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A DUALITY THEOREM FOR HOMOGENEOUS MANIFOLDS OF COMPACT LIE GROUPS

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Introduction

In 1938, T. Tannaka [10] found a duality theorem for compact groups. The theorem was deepened by C. Chevalley [2] for compact Lie groups. In fact his theory establishes an intimate connection between compact Lie groups and reductive linear algebraic groups. More precisely, he associated an affine algebraic group G^c defined over $\boldsymbol{R}$ with each compact Lie group G and proved that G is canonically isomorphic to the subgroup $G_{\boldsymbol{R}}$ of all $\boldsymbol{R}$-rational points of G^c. The existence of this isomorphism amounts to the Tannaka duality theorem in this case. Conversely, for every reductive algebraic linear group G^*, there exists a compact Lie group G such that G^c is isomorphic to G^* (Proposition 1). The purpose of this paper is to prove a duality theorem for the homogeneous spaces of compact Lie groups which is analogous to Chevalley's theorem. For each homogeneous space $M=G/H$ of a compact Lie group G, we construct a complex affine algebraic set M^c which will be called the complexification of M and prove that the associated algebraic group G^c of G acts on M^c rationally and transitively (Proposition 3). Moreover the isotropy subgroup of G^c at the origin is identified with the associated algebraic group H^c of H (Proposition 3). This proves that the quotient space $M^c=G^c/H^c$ of a complex reductive algebraic group G^c over a complex reductive group H^c is an affine algebraic set (corollary to Proposition 3). This fact is proved by Borel and Harish-Chandra [1] when G^c is connected. There is a natural bijection of M onto the subset M_1 of M^c consisting of all real points of M^c. This is our duality theorem (Theorem 1). By introducing the notion of linear representation of a homogeneous space, this duality theorem can be formulated as a theorem of classical Tannaka-Chevalley type (Theorem 2). As the consequences of our duality theorem, we obtain the following results (Theorem 3).

1) The homogeneous space G/H of a compact Lie group G has the structure of a real affine algebraic set. 2) G/H has a faithful linear

representation. 2) was obtained earlier by G.D. Mostow [7]. The results of this paper were obtained in the autumn of 1959 and read at the spring meeting of Mathematical Society of Japan in 1960. The paper has not been published earlier because of several reasons on the side of the authors. Meanwhile A.L. Oniščik [9] introduced the complexification G^c/H^c of G/H in somewhat different context.

1. Notations and conventions

Throughout this paper, we use the following notations:

$C(X)$: the algebra of all complex valued continuous functions on a compact space X with the uniform norm

$$||f|| = \max_{x \in X} |f(x)|,$$

G: a compact Lie group, e: the identity element of G,

M: a C^∞-manifold on which G acts differentiablly and transitively,

p_0: a fixed point (the origin) of M,

H: the isotropy subgroup of G at p_0,

L_g: the left translation induced by $g \in G$ on G, $(L_g f)(g_0) = f(gg_0)$,

R_g: the right translation induced by $g \in G$ on G; $(R_g f)(g_0) = f(g_0 g)$,

T_g: the transformation induced by $g \in G$ on M; $(T_g f)(p) = f(gp)$,

$\mathfrak{o}(G) = \{f \in C(G); \dim [L_g; g \in G] < \infty\}$: the representative algebra of G,

$\mathfrak{o}(M) = \{f \in C(M); \dim [T_g f; g \in G] < \infty\}$:

ι: the canonical injection of $\mathfrak{o}(M)$ into $\mathfrak{o}(G)$

$$(\iota f)(g) = f(gH),$$

$dg(dh)$: the normalized Haar measure of $G(H)$,

dp: the normalized G-invariant measure of M,

$\bar{f}$: the complex conjugate function of $f \in C(X)$.

A representation of G means always a finite dimentional continuons (so analytic) representation over the field $\boldsymbol{C}$ of complex numbers unless the contrary is explicitly stated. A homomorphism ω of an algebra with the unit 1 into an algebra B with 1 is always assumed to be unitary, i.e., $\omega(1) = 1$.

2. Duality theorem of Chevalley-Tannaka

In this section we summarize, using the approach of Hochschild-Mostow [3], the known results about the duality theorem of compact Lie groups due to C. Chevalley and T. Tannaka for our later use.

