Noda, R. and Yamaki, H.
Osaka J. Math.
7 (1970), 313-319

A CHARACTERIZATION OF THE ALTERNATING GROUPS OF DEGREES SIX AND SEVEN[1)]

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(Received April 10, 1970)

1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem. *Let $\mathfrak{G}$ be a doubly transitive group on the set $\Omega=\{1, 2, \cdots, n\}$. If the stabilizer $\mathfrak{K}$ of the set of points* 1 *and* 2 *is isomorphic to the alternating group of degree four, then one of the followings holds:*

(1) $n=6$ *and* $\mathfrak{G}$ *is* $\mathfrak{A}_6$,
(2) $n=15$ *and* $\mathfrak{G}$ *is* $\mathfrak{A}_7$,
(3) $n=16$ *and* $\mathfrak{G}$ *is* $A(2, 4)$,
(4) $\mathfrak{G}=C_{\mathfrak{G}}(J)O(\mathfrak{G})$ *for some involution* J.

Here $\mathfrak{A}_m$ denotes the alternating group of degree m and $A(t, q)$ denotes the group of all affine transformations of the t-dimensional affine geometry $AG(t, q)$ over the field of q-elements.

Notation. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the subset of $\mathfrak{G}$. $\mathfrak{F}(\mathfrak{X})$ will denote the set of all the fixed points of $\mathfrak{X}$ and $\alpha(\mathfrak{X})$ is the number of points in $\mathfrak{F}(\mathfrak{X})$. $\mathfrak{X}\sim\mathfrak{Y}$ means that $\mathfrak{X}$ is conjugate to $\mathfrak{Y}$ in $\mathfrak{G}$. All other notation is standard.

2. Preliminaries

Since $\mathfrak{K}$ is $\mathfrak{A}_4$, $\mathfrak{K}$ is generated by the elements K and τ subject to the following relations:

$$K^3=\tau^2=(K\tau)^3=1 \tag{2.1}$$

Put $\tau_1=K^{-1}\tau K$ and $\mathfrak{B}=\langle\tau, \tau_1\rangle$. Then $\mathfrak{B}$ is a four group and a Sylow 2-subgroup of $\mathfrak{K}$. Let $\mathfrak{H}$ be the stabilizer of the point 1. Since $\mathfrak{G}$ is doubly transitive on Ω, it contains an involution I with the cycle structure $(1, 2)\cdots$ which normalizes $\mathfrak{K}$ and we may assume that $[I, \tau]=1$. Thus we have the following decomposition of $\mathfrak{G}$:

$$\mathfrak{G}=\mathfrak{H}\cup\mathfrak{H}I\mathfrak{H} \tag{2.2}$$

1) We thank Professor H. Nagao for pointing out a gap of our original proof.

Let $g(2)$, $h(2)$ and d denote the number of involutions in $\mathfrak{G}$, $\mathfrak{H}$, and the coset $\mathfrak{H}IH$ for $H\in\mathfrak{H}$, respectively. Then d is the number of elements in $\mathfrak{K}$ inverted by I, that is, the number of involutions in $\mathfrak{G}$ with the cycle structure $(1, 2)\cdots$ and the following equality is obtained from (2.2):

$$g(2)=h(2)+d(n-1) \tag{2.3}$$

Lemma 1. *One of the followings holds:*

(1) $I\tau_1 I=\tau\tau_1$, $IKI=K^{-1}$, $d=6$,
$I\sim IK\sim IK^2\sim I\tau K\tau\sim I\tau K^2\tau\sim I\tau$.

(2) $[I, \mathfrak{V}]=1$, $IKI=\tau K\tau$, $d=4$,
$I\tau\sim I\tau_1\sim I\tau\tau_1$.

(3) $[I, \mathfrak{K}]=1$, $d=4$,
$I\tau\sim I\tau_1\sim I\tau\tau_1$.

Proof. Since the automorphism group of $\mathfrak{K}$ is the symmetric group of degree four we may assume that the action of I on $\mathfrak{K}$ is (1), (2), or (3) by (2.1). Assume that the case (1) holds. Now $\langle I, K\rangle$ and $\langle I, \tau K\tau\rangle$ are dihedral groups of order 6. Therefore $I\sim IK\sim IK^2$ in $\langle I, K\rangle$ and $I\sim I\tau K\tau\sim I\tau K^2\tau$ in $\langle I, \tau K\tau\rangle$. Thus the result follows in this case. Note that in this case every involution is conjugate to τ in $\mathfrak{G}$. The cases (2) and (3) are trivial. This proves our lemma.

