functorial, the lower rectangle is commutative. Regard $S(E\times X^2)^2$ and $(S(E)\otimes S(X)^2)^2$ as π-complexes by the diagonal action of the actions of $S(E\times X^2)$ and $S(E)\otimes S(X)^2$ respectively. Then it follows that the maps in the 1-st and the 2-nd rows are equivariant. Furthermore the argument similar to the proof of Theorem 1 shows that there exists a chain homotopy of $\Delta\circ\rho'$ to $\rho'^2\circ\tau\circ(\Delta\otimes\Delta^2)$ which is equivariant and functorial. Therefore we have a diagram

$$\begin{array}{ccc} S(E)\underset{\pi}{\otimes}S(X)^2 & \xrightarrow{\tau\circ(\Delta\otimes\Delta^2)} & (S(E)\underset{\pi}{\otimes}S(X)^2)^2 \\ \downarrow{\scriptstyle p_\sharp\circ\rho'} & & \downarrow{\scriptstyle (p_\sharp\circ\rho')^2} \\ S(E\underset{\pi}{\times}X^2) & \xrightarrow{\Delta} & S(E\underset{\pi}{\times}X^2) \end{array}$$

which is commutative up to chain homotopy.

Recall now the definition of cup product (cap product) in terms of Δ and cross (slant) product. Then the above diagram yields the desired property. This completes the proof of Theorem 3.

4. Steenrod theorem

In §4–§8, we assume that the ground ring R is $\boldsymbol{Z}_2$, the field of integers mod 2. We assume also that $H_*(X)$ is of finite type.

As is well known, a π-free acyclic complex W can be constructed as follows: For each $i \geqq 0$, W has one cell e_i and its transform Te_i, and $\partial(e_i)=e_{i-1}+Te_{i-1}$ $(i>0)$. Moreover there is a diagonal approximation $d_\sharp$: $W \to W \otimes W$ which is given by

$$d_\sharp(e_i) = \sum_{j=0}^{[i/2]} (e_{2j} \otimes e_{i-2j} + e_{2j+1} \otimes Te_{i-2j-1}) .$$

Therefore we can determine the structure of $H_*(\pi; H_*(X)^2)$ and $H^*(\pi; H^*(X)^2)$, and hence by Theorems 2 and 3 the structure of $H_*(E \underset{\pi}{\times} X^2)$ and $H_*(E \underset{\pi}{\times} X^2)$ as soon as we know the structure of $H_*(X)$. To state the result, we shall first prepare some notations.

For an element $a \in H_*(X)$, let $Q_i(a) \in H_*(\pi; H_*(X)^2)$ $(i \geqq 0)$ denote the element represented by the cycle $e_i \otimes a \otimes a \in W \otimes H_*(X)^2$. Given an element $\alpha \in H^*(X)$ a cocycle $u_i(\alpha) \in \mathrm{Hom}_\pi (W, H^*(X)^2)$ is defined by $\langle u_i(\alpha), e_j \rangle = \alpha \otimes \alpha$ $(i=j)$, $=0$ $(i \neq j)$. Let $Q_i(\alpha) \in H^*(\pi; H^*(X)^2)$ denote the element respresented by $u_i(\alpha)$.

For two elements a, $b \in H_*(X)$, let $Q(a, b) \in H_*(\pi; H_*(X)^2)$ denote the element represented by the cycle $e_0 \otimes a \otimes b \in W \underset{\pi}{\otimes} H_*(X)^2$. Given two elements α, $\beta \in H_*(X)$, a cocycle $u(\alpha, \beta) \in \mathrm{Hom}_\pi (W, H^*(X)^2)$ is defined by $\langle u(\alpha, \beta), e_j \rangle$ $= \alpha \otimes \beta + \beta \otimes \alpha$ $(j=0)$, $=0$ $(j \neq 0)$. Let $Q(\alpha, \beta) \in H^*(\pi; H^*(X)^2)$ denote the element represented by $u(\alpha, \beta)$.

We shall put

$$P_i(a) = \kappa(Q_i(a)) , \quad P(a, b) = \kappa(Q(a, b)) ,$$
$$P_i(\alpha) = \kappa(Q_i(\alpha)) , \quad P(\alpha, \beta) = \kappa(Q(\alpha, \beta)) .$$

Obviously we have $P(a, b) = P(b, a)$, $P(a, a) = P_0(a)$, $P(\alpha, \beta) = P(\beta, \alpha)$ and $P(\alpha, \alpha)=0$.

The following theorem is proved easily.

