Murata, K.
Osaka J. Math.
6 (1969), 229-234

DECOMPOSITION OF RADICAL ELEMENTS OF A COMPACTLY GENERATED *cm*-LATTICE

Dedicated to Prof. Atuo Komatu for his 60th birthday

KENTARO MURATA

(Received July 29, 1968)
(Revised September 11, 1968)

The aim of the present paper is to define a completely prime element and a new type of radicals of elements in some compactly generated *cm*-lattice[1], and to obtain meet-decompositions of radical elements of the *cm*-lattice.

1. Preliminaries

Let L be a complete (upper and lower) lattice. A non-void subset Σ of L is called a compact set of L if, whenever $x \leq \sup N$ for an element x of Σ and a subset N of Σ, there exists a finite number of elements $x_1, \cdots, x_n$ of N satisfying $x_1 \cup \cdots \cup x_n \geq x$. Every element of Σ is said to be compact. If every element of L is a join of a finite or infinite number of elements of the compact set Σ of L, then L is said to be compactly generated, and Σ is called a compact generator of L[2].

We can prove easily that a complete lattice has a compact generator if and only if it is compactly generated in the sense of Dilworth-Crawley[3]. In this case, the set of the compact elements[4] of the lattice is the unique maximal compact generator under the set-inclusion.

Now we shall consider, throughout this paper, a *cm*-lattice L which is compactly generated as a lattice. In what follows, we suppose that L has a compact generator Σ which satisfies the following condition.

(*) *If* $\inf S \leq a$ *for a subset* S *of* Σ *and an element* a *of* L, *then* $a = \inf (S \cup a)$, *where* $S \cup a$ *means the set of the elements* $s \cup a$ $(s \in S)$.

It is easily verified that the condition (*) holds for any infinitely meet-

1) Cf. [1; pp. 200–201].
2) Cf. [5; p. 105] and [6; p. 54].
3), 4) Cf. [2; p. 2] and [3; p. 11].

distributive *cm*-lattice[5] with a compact generator. But, there exists a *cm*-lattice which is not infinitely meet-distributive, and satisfies the condition $(*)$. Let $\mathfrak{L}$ be the *cm*-lattice of all ideals of any Dedekind domain $\mathfrak{O}$, and Σ the compact generator consisting of the principal ideals of $\mathfrak{O}$. Then, of course, $\mathfrak{L}$ is not necessarily infinitely meet-distributive. But, it can be proved easily that the condition $(*)$ holds for $\mathfrak{L}$ and Σ.

Let Σ be any fixed compact generator of L and let $a\in L$. By the symbol Σ_a we shall mean the set $\{x\in\Sigma\,|\,x\leq a\}$. Then we can prove that $\sup A=\sup(\vee_{a\in A}\Sigma_a)$ for every non-void subset A of L, where $\vee$ denotes the set-theoretical union.

2. Completely prime elements

For a non-void subset A of a *cm*-lattice L, $\bar{A}$ will denote the multiplicative system (monoid) which is generated by A under the multiplication of L.

DEFINITION 1. An element p of a *cm*-lattice is said to be *completely prime* if whenever $\inf\bar{A}$ is contained in p, then at least one of the element of A is contained in p.

We can prove easily that completely primes are primes. But the converse is not true. In fact, we can find an example of *cm*-lattices with an element which is prime but not completely prime.

DEFINITION 2. A non-void subset Γ of the compact set Σ of L is called a *c-system*, if $\inf\bar{\Gamma}$ is a member of Γ. The void set is a *c*-system.