Let τ keep i $(i\geqslant 2)$ points of Ω unchanged. So we may put $\mathfrak{J}(\tau)=\{1, 2, \cdots, i\}$. The group $C_{\mathfrak{G}}(\tau)$ acts on $\mathfrak{J}(\tau)$ and the kernel of this permutation representation is $\mathfrak{V}$ or $\langle\tau\rangle$ because $C_{\mathfrak{G}}(\tau)\cap\mathfrak{K}=\mathfrak{V}$. By a theorem of Witt [6; p. 105], $|C_{\mathfrak{G}}(\tau)|=4i(i-1)$. Hence there exist $(\mathfrak{G}: C_{\mathfrak{G}}(\tau))=3(n-1)n/(i-1)i$ involutions in $\mathfrak{G}$ each of which is conjugate to τ.

At first, let us assume that n is odd. Let $h^*(2)$ be the number of involutions in $\mathfrak{H}$ leaving only the point 1 fixed. Then from (2.3) the following equality is obtained:

$$h^*(2)n+3(n-1)n/(i-1)i=3(n-1)/(i-1)+h^*(2)+d(n-1) \tag{2.4}$$

Put $\beta=d-h^*(2)$. It follows from (2.4) that $n=i(\beta i-\beta+3)/3$. This implies that i is odd.

Next let us assume that n is even. Let $g^*(2)$ be the number of involutions in $\mathfrak{G}$ which are semi-regular on Ω. Then corresponding to (2.4) the follwing equality is obtained from (2.3):

$$g^*(2)+3(n-1)n/(i-1)=3(n-1)/(i-1)+d(n-1) \tag{2.5}$$

Put $\beta=d-g^*(2)/(n-1)$. Then by (2.5) we have $n=i(\beta i-\beta+3)/3$. This implies that i is even.

In both cases, by the definition of β, β is the number of involutions with the cycle structure $(1, 2)\cdots$ each of which is conjugate to τ. Since $\mathfrak{G}$ is doubly transitive on Ω, we have $\beta>0$.

Lemma 2. *$\beta=1, 3, 4,$ or 6.*

Proof. The result follows immediately from Lemma 1.

Lemma 3. *If $\alpha(\tau)=\alpha(\mathfrak{V})$, then $\mathfrak{V}\cap G^{-1}\mathfrak{V}G=1$ or $\mathfrak{V}$ for every element G in $\mathfrak{G}$.*

Proof. If $\mathfrak{V}\cap G^{-1}\mathfrak{V}G$ contains τ, then $\mathfrak{J}(\tau)$ contains $\mathfrak{J}(\mathfrak{V})$ and $\mathfrak{J}(G^{-1}\mathfrak{V}G)$, and thus $\mathfrak{J}(\tau)=\mathfrak{J}(\mathfrak{V})=\mathfrak{J}(G^{-1}\mathfrak{V}G)$. This implies that $\mathfrak{V}$ and $G^{-1}\mathfrak{V}G$ are contained in $\mathfrak{K}$ and so $\mathfrak{V}=G^{-1}\mathfrak{V}G$. This proves our lemma.

Lemma 4. *If $\alpha(\tau)>\alpha(\mathfrak{V})$, then one of the followings holds:*

(1) $i=6$ and $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is $\mathfrak{A}_5$,

(2) $i=28$, $\alpha(\mathfrak{V})=4$ and $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is $P\Gamma L(2, 8)$,

(3) $i=p^{2m}$ for some prime p, $\alpha(\mathfrak{V})=p^m$ and $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ contains a regular normal subgroup. Moreover if p is odd, then there exists a unique involution in $C_{\mathfrak{H}}(\tau)/\langle\tau\rangle$ which fixes only one point on $\mathfrak{J}(\tau)$.

Proof. Since $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is doubly transitive on $\mathfrak{J}(\tau)$ of degree i and order $2(i-1)i$, the results follow from Ito's theorem [7] and its proof.

Lemma 5. *If $\alpha(\tau)<\alpha(\mathfrak{V})$, then $\beta=3, 4,$ or 6.*

Proof. There exist two points j and k in $\mathfrak{J}(\tau)-\mathfrak{J}(\mathfrak{V})$ such that $\tau_1=(j, k)\cdots$. Hence $\tau\tau_1=(j, k)\cdots$ and Lemma 2 yields $\beta=3$, 4, or 6 since $\mathfrak{G}$ is doubly transitive on Ω.