Theorem 4. (i) *If $\{a_j \mid j \in J\}$ is an ordered basis of the module $H_*(X)$, then $\{P_i(a_j), P(a_j, a_k)\}$ $i \geqq 0, j, k \in J, j>k\}$ is a basis of the module $H_*(E \underset{\pi}{\times} X^2)$. Similarly, if $\{\alpha_j \mid j \in J\}$ is an ordered basis of the module $H^*(X)$, then $\{P_i(\alpha_j)$, $P(\alpha_j, \alpha_k) \mid i \geqq 0, j, k \in J, j>k\}$ is a basis of the module $H^*(E \underset{\pi}{\times} X^2)$.*

(ii) *For α, β, α', $\beta' \in H^*(X)$ and a, $b \in H_*(X)$, we have*

$$P_i(\alpha) \smile P_j(\beta) = P_{i+j}(\alpha \smile \beta);$$

$$P_i(\alpha) \smile P(\alpha', \beta') = \begin{cases} P(\alpha \smile \alpha', \alpha \smile \beta') & \text{if } i=0 , \\ 0 & \text{if } i>0 ; \end{cases}$$

$$P(\alpha,\beta)\smile P(\alpha',\beta')=P(\alpha\smile\alpha',\beta\smile\beta')+P(\alpha\smile\beta',\beta\smile\alpha'),$$

$$P_j(\alpha)\frown P_i(a)=\begin{cases} P_{i-j}(\alpha\frown a) & \text{if } i\geqq j,\\ 0 & \text{if } i<j;\end{cases}$$

$$P(\alpha,\beta)\frown P_i(a)=0;$$

$$P_j(\alpha)\frown P(a,b)=\begin{cases} P(\alpha\frown a,\alpha\frown b) & \text{if } j=0,\\ 0 & \text{if } j>0;\end{cases}$$

$$P(\alpha,\beta)\frown P(a,b)=P(\alpha\frown a,\beta\frown b)+P(\beta\frown a,\alpha\frown b),$$

where it is to be understood that $\alpha\frown a=0$ *if* $\deg\alpha>\deg a$.

(iii) *If* $\{a_j\}$ *and* $\{\alpha_j\}$ *are dual bases, then so are* $\{P(a_j), P(a_j, a_k)\}$ *and* $\{P_i(\alpha_j), P(\alpha_j,\alpha_k)\}$.

Define a continuous map $d\colon E\times X\to E\times X^2$ by $d(y,x)=(y,x,x)$ ($y\in E$, $x\in X$). Then d induces a homomorphism

$$d^*\colon H^*(E\underset{\pi}{\times}X^2)\to H^*(E_\pi\times X).$$

We have the following theorem due to Steenrod (see [8], p. 103).

Theorem 5. *For* $\alpha\in H^q(X)$ *we have*

$$d^*P_0(\alpha)=\sum_{k=0}^{q}\omega^k\times Sq^{q-k}\alpha,$$

where $\omega^k\in H^k(E_\pi)$ *is the generator, and* $Sq^i\colon H^q(X)\to H^{q+i}(X)$ *is the squaring operation.*

Proof. Let $\lambda\colon S(E)\otimes S(X)\to S(X)^2$ be a functorial equivariant chain map defined for each topological space X, and $\varphi\colon S(E\times X)\to S(E)\otimes S(X)$ be a chain equivalence in the Eilenberg-Zilber theorem. Let $u\in S(X)^*$ be a cocycle representing α. Then it follows from the definition of Sq^i that $\varphi^\sharp\lambda^\sharp\mu(u\otimes u)$ $\in(S(E)\otimes S(X))^*$ is an equivariant cocycle representing $\sum_{k=0}^{q}\omega^k\times Sq^{q-k}\alpha\in H^{2q}(E_\pi\times X)$, where $\mu\colon S(X)^{*2}\to S(X)^{2*}$ is the canonical cochain map. Consider the composition of chain maps

$$S(E)\otimes S(X)\xrightarrow{\varphi'}S(E\times X)\xrightarrow{d_\sharp}S(E\times X^2)$$
$$\xrightarrow{\rho}S(E)\otimes S(X)^2\xrightarrow{\varepsilon\otimes 1}S(X)^2,$$

where φ' is an inverse of φ and ε is the augmentation. Since each map is functorial and equivariant, we can take the composition as λ. It is easily seen that $\rho^\sharp(\varepsilon\otimes 1)^\sharp\mu\,(u\times u)\in S(E\times X^2)^*$ is an equivariant cocycle representing $P_0(\alpha)$. Therefore we have the desired result.