Lemma 1. *The following conditions are equivalent to one another.*

(1) *p is completely prime.*

(2) *If $\inf\bar{\Delta}\leq p$ (ϕ[6] $\neq\Delta\subseteq\Sigma$), then there exists an element $x\in\Delta$ such that $x\leq p$.*

(3) *$\Sigma\backslash\Sigma_p$[7] is a c-system.*

Proof. (1)$\Rightarrow$(2) is evident. (2)$\Rightarrow$(1): Suppose that p is not completely prime. Then, there exists a subset A of L such that $\inf\bar{A}\leq p$ and $a\not\leq p$ for every $a\in A$. Therefore we can take an element $x_a\in\Sigma_a$ such as $x_a\not\leq p$. Put $\Delta=\{x_a\,|\,a\in A\}$. Then it is easy to see that $\inf\bar{\Delta}\leq p$. (1)$\Rightarrow$(3): Suppose that p is completely prime. It is then easily verified that p does not contain $\inf\overline{\Sigma\backslash\Sigma_p}$. Hence we can take an element u of $\Sigma\backslash\Sigma_p$ such that $u\leq\inf\overline{\Sigma\backslash\Sigma_p}$. (3)$\Rightarrow$(2): Let $\Sigma\backslash\Sigma_p$ be a *c*-system, and Γ a non-void subset of Σ. If we suppose that p con-

5) =the dual of a relatively pseudo-complemeted *cm*-lattice.

6) ϕ will mean the void set.

7) $\backslash$ will denote the set-defference.

tains no element of Γ, then $\Gamma \subseteq \Sigma \backslash \Sigma_p$, $\overline{\Gamma} \subseteq \overline{\Sigma \backslash \Sigma_p}$. Hence we have $\inf \overline{\Sigma \backslash \Sigma_p} \leq \inf \overline{\Gamma}$. On the other hand, $\inf \overline{\Sigma \backslash \Sigma_p}$ is not contained in p, since $\Sigma \backslash \Sigma_p$ is a c-system.

Lemma 2. *Let Γ be a c-system such that $\Gamma \wedge \Sigma_a = \phi$[8] for an element a of L. Then there exists a completely prime element p such that $p \geq a$ and $\Gamma \wedge \Sigma_p = \phi$.*

Proof. First we show that the set $S = \{c \in L \mid a \leq c, \Gamma \wedge \Sigma_c = \phi\}$ is inductive. Let $\{c_\lambda\}$ be any chain in S, and let $c^* = \sup\{c_\lambda\}$. If we suppose that $\Gamma \wedge \Sigma_{c^*}$ contains an element x, then $x \leq c^* = \sup(\vee_\lambda \Sigma_{c_\lambda})$. Hence there exists a finite number of elements $x_1, \cdots, x_n$ such that $x \leq x_1 \cup \cdots \cup x_n$ and $x_\nu \in \vee_\lambda \Sigma_{c_\lambda}$. Since there exists c_m such that $x_\nu \leq c_m$ $(\nu = 1, \cdots, n)$, we have that $\Gamma \wedge \Sigma_{c_m} \ni x$, which is a contradiction. Zorn's lemma assures therefore the existence of a maximal element p in S. We now prove that p is completely prime. Let Δ be a non-void subset of Σ such that $\inf \overline{\Delta} \leq p$. If we suppose that p contains no element of Δ, then since $p < p \cup u$ for every $u \in \Delta$, we can find an element $v(u)$ (depending on u) of Σ such that $v(u) \in \Gamma \wedge \Sigma_{p \cup u}$. Now we put $V = \{v(u) \mid u \in \Delta\}$. Then $\overline{V}$ is contained in $\overline{\Gamma}$. We have therefore that $\inf \overline{V} \geq \inf \overline{\Gamma}$. Now let $u^* = \mathfrak{P}(u_1, \cdots, u_n)$ be an arbitrary element of $\overline{\Delta}$, where $u_i \in \Delta$ $(i = 1, \cdots, n)$, and $\mathfrak{P}$ denotes a product-form (product-polynomial) of $u_1, \cdots, u_n$. Then we have that

$$\mathfrak{P}(v(u_1), \cdots, v(u_n)) \leq \mathfrak{P}(p \cup u_1, \cdots, p \cup u_n) \leq p \cup \mathfrak{P}(u_1, \cdots, u_n) = p \cup u^* .$$

Since $\mathfrak{P}(v(u_1), \cdots, v(u_n)) \in \overline{V}$, we obtain that $\inf \overline{V} \leq \inf \overline{\Delta} \leq \inf\{p \cup u^* \mid u^* \in \overline{\Delta}\} = p \cup \inf \overline{\Delta} = p$. Hence we have that $\inf \overline{\Gamma} \leq p$. Therefore we obtain that $\inf \overline{\Gamma} \in \Gamma \wedge \Sigma_p$, which is a contradiction.