Let $\mathfrak{o}(G)$ be the representative ring of a compact Lie group G. $\mathfrak{o}(G)$ is, by definition, the algebra consisting of the finite linear combinations of the matricial coefficients of the representations of G. The associated algebraic group (the complexification) G^c of a compact Lie group G is defined as follows. As a group, G^c is a subgroup of $\mathrm{Aut}(\mathfrak{o}(G))$ defined as follows:

$$G^c = \{\sigma \in \mathrm{Aut}(\mathfrak{o}(G))\,;\ \sigma \circ L_g = L_g \circ \sigma \qquad \text{for all } g \in G\}\ ,$$

where $\mathrm{Aut}(\mathfrak{o}(G))$ denotes the automorphism group of the algebra $\mathfrak{o}(G)$. On the other hand G^c has the structure of an affine algebraic set. The algebra $\mathfrak{o}(G)$ is a finitely generated commutative algebra without nilpotent element. To every σ in G^c corresponds the homomorphism $\omega = \omega_\sigma \in \mathrm{Hom}(\mathfrak{o}(G), \boldsymbol{C})$ defined by

$$\omega_\sigma(f) = (\sigma f)(e) \qquad \text{for every } f \text{ in } \mathfrak{o}(G)\,,$$

where e is the identity element of G. The mapping $\sigma \to \omega_\sigma$ is a bijection of G^c onto $\mathrm{Hom}(\mathfrak{o}(G), \boldsymbol{C})$. The group G^c identified with $\mathrm{Hom}(\mathfrak{o}(G), \boldsymbol{C})$ by this bijection has the structure of an affine algebraic set. And the group G^c now becomes a complex affine algebraic group and hence a complex Lie group. For every element $g \in G$, the right translation R_g belongs to the associated algebraic group G^c of G. The mapping $g \to R_g$ is an isomorphism of the group G onto the subgroup $G_1 = \{\sigma \in G^c\,;\ \sigma(\bar{f}) = \overline{\sigma(f)}$ for every $f \in \mathfrak{o}(G)\}$ of G^c. This fact is the essential part of the Tannaka duality theorem for the compact Lie group G (see [4] for a proof). In the following we identify $g \in G$ with R_g and regard G as a subgroup of G^c via the injective homomorphism $g \to R_g$ from G into G^c.

Let $\mathfrak{R}$ be the set of all matricial representations of G. A representation of $\mathfrak{R}$ is, by definition, a mapping ζ which assigns to every $\rho \in \mathfrak{R}$ a regular matrix $\zeta(\rho)$ of degree equal to the degree $d(\rho)$ of ρ in such a way that the equalities

$$\begin{aligned} \zeta(\rho_1 \oplus \rho_2) &= \zeta(\rho_1) \oplus \zeta(\rho_2) \\ \zeta(\rho_1 \otimes \rho_2) &= \zeta(\rho_1) \otimes \zeta(\rho_2) \\ \zeta(\gamma \rho \gamma^{-1}) &= \gamma \zeta(\rho) \gamma^{-1} \end{aligned}$$

hold for any representations ρ_1, ρ_2, ρ of G and any regular matrix γ of degree $d(\rho)$. The set of all representations of $\mathfrak{R}$ is denoted by G^*. G^* has the structure of a group. The group operation is defined by $\zeta_1\zeta_2^{-1}(\rho) = \zeta_1(\rho)\zeta_2(\rho)^{-1}$ for any ζ_1, $\zeta_2 \in G^*$ and $\rho \in \mathfrak{R}$. Every $\sigma \in G^c$ corresponds to an element ζ_σ of G^* defined by the equality

$$\zeta_\sigma(\rho) = (\sigma\rho)(e) \qquad \text{for any } \rho \in \mathfrak{R}$$