Corollary. *For* $\alpha \in H^q(X)$ *we have*

$$d^*P_i(\alpha) = \sum_{k=0}^{q} \omega^{k+i} \times Sq^{q-k}\alpha\,.$$

Proof. Since $P_i(\alpha)=P_i(1)\smile P_0(\alpha)$ by Theorem 4, we have $d^*P_i(\alpha)=d^*P_i(1)\smile d^*P_0(\alpha)$. Let P be a single point, and consider the commutative diagram

$$\begin{array}{ccccc} H^*(\pi;\, H^*(P)^2) & \xrightarrow{\kappa} & H^*(E \underset{\pi}{\times} P^2) & \xrightarrow{d^*} & H^*(E_\pi \times P) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(\pi;\, H^*(X)^2) & \xrightarrow{\kappa} & H^*(E \underset{\pi}{\times} X^2) & \xrightarrow{d^*} & H^*(E_\pi \times X)\,, \end{array}$$

where the vertical maps are induced by the map $X \to P$. Then it follows that $d^*P_i(1)=\omega^i \times 1$. Therefore the corollary follows from the theorem.

5. Homology of $N \underset{\pi}{\times} X^2$

Let Y be a topological space, and consider the suspension SY of Y. We regard SY as the quotient space of $Y\times[0, 1]$, and identify Y with the subspace $Y\times 1/2$ of SY. If T is an involution on Y, it is extended to an involution T' on SY by putting

$$T'(y, s) = (Y(y), 1-s) \qquad (y \in Y,\ 0 \leqq s \leqq 1).$$

If T has no fixed point, so does T'.

Let N be a compact Hausdorff space having the mod 2 homology of the n-sphere, and suppose that there is given on N an involution without fixed points. Define now, for each integer $i \geqq 0$, a compact Hausdorff space N^i by

$$N^0 = N, \quad N^i = SN^{i-1} \quad (i \geqq 1),$$

and let N^∞ denote the inductive limit of the sequence $N \subset N^1 \subset \cdots \subset N^i \subset \cdots$. Then, by the fact stated above, there exists an involution T: $N^\infty \to N^\infty$ without fixed points such that $T(N^i)=N^i$ $(i \geqq 0)$ and $T \mid N$ is the given involution. Moreover, since $\widetilde{H}_q(N^i) \cong \widetilde{H}_{q-i}(N)=0$ $(i>q-n)$, we have $\widetilde{H}_q(N^\infty)=\varinjlim \widetilde{H}_q(N^i)=0$ for any q. Thus N^∞ has the properties assumed for E.

Assuming next that X is an arcwise connected topological space, we shall consider the space $N^\infty \underset{\pi}{\times} X^2$ and its subspace $N \underset{\pi}{\times} X^2$.

Theorem 6. *The homomorphism* i_*: $H_k(N \underset{\pi}{\times} X^2) \to H_k(N^\infty \underset{\pi}{\times} X^2)$ *induced by the inclusion is an isomorphism if* $k \leqq n$.

Proof. The projection $N^\infty \times X^2 \to N^\infty$ to the first factor defines a fibration p: $N^\infty \underset{\pi}{\times} X^2 \to N^\infty_\pi$ with fiber X^2, and we have $p^{-1}(N^i_\pi)=N^i \underset{\pi}{\times} X^2$ for each i.

Consider the Serre spectral sequence E for the relative fibration p: $(N^\infty \underset{\pi}{\times} X^2, N \underset{\pi}{\times} X^2) \to (N_\pi^\infty, N_\pi)$. Then we have

$$E_{p,q}^2 \cong H_p(N_\pi^\infty, N_\pi; \{H_q(X^2)\}),$$

and E^∞ is the graded module associated to some filtration of $H_*(N^\infty \underset{\pi}{\times} X^2, N \underset{\pi}{\times} X^2)$. By the properties of homology, it holds that

$$\begin{aligned} & H_p(N_\pi^i, N_\pi^{i-1}; \{H_q(X^2)\}) \cong H_p(CN^{i-1}, N^{i-1}, H_q(X^2)) \\ \cong\ & \widetilde{H}_{p-1}(N^{i-1}; H_q(X^2)) \cong \widetilde{H}_{p-i}(N) \otimes H_q(X^2) \\ \cong\ & \begin{cases} H_q(X^2) & \text{if } p-i=n, \\ 0 & \text{if } p-i \neq n, \end{cases} \end{aligned}$$