3. Radicals of elements

Let a be an arbitrary element of L, and let X_a be the set of the elements x of Σ such that every c-system containing x contains an element of Σ_a. We now put

DEFINITION 3. The suplemum of X_a is called a *radical* of a, and is denoted by $r(a)$. An element a of L is said to be *radical* if $r(a) = a$.

Therorem 1. *Let L be a compactly generated cm-lattice with the condition $(*)$. Then the radical of any element a of L is decomposed into the meet of the completely prime elements containing a. In particular, so is any radical element of L.*

Proof. Let p be any completely prime element containing a. Then $r(a) \leq p$. For, if contrary, the c-system $\Sigma \backslash \Sigma_p$ contains an element $x \in \Sigma$ such that

8) $\wedge$ will denote the intersection. We say, following McCoy [4], that Γ does not meet Σ_a, if $\Gamma \wedge \Sigma_a = \phi$.

$x \nleq p$ and $x \leq r(a)$. Hence $(\Sigma \backslash \Sigma_p) \wedge \Sigma_a$ is not vacuous. Since Σ_a is contained in Σ_p, this is a contradiction. Therefore we obtain that $r(a) \leq p$, $r(a) \leq \cap_{a \leq p} p$. For the proof of the converse inclusion, it is sufficient to show that $x \nleq r(a)$ $(x \in \Sigma)$ implies $x \nleq \cap_{a \leq p} p$. Since $x \nleq r(a)$, there exists a c-system Γ such that $\Gamma \ni x$ and it does not meet Σ_a. Then, by Lemma 2, we can take a completely prime element p satisfying $a \leq p$ and Γ does not meet Σ_p. Hence x is not contained in $\cap_{a \leq p} p$. This completes the proof.

Corollary 1. *Let L be a compactly generated cm-lattice with the condition $(*)$, and I the greatest element of L. If the ascending chain condition holds for the elements of the interval I/a, the radical of a has a unique irredundant meet decomposition into completely prime elements. In particular, a is so if it is radical.*

Lemma 3. *A completely prime element of any cm-lattice is completely irreducible.*[9]

Proof. Let p be a completely prime element of any cm-lattice. If $p = \inf Q$, then $p \geq \inf \bar{Q}$. Hence there exists an element q in Q such that $q \leq p$. If $q < p$, then $p = \inf Q \leq q < p$. This is a contradiction.

Corollary 2. *Let D be an infinitely meet distributive lattice with a compact generator. Then every element of D, which is different from the greatest element, is decomposed into completely irreducible elements of D.*

Proof. Let a be any element of D, and let Σ be any complact generator of D. Then the set $\Gamma = \{x\}$ consisting of the single element x of $\Sigma_{r(a)}$, is a c-system satisfying $\inf \bar{\Gamma} = x$. Hence a is radical. By Theorem 1 and Lemma 3, we complete the proof.

4. Completely minimal primes

Let Γ_0 be any fixed c-system of L, and let a be an element of L such that Σ_a does not meet Γ_0. Then it is easily verified that the family of c-systems Γ each of which contains Γ_0 and does not meet Σ_a is inductive. Zorn's lemma assures therefore the existence of maximal c-systems. Let $\mathfrak{M} = \{\Gamma_\lambda\}$ be the family of the maximal c-systems based on Γ_0. Then we can see that $p_\lambda = \sup (\Sigma \backslash \Gamma_\lambda)$ is completely prime, and the completely prime element p satisfying $a \leq p$ and $\Gamma_\lambda \wedge \Sigma_p = \phi$ coincides with p_λ. Moreover

$$\Gamma_\lambda \to p_\lambda = \sup (\Sigma \backslash \Gamma_\lambda) ,$$
$$p_\lambda \to \Gamma_\lambda = \Sigma \backslash \Sigma_{p_\lambda}$$

give a one-to-one correspondence between $\mathfrak{M}$ and the set of the completely primes $\{p\}$ which satisfy $a \leq p$ and $\Gamma_\lambda \wedge \Sigma_p = \phi$.