where $\sigma\rho=(\sigma\rho_{ij})$ if ρ_{ij} is the (i, j)-coefficient of ρ. Then the mapping $\sigma\rightarrow\zeta_\sigma$ is an isomorphism of G^c onto G^*. The above defined isomorphism $g\rightarrow R_g$ of G onto G_1 induces an isomorphism $g\rightarrow\zeta_{R_g}$ of G onto $G_1^*=\{\zeta\in G^*;\ \zeta(\bar{\rho})=\overline{\zeta(\rho)}\}$. This is the original form of the duality theorem of Chevalley-Tannaka. Every representation ρ of G can be uniquely extended to a rational representation of the associated algebraic group G^c of G. ρ^c is defined by

$$\rho^c(\sigma) = \zeta_\sigma(\rho).$$

Let τ be the automorphism of $\mathfrak{o}(G)$ regarded as the algebra over $\boldsymbol{R}$ defined by $\tau(f)=\bar{f}$ $(f\in\mathfrak{o}(G))$. Then the mapping $S: \sigma\rightarrow\tau\sigma\tau(\sigma\in G^c)$ is an involutive automorphism of the group G^c. Let $(\rho, \boldsymbol{C}^n)$ be a unitary representation of G whose coefficients generate the algebra $\mathfrak{o}(G)$. Then ρ^c is a faithful representation of G^c and

$$\rho^c(\tau\sigma^{-1}\tau) = \overline{{}^t\rho^c(\sigma)}. \tag{1}$$

Therefore $\rho^c(G^c)$ is a self-adjoint algebraic subgroup of $GL(n, \boldsymbol{C})$. We identify $\sigma\in G^c$ with $\rho^c(\sigma)$ and G^c with $\rho^c(G^c)$. Then G^c is a self-adjoint algebraic subgroup of $GL(n, \boldsymbol{C})$ and the intersection of G^c with the unitary group $U(n)$ coincides with G by the identity (1): $G^c\cap U(n)=G$. Every element σ of G^c can be expressed uniquely as

$$\sigma = g\exp\sqrt{-1}\,X, \quad g\in G \quad \text{and} \quad X\in\mathfrak{g}$$

where $\mathfrak{g}$ is the Lie algebra of G. $\exp\sqrt{-1}\,\mathfrak{g}$ is the intersection of G^c with the set $P(n)$ of all positive definite hermitian matrices of degree n. G^c is the smallest algebraic subgroup of $GL(n, \boldsymbol{C})$ containing G and G is a maximal compact subgroup of G^c. Using the canonical $\boldsymbol{R}$-isomorphism between $\boldsymbol{C}^n$ and $\boldsymbol{R}^{2n}$, we may assume that the above representation ρ of G is a real representation, i.e., $\rho(G)$ consists of real matrices. So the algebraic set G^c is defined over $\boldsymbol{R}$. By means of the Tannaka duality theorem, G is equal to the subgroup of G^c consisting of all real matrices in G^c. Therefore every compact Lie group G is an $\boldsymbol{R}$-algebraic group.

Proposition 1. *For an algebraic subgroup G^* of $GL(n, \boldsymbol{C})$, the following two conditions,* 1) *and* 2), *are mutually equivalent*:

1) *G^* is the associated algebraic group of a compact Lie group,*
2) *G^* is self-adjoint for some positive definite hermitian form on $\boldsymbol{C}^n$.*

When these conditions are satisfied, G^* is the associated algebraic group of a maximal compact subgroup G of G^* which is the intersection of G^* with the unitary group of the above hermitian form.

Proof. 1) implies 2) as explained above (cf. Chevalley [2]). Let G^* be a self-adjoint algebraic subgroup of $GL(n, \boldsymbol{C})$. Then G^* can be decomposed uniquely as

$$G^* = G \exp \sqrt{-1}\, \mathfrak{g}$$

([2] Ch. VI. § IX Lemma 2), where $G=G^* \cap U(n)$ and $\mathfrak{g}$ is the Lie algebra of G. The identity mapping $\rho: g \to g$ is a faithful representation of G. ρ can be extended uniquely to a rational representation ρ^c of the associated algebraic group G^c of G. ρ^c is a homomorphism of G^c into $GL(n, \boldsymbol{C})$ and $\rho^c(G^c)=\rho(G) \exp \sqrt{-1}\, d\rho(\mathfrak{g})=G^*$. Let $\sigma=g \exp \sqrt{-1}\, X (g\in G,\ X\in \mathfrak{g})$ be an arbitrary element in the kernel of ρ^c. Then we have $1=\rho^c(\sigma)=\rho(g) \exp \sqrt{-1}\, d\rho(X)=\sigma$. So ρ^c is an isomorphism of G^c onto G^*.