where $i \geqq 1$ and CN^{i-1} denote the cone over N^{i-1}. Therefore the homomorphism $H_p(N_\pi^{i-1}, N_\pi; \{H_q(X^2)\}) \to H_p(N_\pi^i, N_\pi; \{H_q(X^2)\})$ is injective if $p-i \neq n-1$, and is surjective if $p-i \neq n$. Hence $H_p(N_\pi^i, N_\pi; \{H_q(X^2)\}) \cong H_p(N_\pi^\infty, N_\pi; \{H_q(X^2)\})$ if $p-i \leqq n-1$. In particular, $H_p(N_\pi, N_\pi; \{H_q(X^2)\}) \to H_p(N_\pi^\infty, N_\pi; \{H_q(X^2)\})$ is surjective if $p \leqq n$, and so we have

$$E_{p,q}^2 = 0 \qquad (p \leqq n).$$

The usual technique in spectral sequence proves now that $H_k(N^\infty \underset{\pi}{\times} X^2, N \underset{\pi}{\times} X^2) = 0$ for $k \leqq n$. Thus i_*: $H_k(N \underset{\pi}{\times} X^2) \to H_k(N^\infty \underset{\pi}{\times} X^2)$ is bijective if $k \leqq n-1$, and is surjective if $k=n$.

We shall next prove that i_*: $H_n(N \underset{\pi}{\times} X^2) \to H_n(N^\infty \underset{\pi}{\times} X^2)$ is injective.

Since $H_{n+1}(N_\pi) = 0$, the homomorphism $H_{n+1}(N_\pi^\infty) \to H_{n+1}(N_\pi^\infty, N_\pi)$ is injective. On the other hand, $\boldsymbol{Z}_2 \cong H_{n+1}(N_\pi^1, N_\pi) \to H_{n+1}(N_\pi^\infty, N_\pi)$ is surjective. Therefore we have $H_{n+1}(N_\pi^\infty) \cong H_{n+1}(N_\pi^\infty, N_\pi)$.

Consider the Serre spectral sequence $'E$ of the fibration p: $N^\infty \underset{\pi}{\times} X^2 \to N_\pi^\infty$. Then we have $'E_{p,q}^2 \cong H_p(N_\pi^\infty; \{H_q(X^2)\})$, and $'E^\infty$ is the graded module associated with some filtration of $H_*(N^\infty \underset{\pi}{\times} X^2)$. Since $H_*(N^\infty \underset{\pi}{\times} X^2) \cong H_*(N_\pi^\infty; \{H_*(X^2)\})$ by Theorem 2, the usual technique in spectral sequence proves that $'E_{p,q}^2 = {'E_{p,q}^\infty}$.

Consider now the commutative diagram

$$\begin{array}{ccc} H_{n+1}(N^\infty \underset{\pi}{\times} X^2) & \longrightarrow & 'E_{n+1,0}^2 = H_{n+1}(N_\pi^\infty) \\ \big\downarrow & & \big\downarrow \cong \\ H_{n+1}(N^\infty \underset{\pi}{\times} X^2, N \underset{\pi}{\times} X^2) & = E_{n+1,0}^2 = & H_{n+1}(N_\pi^\infty, N_\pi). \end{array}$$

Then it follows that the map in the left is surjective. Therefore i_*: $H_n(N \underset{\pi}{\times} X^2) \to H_n(N^\infty \underset{\pi}{\times} X^2)$ is injective. This completes the proof of Theorem 6.

Lemma 3. *Let $i \leqq n$ and $a \in H_*(X)$. Then $P_i(a)$ is in the image of the homomorphism i_*: $H_*(N \underset{\pi}{\times} X^2) \to H_*(N^\infty \underset{\pi}{\times} X^2)$.*

Proof. Since $i \leqq n$, we have $H_i(N_\pi^\infty, N_\pi; \{H_*(X^2)\})=0$. Hence the homomorphism $H_i(N_\pi; \{H_*(X^2)\}) \to H_i(N_\pi^\infty; \{H_*(X^2)\})$ is surjective. This shows that $Q_i(a)$ is represented by a cycle of $S(N)\otimes S(X)^2$. Since $\tilde{H}_i(N)=0$ $(i<n)$ and $T(N)=N$, it follows from Corollary to Theorem 1 that $P_i(a)$ is represented by a cycle of $S(N\times X^2)$. Therefore we get the desired result.

6. The element θ

Let N be a closed topological manifold having the mod 2 homology of the n-sphere, and let M be a connected closed topological manifold of dimension m. Suppose that there is given on N an involution without fixed points. Then $N \underset{\pi}{\times} M^2$ is a connected closed topological manifold of dimension $n+2m$. Let $\mu \in H_m(M)$ and $\lambda \in H_{2m+n}(N \underset{\pi}{\times} M^2)$ denote the mod 2 fundamental class of M and $N \underset{\pi}{\times} M^2$ respectively.