3. The case n is odd

In the following if $h^*(2)>0$, then without loss of generality we may assume that $\alpha(I)=1$.

Lemma 6. *If $h^*(2)=1$, then $\mathfrak{G}=C_{\mathfrak{G}}(I)O(\mathfrak{G})$.*

Proof. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $\mathfrak{G}$ containing I. By our assumption $\{G^{-1}IG;\ G\in\mathfrak{G}\}\cap\mathfrak{S}=\{I\}$. It follows from the Z^*-theorem of Gluaberman [6; p. 628] that $\langle I\rangle O(\mathfrak{G})$ is a normal subgroup of $\mathfrak{G}$ and then Frattini argument implies that $\mathfrak{G}=C_{\mathfrak{G}}(I)O(\mathfrak{G})$. This proves our lemma.

Lemma 7. *If $\alpha(\tau)=\alpha(\mathfrak{V})$ and $h^*(2)=3$, then there exists no group satisfying the condition of our theorem.*

Proof. Put $\mathfrak{J}(I)=\{k\}$ and let $\mathfrak{G}_k$ be the stabilizer of a point k in $\mathfrak{G}$. Then $\mathfrak{G}_k$ contains $C_{\mathfrak{G}}(I)$ and $\mathfrak{J}(I\tau_1)=\mathfrak{J}(J\tau\tau_1)=\{k\}$. It follows from $h^*(2)=3$

that $\langle I\tau_1, I\tau\tau_1, I\rangle=\langle I, \mathfrak{V}\rangle$ is normal in $\mathfrak{G}_k$. Now $\langle I, \mathfrak{V}\rangle$ is half transitive on $\Omega-\{k\}$. On the other hand I acts on $\mathfrak{J}(\mathfrak{V})$, and I-orbits on $\mathfrak{J}(\mathfrak{V})$ are of length 2 and $\mathfrak{V}$-orbits on $\Omega-\mathfrak{J}(\mathfrak{V})$ are of length 4 by our assumption. Thus we get a contradiction. The proof is complete.

Now $h^*(2)=0$ and then τ is a central involution in some Sylow 2–subgroup of $\mathfrak{G}$. Let $\mathfrak{S}$ be a Sylow 2–subgroup of $\mathfrak{G}$ containing $\langle I, \mathfrak{V}\rangle$ and contained in $C_{\mathfrak{G}}(\tau)$.