9) Cf. [2; p. 3].

DEFINITION 4. An completely prime element p of a *cm*-lattice is said to be *completely minimal prime belonging to* a, if it satisfies (1) $p \geq a$ and (2) there exists no completely prime p' such that $a \leq p' < p$.

Lemma 4. *In order that an element* p *of* L *is a completely minimal prime belonging to* a, *it is necessary and sufficient that* $\Sigma \backslash \Sigma_p$ *is a maximal c-system which does not meet* Σ_a.

Proof. We suppose that p is completely minimal prime, and a is not completely prime. Then we can assume that $a < p$. Now, by Zorn's lemma, there exists a maximal c-system Γ such that it does not meet Σ_a. Then we have that $p_0 = \sup(\Sigma \backslash \Gamma) \leq \sup(\Sigma \backslash (\Sigma \backslash \Sigma_p)) = p$. Since p_0 is completely prime, we obtain $p = p_0$. This implies that $\Sigma \backslash \Gamma = \Sigma_p$. Therefore $\Gamma = \Sigma \backslash \Sigma_p$ is a maximal c-system which does not meet Σ_a. The converse is easy to see.

Theorem 2. *Let* L *be a compactly generated cm-lattice with the condition* $(*)$. *Then the radical of any element* a *of* L *is decomposed into the meet of the completely minimal primes belonging to* a. *In particular, so is any radical element of* L.

Proof. Let $\{p_\lambda\}$ be the completely minimal primes belonging to a. In order to prove that $r(a) \geq \cap_\lambda p_\lambda$, it is sufficient to show that $x \nleq r(a)$ implies $x \nleq \cap_\lambda p_\lambda$, where $x \in \Sigma$. Now by the definition of $r(a)$, we can take a c-system Γ such that $x \in \Gamma$ and it does not meet Σ_a. Hence there exists a completely minimal prime p belonging to a such that Σ_p does not meet Γ. Evidently $x \nleq p$. We have therefore $x \nleq \cap_\lambda p_\lambda$. The converse inclusion is evident.

REMARK 1. Suppose that p is any completely prime containing a. Then there exists a completely minimal prime belonging to a which is contained in p. Because, $\Sigma \backslash \Sigma_p$ is a c-system which does not meet Σ_a; and we can take a maximal c-system which contains $\Sigma \backslash \Sigma_p$ and does not meet Σ_a.

REMARK 2. If the ascending chain condition holds for elements in the interval I/a, then the meet-decomposition mentioned in Corollary 1 to Theorem 1 is the irredundant meet-decomposition into the completely minimal primes belonging to a.

YAMAGUCHI UNIVERSITY

References

[1] G. Birkhoff: Lattice Theory, Amer. Math. Colloq. Publ., 25, (2nd ed.) 1968.

[2] R.P. Dilworth and Peter Crawley: *Decomposition theory for lattices without chain conditions,* Trans. Amer. Math. Soc. **96** (1960), 1–22.

[3] R.P. Dilworth: *Structure and decomposition theory of lattices,* Proceedings of Symposia in Pure Mathematics II, Amer. Math. Soc. (1961), 1–16.

[4] N.H. McCoy: *Prime ideals in general rings,* Amer. J. Math. **71** (1948) 823–838.

[5] K. Murata: *Additive ideal theory in multiplicative systems,* J. Inst. Polytec. Osaka City Univ. **10** (1959), 91–115.

[6] ———: *On nilpotent-free multiplicative systems,* Osaka Math. J. **14** (1962), 53–70.