By Lemma 3, $P_n(\mu)\in H_{n+2m}(N^\infty \underset{\pi}{\times} M^2)$ is in the image of the homomorphism $i_*: H_{n+2m}(N \underset{\pi}{\times} M^2) \to H_{n+2m}(N^\infty \underset{\pi}{\times} M^2)$. Therefore we have

$$P_n(\mu)=i_*(\lambda)\,.$$

Define a continuous map $d_0: N\times M \to N\times M^2$ by $d_0(y, x)=(y, x, x)(y\in N, x\in M)$. Then d_0 induces a homomorphism $d_{0*}: H_*(N_\pi\times M) \to H_*(N \underset{\pi}{\times} M^2)$. Define

$$\theta_0 \in H^m(N \underset{\pi}{\times} M^2)$$

to be the Poincaré dual of $d_{0*}(\nu_n\times\mu)$, where $\nu_n \in H_n(N_\pi)$ is the generator. We have

$$\theta_0 \frown \lambda = d_{0*}(\nu_n\times\mu)\,.$$

Assume now that $m \leqq n$. Since $i^*: H^k(N^\infty \underset{\pi}{\times} M^2) \simeq H^k(N \underset{\pi}{\times} M^2)$ for $k \leqq n$ by Theorem 6, there exists a unique element $\theta \in H^m(N^\infty \underset{\pi}{\times} M^2)$ such that

$$i^*(\theta) = \theta_0\,.$$

For the homomorphism $d_*: H_*(N_\pi^\infty\times M) \to H_*(N^\infty \underset{\pi}{\times} M^2)$ induced by the 'diagonal' map, we have

$$d_*(i_*(\nu_n)\times\mu) = \theta \frown P_n(\mu)\,.$$

In fact,

$$\begin{aligned} d_*(i_*(\nu_n)\times\mu) &= d_* i_*(\nu_n\times\mu) \\ &= i_* d_{0*}(\nu_n\times\mu) = i_*(\theta_0\frown\lambda) \\ &= i_*(i^*(\theta)\frown\lambda) = \theta\frown i_*(\lambda) \\ &= \theta \frown P_n(\mu)\,. \end{aligned}$$

Let $U_i \in H^i(M)$ denote the *Wu* class, i.e. the element defined by

$$U_i \cap \mu = Sq^{i*}(\mu),$$

where $Sq^{i*}: H_m(M) \rightarrow H_{m-i}(M)$ is the transpose of $Sq^i: H^{m-i}(M) \rightarrow H^m(M)$.

Theorem 7. *If $m \leqq n$, we have*

$$\theta = \sum_{i=0}^{[m/2]} P_{m-2i}(U_i) + \delta,$$

where δ is a linear combination of elements of the type $P(\alpha, \beta)$.

Proof. For any $\alpha \in H^q(M)$ with $2q \geqq m \geqq q$, we have

$$\begin{aligned}
&\langle P_{n+m-2q}(\alpha), d_*(i_*(\nu_n) \times \mu)\rangle \\
&= \langle P_{n+m-2q}(\alpha), \theta \frown P_n(\mu)\rangle \\
&= \langle P_{n+m-2q}(\alpha) \smile \theta, P_n(\mu)\rangle \\
&= \langle \theta, P_{n+m-2q}(\alpha) \frown P_n(\mu)\rangle \\
&= \langle \theta, P_{2q-m}(\alpha \frown \mu)\rangle \qquad \text{(by (ii) of Theorem 4).}
\end{aligned}$$

We have also

$$\begin{aligned}
&\langle P_{n+m-2q}(\alpha), d_*(i_*(\nu_n) \times \mu)\rangle \\
&= \langle d^* P_{n+m-2q}(\alpha), i^*(\nu_n) \times \mu\rangle \\
&= \langle \sum_{i=0}^{q} \omega^{n+m-2q+i} \times Sq^{q-i}(\alpha), i_*(\nu_n) \times \mu\rangle \\
&\qquad\qquad \text{(by Corollary of Theorem 5)} \\
&= \langle \omega^n \times Sq^{m-q}(\alpha), i_*(\nu_n) \times \mu\rangle \\
&= \langle Sq^{m-q}(\alpha), \mu\rangle = \langle \alpha, U_{m-q} \frown \mu\rangle \\
&= \langle U_{m-q}, \alpha \frown \mu\rangle \\
&= \langle \sum_{i=0}^{[m/2]} P_{m-2i}(U_i), P_{2q-m}(\alpha \frown \mu)\rangle \qquad \text{(by (ii) of Theorem 4).}
\end{aligned}$$

Therefore we get the desired result by (iii) of Theorem 4.

7. Proof of the main theorem

In this section we shall prove the main theorem.