REMARK. G. D. Mostow [6] showed that the condition 2) in proposition 1 is equivalent to the following condition: 3) G^* is fully reducible on $\boldsymbol{C}^n$. M. Nagata [8] showed that the condition 3) is equivalent to the following condition: 4) the radical of G^* is a complex torus $\boldsymbol{C}^{*m}$.

3. The algebra $\mathfrak{o}(M)$

Let G be a compact Lie group acting transitively and differentiably on a C^∞-manifold M and H be the isotropic subgroup of G at a point p_0. H consists of the elements of G leaving p_0 fixed. M is identified with the coset space G/H by the homeomorphism $gp_0 \to gH$.

DEFINITION. A continuous function f on M is called a spherical function on M if

$$\dim \{T_g f;\ g\in G\} < \infty$$

The set $\mathfrak{o}(M)$ of all spherical functions on M is an algebra over $\boldsymbol{C}$. The structure of an algebra is defined by

$$(af)(p) = af(p),\ a\in \boldsymbol{C};\ (f+g)(p) = f(p)+g(p);\ (fg)(p) = f(p)g(p).$$

Then the canonical projection $\pi: g \to gH$ of G onto M induces an injection $\iota: f \to f\circ\pi$ of $\mathfrak{o}(M)$ into $\mathfrak{o}(G)$, $\iota\mathfrak{o}(M)$ consists of the elements in $\mathfrak{o}(G)$ which are invariant under every right translation $R_h (h\in H)$. In the following, we shall identify f in $\mathfrak{o}(M)$ with $\iota(f)$ in $\mathfrak{o}(G)$ and regard $\mathfrak{o}(M)$ as a subalgebra of $\mathfrak{o}(G)$. Let (ρ, V) be a representation of G whose restriction to the subgroup H has an invariant vector $x_1 \neq 0$ in V, and $(x_1, \cdots, x_n)$ be a basis of V containing the invariant vector x_1. Then the linear transformation $\rho(g)$ can be represented by a matrix $(\rho_{ij}(g))$ using the basis $(x_1, \cdots, x_n)$. The coefficients ρ_{i1}'s in the first column of this matrix belong to $\mathfrak{o}(M)$. $\mathfrak{o}(M)$ consists of the finite linear combinations of all such ρ_{i1}'s varying the representation ρ and the invariant vector x_1. Now we shall prove that the algebra $\mathfrak{o}(M)$ is finitely generated. Since the algebra $\mathfrak{o}(G)$ is finitely generated ([2] Ch. VI), G has a representation (ρ, V) whose coefficients generate $\mathfrak{o}(G)$. Let E be the vector space consisting of all linear transformations of the vector space V and $\mathfrak{P}(E)$ be the algebra of all polynomial functions on E. Then every element g in G induces an automorphism R_g of the algebra $\mathfrak{P}(E)$. R_g is defined by

$$(R_gF)(X) = F(X\rho(g)), \quad F\in\mathfrak{P}(E) \text{ and } X\in E.$$

Every polynomial function F in $\mathfrak{P}(E)$ defines a function $\hat{F}=F\circ\rho$ on G. Since the coefficients of ρ generate the algebra $\mathfrak{o}(G)$, the mapping Φ: $F\rightarrow\hat{F}$ is a homomorphism of the algebra $\mathfrak{P}(E)$ *onto* $\mathfrak{o}(G)$. Moreover we have

$$\widehat{(R_gF)} = R_g\hat{F} \qquad \text{for every} \quad g\in G \text{ and } F\in\mathfrak{P}(E). \tag{2}$$

Lemma 1. *Let*

$$\mathfrak{P}^H(E) = \{F\in\mathfrak{P}(E)\,;\, R_hF = F \qquad \text{for every} \quad h\in H\}.$$

Then the homomorphism $\Phi: F\rightarrow\hat{F}=F\circ\rho$ *maps* $\mathfrak{P}^H(E)$ *onto* $\mathfrak{o}(M)$.