Lemma 8. *If $h^*(2)=0$, then $\alpha(\tau)=\alpha(\mathfrak{V})$.*

Proof. Assume by way of contradiction that $\alpha(\tau)<\alpha(\mathfrak{V})$. Then $|C_{\mathfrak{G}}(\tau)|=4i(i-1)$ and $|N_{\mathfrak{G}}(\mathfrak{V})|=4\sqrt{i}(\sqrt{i}-1)$ by Lemma 4 since $C_{\mathfrak{G}}(\tau)$ and $N_{\mathfrak{G}}(\mathfrak{V})$ is doubly transitive on $\mathfrak{J}(\tau)$ and $\mathfrak{J}(\mathfrak{V})$, respectively. Hence $N_{\mathfrak{G}}(\mathfrak{V})$ does not contain a Sylow 2–subgroup of $\mathfrak{G}$. We first show that (1) in Lemma 1 holds in this case. Suppose that (2) or (3) in Lemma 1 holds. Then $\mathfrak{G}_{\{1,2\}}=\langle I\rangle\mathfrak{K}$ has no element of order four and hence any 2–element with a 2–cycle must be an involution since $\mathfrak{G}$ is doubly transitive. Then since $h^*(2)=0$ $\mathfrak{G}$ has **no** element of order four and so $\mathfrak{S}$ is elementary abelian. Then $N_{\mathfrak{G}}(\mathfrak{V})$ contains $\mathfrak{S}$, a contradiction. Thus (1) in Lemma 1 must hold. Then $\mathfrak{G}_{\{1,2\}}$ is the symmetric group of degree four, $\beta=d=6$ and $n=i(2i-1)$. In particular, $O(\mathfrak{G})=1$. Four groups in $\mathfrak{G}_{\{1,2\}}$ form two conjugate classes and their representatives are $\mathfrak{V}$ and $\langle I, \tau\rangle$. Now we regard $\mathfrak{G}$ as a transitive permutation group on the set of the unordered pairs of the points of Ω. Then $\mathfrak{G}_{\{1,2\}}$ is the stabilizer of the pair $\{1, 2\}$. If $\mathfrak{V}$ is not conjugate to $\langle I, \tau\rangle$ in $\mathfrak{G}$, $\mathfrak{V}$ satisfies the assumption of a theorem of Witt [6; p. 150], and hence $N_{\mathfrak{G}}(\mathfrak{V})$ is transitive on the pairs which $\mathfrak{V}$ fixes. This forces $\mathfrak{V}$ to have no orbit of length 2 on Ω since $N_{\mathfrak{G}}(\mathfrak{V})$ fixes $\mathfrak{J}(\mathfrak{V})$ as a whole. This implies that $\alpha(\tau)=\alpha(\mathfrak{V})$, contrary to the assumption. Thus $\mathfrak{V}\sim\langle I, \tau\rangle$ in $\mathfrak{G}$. On the other hand since $h^*(2)=0$, any four group has an orbit of length 2 and hence is conjugate to $\mathfrak{V}$. Then if $\mathfrak{S}$ is not of a maximal class, $N_{\mathfrak{G}}(\mathfrak{V})$ contains a Sylow 2–subgroup of $\mathfrak{G}$ (See [2; p. 215]) which is a contradiction. Thus $\mathfrak{S}$ must be of a maximal class and hence dihedral or semi-dihedral. Since $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ has a dihedral Sylow 2–subgroup $\mathfrak{S}/\langle\tau\rangle$, a result of Gorenstein-Walter [4] and Lemma 4 imply that $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is $2'$–closed and so is $C_{\mathfrak{G}}(\tau)$. By theorems of Gorenstein [3] and Lüneburg [9] we get a contradiction. The proof is complete.

Lemma 9. *If $h^*(2)=0$, then $n=15$ and $\mathfrak{G}$ is $\mathfrak{A}_7$.*

Proof. Since $\alpha(\tau)=\alpha(\mathfrak{V})$ by Lemma 8, $C_{\mathfrak{G}}(\tau)/\mathfrak{V}$ is a complete Frobenius group of odd degree i and then $\mathfrak{S}/\mathfrak{V}$ is cyclic or generalized quaternion. Assume that $[I, \mathfrak{V}]\neq 1$. It follows that $(\mathfrak{S}: C_{\mathfrak{S}}(\mathfrak{V}))=2$ and $I\mathfrak{V}$ is a unique involution in $\mathfrak{S}/\mathfrak{V}$. Now $C_{\mathfrak{S}}(\mathfrak{V})=\mathfrak{V}$ and $\mathfrak{S}=\langle I, \mathfrak{V}\rangle$ is a dihedral group of order 8. Since $\beta=d=6$ by Lemma 1 and $n=i(2i-1)$, $\mathfrak{G}$ contains no regular normal subgroup.

Applying theorems of Gorenstein-Walter [4] and Lüneburg [9], $\mathfrak{G}$ is $\mathfrak{A}_7$. On the other hand it is well known that $\mathfrak{A}_7$ has a doubly transitive permutation representation of degree 15 in which the stabilizer of two points is $\mathfrak{A}_4$ (See [6; p. 157]). Thus $n=15$ and $\mathfrak{G}$ is $\mathfrak{A}_7$. Next assume that $[I, \mathfrak{V}]=1$. Then by the same way as in the proof of Lemma 8, $\mathfrak{S}$ is elementary abelian and hence $\mathfrak{S}=\langle I, \mathfrak{V}\rangle$. Now $C_{\mathfrak{G}}(\tau)$ is solvable and so theorems of Gorenstein [1] and Lüneburg [9] yield a contradiction. This proves our Lemma.