For a continuous map $f: N \rightarrow M$, a continuous $s: N_\pi \rightarrow N \underset{\pi}{\times} M^2$ can be defined by

$$s(y) = (y, f(y), f(Ty)) \qquad (y \in N).$$

For the homomorphism $s^*: H^m(N \underset{\pi}{\times} M^2) \rightarrow H^m(N_\pi)$, we have

Lemma 4. *If $m \leqq n$ and $f_*: H_n(N) \rightarrow H_n(M)$ is trivial, it holds that $s^*(\theta_0) \neq 0$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} H^m(N^\infty \underset{\pi}{\times} M^2) & \xrightarrow{i^*} & H^m(N \underset{\pi}{\times} M^2) \\ \Big\downarrow{\scriptstyle (1\times f^2)} & & \Big\downarrow{\scriptstyle s^*} \\ H^m(N^\infty \underset{\pi}{\times} N^2) & \xrightarrow{k^*} & H^m(N_\pi), \end{array}$$

where $k\colon N_\pi \to N^\infty \underset{\pi}{\times} N^2$ is given by $k(y)=(y, y, Ty)$ $(y\in N)$. Therefore, by Theorem 7 we have

$$\begin{aligned} s^*(\theta_0) &= s^*i^*(\theta) = k^*(1\times f^2)^*(\theta) \\ &= k^*(1\times f^2)^*(\sum_{i=0}^{[m/2]} P_{m-2i}(U_i)+\delta)\,. \end{aligned}$$

From this and the assumption it follows that

$$s^*(\theta_0) = k^*(P_m(1))\,.$$

If P is a single point and $g\colon N\to P$ is the map, the diagram

$$\begin{array}{ccc} H^m(N^\infty_\pi) & \xrightarrow{p^*} & H^m(N^\infty \underset{\pi}{\times} P^2) \\ \Big\downarrow{\scriptstyle i^*} & \searrow{\scriptstyle p^*} & \Big\downarrow{\scriptstyle (1\times g^2)^*} \\ H^m(N_\pi) & \xleftarrow{k^*} & H^m(N^\infty \underset{\pi}{\times} N^2) \end{array}$$

is commutative. Therefore we have

$$k^*(P_m(1)) = k^*p^*(\omega^m) = i^*(\omega^m) \neq 0\,,$$

which completes the proof of Lemma 4.

For a continuous map $f\colon N\to M$, put

$$A(f) = \{y\in N \mid f(y) = f(Ty)\}\,.$$

Lemma 5. *If $s^*(\theta_0)\neq 0$, then the covering dimension of $A(f)$ is at least $n-m$.*

Proof. By the Thom isomorphism, we have

$$H^0(N \underset{\pi}{\times} M^2) \cong H^{n+2m}((N \underset{\pi}{\times} M^2)^2,\ (N \underset{\pi}{\times} M^2)^2 - \Delta(N \underset{\pi}{\times} M^2))\,,$$

where $\Delta\colon N \underset{\pi}{\times} M^2 \to (N \underset{\pi}{\times} M^2)^2$ is the diagonal map. Let $\gamma\in H^{n+2m}((N \underset{\pi}{\times} M^2)^2,$ $(N \underset{\pi}{\times} M^2)^2 - \Delta(N \underset{\pi}{\times} M^2))$ be the generator, and put

$$\begin{aligned} \gamma_1 &= \gamma \mid (N \underset{\pi}{\times} M^2)^2\,, \\ \gamma_2 &= \gamma \mid d_0(N_\pi \times M) \times (N \underset{\pi}{\times} M^2,\ N \underset{\pi}{\times} M^2 - d_0(N_\pi \times M))\,. \end{aligned}$$

Write $B(f)$ for the image of $A(f)$ under the projection $N \to N_\pi$. Then the following commutative diagram holds:

$$\begin{array}{ccc}
H_{n+m}(d_0(N_\pi \times M)) & \xrightarrow{j_*} & H_{n+m}(N \underset{\pi}{\times} M^2) \\
\downarrow \backslash\gamma_2 & & \downarrow \backslash\gamma_1 \\
H^m(N \underset{\pi}{\times} M^2, N \underset{\pi}{\times} M^2 - d_0(N_\pi \times M)) & \xrightarrow{j^*} & H^m(N \underset{\pi}{\times} M^2) \\
\downarrow s^* & & \downarrow s^* \\
H^m(N, N_\pi - B(f)) & \xrightarrow{j^*} & H^m(N_\pi),
\end{array}$$

where j are the inclusion maps, and $\backslash$ denotes the slant product (see [7], p. 351). Since $\backslash\gamma_1$ is the inverse of the Poincaré duality isomorphism, the image of the generator of $H_{n+m}(d_0(N_\pi \times M))$ under the composition of j_* and $\backslash\gamma_1$ is θ_0. Therefore Lemma 4 implies $H^m(N_\pi, N_\pi - B(f)) \neq 0$. Since this shows $H_m(N_\pi, N_\pi - B(f)) \neq 0$, it follows that the Čech cohomology group $\check{H}^{n-m}(B(f))$ is not zero (see Theorem 17 in p. 296, Corollary 8 in p. 334 and Corollary 9 in p. 341 of [7]). Therefore dim $(B(f)) \geqq n-m$, and hence dim $(A(f)) \geqq n-m$. This complestes the proof of Lemma 5.

We are now ready for proving the main theorem.

Proof of Main Theorem. By Lemma 4 and Lemma 5, we have the main theorem for a connected closed topological manifold M. From this the result for any compact manifold M is obtained easily (see [4]).

8. Corollaries of the main theorem

The following corollary is obtained immediately from the main theorem.

Corollary 1. *Let N, T and M be the same as in the main theorem, and T' be a fixed point free involution on M.*

(i) *If $n > m$, there exists no continuous map $f: N \to M$ equivariant with T and T'.*

(ii) *If $n = m$ and $f: N \to M$ is a continuous map equivariant with T and T', then $f_*: H_n(N) \to H_n(M)$ is not trivial.*

The following corollaries are obtained by the same way as in [1], p. 89.

Corollary 2. *Let N be a closed topological manifold which is a mod 2 homology n-sphere. Then any pair of fixed point free involution T_1 and T_2 on N have a co-incidence.*

Corollary 3. *If G is a group acting freely on a closed topological manifold N having the mod 2 homology of the n-sphere. Then any element of G order 2 lies in the center of G.*

REMARK. This corollary was first proved by Milnor [3] in a different method.

9. The corresponding theorem for Z_p-actions

The main theorem has the following corresponding result for Z_p-actions on mod p homology spheres, where p is an odd prime.

Theorem 8. *Let N be a closed topological manifold which is a mod p homology n-sphere, and T: $N \to N$ be a continuous map of period p without fixed points, where p is an odd prime. Let f: $N \to M$ be a continuous map of N to a compact orientable topological manifold of dimension m, where $n \geqq (p-1)m$. Then the covering dimension of*

$$A(f) = \{y \in N \mid f(y) = f(Ty) = \cdots = f(T^{p-1}y)\}$$

is at least $n-(p-1)m$.

REMARK. Munkholm [5] proved this theorem under a hypotheses that f is a 'nice' map.

Theorem 8 is proved in the similar way to the proof of the main theorem. We shall give outlines of the proof in the following and leave details to the reader.

Let E be a Hausdorff space such that $\widetilde{H}_*(E)=0$, and T: $E \to E$ be a continuous map of period p without fixed points. Let π denote the cyclic group of order p whose generator is T. Then there exist functorial isomorphisms

$$\kappa\colon\ H_*(\pi;\, H_*(X)^p) \cong H_*(E \underset{\pi}{\times} X^p)\,,$$
$$\kappa\colon\ H^*(\pi;\, H^*(X)^p) \cong H^*(E \underset{\pi}{\times} X^p)\,,$$

defined for each topological space X such that $H_*(X)$ is free and of finite type (see Theorem 2), and κ preserve the cup products and the cap products (see Theorem 3). In virtue of these results, the elements $P_i(a)$, $P(a_1, \cdots, a_p) \in H_*(E \underset{\pi}{\times} X^p;\, Z_p)$ can be defined for $a, a_1, \cdots, a_p \in H_*(X;\, Z_p)$, and the elements $P_i(\alpha)$, $P(\alpha_1, \cdots, \alpha_p) \in H^*(E \underset{\pi}{\times} X^p;\, Z_p)$ can be defined for $\alpha, \alpha_1, \cdots, \alpha_p \in H^*(X;\, Z_p)$. As for these, the theorem similar to Theorem 4 holds. Let $\omega^k \in H^k(E_\pi;\, Z_p)$ be the canonical generator, and d^*: $H^*(E \underset{\pi}{\times} X^p;\, Z_p) \to H^*(E_\pi \times X;\, Z_p)$ denote the homomorphism induced by the 'diagonal' map d: $E \times X \to E \times X^p$. Then, for $\alpha \in H^q(X;\, Z_p)$ we have

$$d^*P_0(\alpha) = c_q \sum_i (-1)^i (\omega^{(p-1)(q-2i)} \times \mathcal{P}^i(\alpha) - \omega^{(p-1)(q-2i)-1} \times \beta\mathcal{P}^i(\alpha))\,,$$

where $\mathcal{P}^i$ is the p-th reduced power, β is the Bockstein homomorphism, and

$c_q=(-1)^{q/2}$ or $(-1)^{(q-1)/2}((p-1)/2)!$ according as q is even or odd (see Theorem 5).