4. The case n is even

Lemma 10. *If $\alpha(\tau)=\alpha(\mathfrak{V})$, then $n=6$ and $\mathfrak{G}$ is $\mathfrak{A}_6$, or $n=16$ and $\mathfrak{G}$ is $A(2, 4)$.*

Proof. Assume that $\mathfrak{H}$ is 2–closed. Then $\mathfrak{H}$ acts on $\mathfrak{J}(\mathfrak{V})$ and since $\mathfrak{H}$ is transitive on $\Omega-\{1\}$, we have $\mathfrak{J}(\mathfrak{V})=\Omega$ which is impossible. By Lemma 3, $\mathfrak{H}$ is a (TI)-group in the sense of Suzuki [11]. Since $\mathfrak{H}/O(\mathfrak{H})$ is also a (TI)-group, Suzuki's result [11; p. 69] implies that $\mathfrak{H}/O(\mathfrak{H})$ is $PSL(2, 4)$ and $O(\mathfrak{H})$ is contained in the center of $\mathfrak{H}$. On the other hand we have $|N_{\mathfrak{G}}(\mathfrak{V})|=12i(i-1)$ and $|C_{\mathfrak{H}}(\mathfrak{V})|=4(i-1)$. Now $|O(\mathfrak{H})|=i-1$ and $|\mathfrak{H}/O(\mathfrak{H})|=4(\beta i+3)=|PSL(2, 4)|=60$. Therefore $\beta i=12$. In our case since $C_{\mathfrak{G}}(\tau)/\mathfrak{V}$ is a complete Frobenius group of even degree i, i is a power of 2 and then $i=2$ or 4. If $i=2$, then $n=6$ and $\mathfrak{G}$ is $\mathfrak{A}_6$. If $i=4$, then $n=16$. Since $\mathfrak{K}=\langle K, \mathfrak{V}\rangle$ and K is of order 3, $\alpha(\mathfrak{V})-\alpha(\mathfrak{K})$ is divisible by 3 and $\mathfrak{J}(\mathfrak{V})=\mathfrak{J}(\mathfrak{K})$. Applying a result of Witt [6; p. 150] to $N_{\mathfrak{G}}(\mathfrak{K})$ and $N_{\mathfrak{G}}(\langle K\rangle)$ we can get easily $\alpha(\mathfrak{K})=4$. Therefore $\mathfrak{K}$ is semi-regular on $\Omega-\mathfrak{J}(\mathfrak{K})$ and thus $\mathfrak{K}$ is transitive on $\Omega-\mathfrak{J}(\mathfrak{K})$. Since n is even it follows from Kantor's theorem [8] that $\mathfrak{G}$ is isomorphic to a subgroup of $A(2, 4)$ or $A(4, 2)$. Assume that $\mathfrak{G}$ is a subgroup of $A(4, 2)$. Let $\mathfrak{N}$ be a regular normal subgroup of $A(4, 2)$. If $\mathfrak{G}\cap\mathfrak{N}=1$, then $\mathfrak{G}$ is isomorphic to a subgroup of $GL(4, 2)$ which is impossible because $GL(4, 2)$ contains no subgroup of index 7. Hence $\mathfrak{G}\cap\mathfrak{N}\neq 1$ and then $\mathfrak{G}$ contains $\mathfrak{N}$ and $\mathfrak{H}$ is isomorphic to $\mathfrak{G}/\mathfrak{N}$. Since $|O(\mathfrak{H})|=3$ and $O(\mathfrak{H})$ is contained in the center of $\mathfrak{H}$, $\mathfrak{H}$ is $GL(2, 4)$. Thus $n=16$ and $\mathfrak{G}$ is $A(2, 4)$. This proves our lemma.

Lemma 11. *If $\alpha(\tau)>\alpha(\mathfrak{V})$ and $i=6$ or 28, then there exists no group satisfying the condition of our theorem.*

Proof. Assume that $i=6$. Since $C_{\mathfrak{G}}(\tau)$ contains $\mathfrak{V}$, Schur's theorem [10] implies that $C_{\mathfrak{G}}(\tau)=\langle\tau\rangle\times\mathfrak{F}$ where $\mathfrak{F}$ is $\mathfrak{A}_5$ by Lemma 4. It follows that $[I, \mathfrak{V}]=1$ and $d=4$. Now $|C_{\mathfrak{H}}(\tau)|=2^2\cdot 5$ and $|\mathfrak{H}|=2^2\cdot 3^3\cdot 5$ or $2^2\cdot 3\cdot 5\cdot 7$. Assume that $i=28$. Then Lemma 4 yields $\alpha(\mathfrak{V})=\alpha(\mathfrak{K})=4$ and using a result of Witt [6; p. 150], $|N_{\mathfrak{G}}(\mathfrak{K})|=|N_{\mathfrak{G}}(\mathfrak{V})|=144$, $N_{\mathfrak{G}}(\mathfrak{K})=\mathfrak{K}\times C_{\mathfrak{G}}(\mathfrak{K})$. It follows that $[I, \mathfrak{V}]=1$ and so $d=4$. Now $|C_{\mathfrak{H}}(\tau)|=2^2\cdot 3^3$ and $|\mathfrak{H}|=2^2\cdot 3^2\cdot 5\cdot 23$ or $2^2\cdot 3^4\cdot 29$. In both cases applying a theorem of Gorenstein-Watler [4], $\mathfrak{H}/O(\mathfrak{H})$