Put $N^0=N$, and define N^{2i} $(i=1, 2, \cdots)$ inductively to be the join of N^{2i-2} and $S^1=\{z\in \boldsymbol{C} \mid |z|=1\}$. We define also N^{2i-1} $(i=1, 2, \cdots)$ to be a subspace of N^{2i} consisting of all $sy\oplus(1-s)e^{2\pi k\sqrt{-1}/p}$, where $0\leqq s\leqq 1$ and $k=0, 1, \cdots, p-1$. Let N^∞ be the limit space of the sequence $N\subset N^1\subset N^2\subset\cdots$. There exists a continuous map $T\colon N^\infty\to N^\infty$ of period p without fixed points such that $T(N^i)\subset N_i$ $(i=0, 1, 2, \cdots)$ and $T\mid N$ is the given map $T\colon N\to N$. In fact, such a map T is defined by

$$T_i(sy\oplus(1-s)e^{2\pi\sqrt{-1}t}) = s(T_{i-1}y)\otimes(1-s)e^{2\pi\sqrt{-1}(t+1/p)}$$

where $T=T\mid N^{2i}$ and $s, t\in[0, 1]$. It follows that N^∞ has the properties assumed for E, and that $i_*\colon H_k(N\underset{\pi}{\times}X^p; \boldsymbol{Z}_p)\to H_k(N^\infty\underset{\pi}{\times}X^p; \boldsymbol{Z}_p)$ is an isomorphism if $k\leqq n$ (see Theorem 6).

Let $d_{0*}\colon H_{n+m}(N_\pi\times M; \boldsymbol{Z}_p)\to H_{n+m}(N\underset{\pi}{\times}M^p; \boldsymbol{Z}_p)$ be the homomorphism induced by the 'diagonal' map, and $\theta_0\in H^{(p-1)m}(N\underset{\pi}{\times}M^p; \boldsymbol{Z}_p)$ be the Poincaré dual of $d_{0*}(\lambda)$, where $\lambda\in H_{n+m}(N_\pi\times M; \boldsymbol{Z}_p)$ is the fundamental class. If $(p-1)m\leqq n$, there exists a unique $\theta\in H^{(p-1)m}(N^\infty\underset{\pi}{\times}M^p; \boldsymbol{Z}_p)$ such that $\theta\mid N\underset{\pi}{\times}M^p=\theta_0$. Similarly to Theorem 7, we have

$$\theta = c_m(\sum_j(-1)^j P_{(p-1)(m-2jp)}(U_j))+\delta\ ,$$

where δ is a linear combination of elements of the type $P(\alpha_1, \cdots, \alpha_p)$, and $U_j\in H^{2j(p-1)}(M; \boldsymbol{Z}_p)$ is the "Wu class' defined in terms of $\mathcal{P}^j$.

Consider now a continuous map $s\colon N_\pi\to N\underset{\pi}{\times}M^p$ defined by $s(y)=(y, f(y), f(Ty), \cdots, f(T^{p-1}y))$ $(y\in N)$, and proceed as in §7. Then we see that $s^*(\theta_0)\neq 0$ and hence $\dim A(f)\geqq n-(p-1)m$.

OSAKA UNIVERSITY

References

[1] P.E. Conner and E.E. Floyd: Differentiable Periodic Maps, Springer-Verlag, Berlin, 1964.

[2] A. Heafliger: *Points multiples d'une application et produit cyclique reduit,* Amer. J. Math. **83** (1961), 57–70.

[3] J. Milnor: *Groups which act on S^n without fixed points,* Amer. J. Math. **79** (1957), 623–630.

[4] H.J. Munkhom: *A Borsuk-Ulam theorem for maps from a sphere to a compact topological manifold,* Illinois J. Math. **13** (1969), 116–124.

[5] H.J. Munkholm: *Borsuk-Ulam type theorems for proper Z_p-actions on (mod p homology) n-spheres*, Math. Scand. **24** (1969), 167–185.
[6] M. Nakaoka: *Homology of the infinite symmetric group*, Ann. of Math. **73** (1961), 229–257.
[7] E.H. Spanier: Algebraic Topology, McGraw-Hill, New York, 1966.
[8] N.E. Steenrod and D.B.A. Epstein: Cohomology Operations, Princeton Univ. Press, 1962.