is isomorphic to a subgroup of $P\Gamma L(2, r)$ containing $PSL(2, r)$. Clearly this is impossible. This proves our lemma.

In the following we assume that the case (3) of Lemma 4 holds.

Lemma 12. *The group $\mathfrak{H}/O(\mathfrak{H})$ is $PSL(2, q)$ for some q.*

Proof. Since $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is a solvable doubly transitive group on $\dot{\mathfrak{I}}(\tau)$ of degree 2^{2m}, it follows from a theorem of Huppert [5] that $C_{\mathfrak{G}}(\tau)/\langle\tau\rangle$ is contained in 1-dimensional semi-linear affine transformation group over the field of 2^{2m}-elements and then $C_{\mathfrak{H}}(\tau)$ has a cyclic normal 2-complement. Thus by a result of Gorenstein-Walter [4], $\mathfrak{H}/O(\mathfrak{H})$ is $PSL(2, q)$. This proves our lemma.

Lemma 13. *There exists no group with $\alpha(\tau)=2^{2m}$ and $\alpha(\mathfrak{V})=2^m$.*

Proof. Put $\bar{\mathfrak{H}}=\mathfrak{H}/O(\mathfrak{H})$. Then $\bar{\mathfrak{H}}$ is $PSL(2, q)$ with $q\equiv 3$ or 5 (mod. 8) by Lemma 12. Hence $C_{\bar{\mathfrak{H}}}(\bar{\mathfrak{V}})=\bar{\mathfrak{V}}$ and $C_{\bar{\mathfrak{H}}}(\bar{\tau})$ has a normal 2-complement of order $q+1/4$ or $q-1/4$ according as $q\equiv 3$ or 5 (mod. 8). On the other hand $|C_{\mathfrak{H}}(\tau)|=4(i-1)$ and $|C_{\mathfrak{H}}(\mathfrak{V})|=4\ (\sqrt{i}-1)$ since $|N_{\mathfrak{H}}(\mathfrak{V})|=12\ (\sqrt{i}-1)$. Then $|O(\mathfrak{H})\cap C_{\mathfrak{H}}(\mathfrak{V})|=(\sqrt{i}-1)$ and hence $|O(\mathfrak{H})\cap C_{\mathfrak{H}}(\tau)|=x(\sqrt{i}-1)$ with some odd integer x dividing $\sqrt{i}+1$. Then the formula of Brauer-Wielandt [12] yields $|O(\mathfrak{H})|=x^3(\sqrt{i}-1)$. Since $|\mathfrak{H}|=12(n-1)=4(\beta-1)(\beta i+3)$, we have

$$|\bar{\mathfrak{H}}| = 4(\sqrt{i}+1)(\beta i+3)/x^3 = (q+1)q(q-1)/2 \tag{4.1}$$

Since $C_{\mathfrak{H}}(\tau)$ has a cyclic normal 2-complement $\langle w\rangle$ of order $i-1$ and $C_{\bar{\mathfrak{H}}}(\bar{\tau})=\overline{C_{\mathfrak{H}}(\tau)}$, we have

$$o(\bar{w}) = i-1/x(\sqrt{i}-1) = q\pm 1/4 \tag{4.2}$$

Then (4.1) yields

$$2(\beta i+3)/x^2 = q(q\mp 1) \tag{4.3}$$

In particular, x is a common divisor of $\sqrt{i}+1$ and $\beta i+3$ where $\sqrt{i}=2^m$ and $\beta=3, 4,$ or 6 by Lemma 5. Assume that $\beta=3$. Then $x=1$ or 3 since $\beta i+3 =3\cdot 2^{2m}+3\equiv 6$ (mod. 2^m+1). Now it is easy to see that (4.2) and (4.3) are incompatible. In the case where $\beta=4$ or 6, the proofs are similar.

The proof of our theorem is complete.

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