Proof. First, $\mathfrak{P}^H(U)^\wedge$ is contained in $\mathfrak{o}(M)=\{f\in\mathfrak{o}(G)\,;\, R_hf=f$ for every $h\in H\}$ by the above identity (2). Conversely let f be any element in $\mathfrak{o}(M)$. Then there exists an F in $\mathfrak{P}(E)$ such that $\hat{F}=f$, since Φ maps $\mathfrak{P}(E)$ onto $\mathfrak{o}(G)$. The polynomial function F_0 defined by

$$F_0 = \int_H R_hFdh$$

belongs to $\mathfrak{P}^H(E)$. Moreover since $R_h\hat{F}=\hat{F}$ for every h in H, we have $\hat{F}_0=\hat{F}=f$. Thus Φ map $\mathfrak{P}^H(E)$ onto $\mathfrak{o}(M)$.

Proposition 2. *Let G be a compact Lie group and H be a closed subgroup of G. Then the algebra* $\mathfrak{o}(M)$ *of the sperical functions on* $M=G/H$ *is finitely generated.*

Proof. The algebra $\mathfrak{P}^H(E)$ of the invariant polynomials for the compact group H is finitely generated (cf. H. Weyl [11] p. 274). So proposition 2 is an immediate consequence of Lemma 1.

4. The complexification M^c of M

Let G, H, $M=G/H$ and $\mathfrak{o}(M)$ be the same as above.

DEFINITION. The complexification M^c of M is defined as

$$M^c = \mathrm{Hom}_G(\mathfrak{o}(M),\ \mathfrak{o}(G)),\ \text{i.e.,}$$
$$M^c = \{\varphi \in \mathrm{Hom}\,(\mathfrak{o}(M),\ \mathfrak{o}(G))\,;\ \varphi \circ T_g = L_g\ \varphi \quad \text{for every} \quad g \in G\},$$

where $\mathrm{Hom}\,(\mathfrak{o}(M), \mathfrak{o}(G))$ is the set of all homomorphism of the algebra $\mathfrak{o}(M)$ into the algebra $\mathfrak{o}(G)$ and

$$(T_g f)(p) = f(gp),\ f \in \mathfrak{o}(M),\ g \in G,\ p \in M,$$
$$(L_g f)(x) = f(gx),\ f \in \mathfrak{o}(G),\ g \in G,\ x \in G\,.$$

The topology of M^c is defined as the weakest topology such that the map $\varphi \to \varphi(f)$ is continuous for every f in $\mathfrak{o}(M)$. Any positive number ε and any finite set $\{f_1, \cdots, f_n\}$ of functions in $\mathfrak{o}(M)$ define a neighbourhood

$$U(\varphi_0\,;\, f_1, \cdots, f_n\,;\, \varepsilon) = \{\varphi \in M^c\,;\ \|\varphi f_i - \varphi_0 f_i\| < \varepsilon \quad (1 \leqq i \leqq n)\}$$

of a point φ_0 in M^c. All such $U(\varphi_0\,;\, f_1, \cdots, f_n\,;\, \varepsilon)$'s form a basis of neighbourhoods of φ_0. Now let $\mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$ be the set of all homomorphisms of the algebra $\mathfrak{o}(M)$ into the field $\boldsymbol{C}$ of complex numbers. The topology of $\mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$ is the weakest topology such that the map $\omega \to \omega(f)$ is continuous for every f in $\mathfrak{o}(M)$. Now let us construct a bijection $M^c \to \mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$ using the idea given in [3].

Lemma 2. *The mapping* $\varphi \to \omega = \omega_\varphi$ *defined by*

$$\omega(f) = (\varphi f)(e), \quad f \in \mathfrak{o}(M) \tag{3}$$

is a homeomorphism of M^c *onto* $\mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$.

Proof. It is clear that ω defined by (3) belongs to $\mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$ if φ is in M^c. Moreover the identity

$$(\varphi f)(g) = \omega(T_g f),\ g \in G,\ f \in \mathfrak{o}(M)$$

shows that the mapping $\varphi \to \omega$ is a bijection from M^c onto $\mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$. Since $|\omega(f) - \omega_0(f)| \leqq \|\varphi f - \varphi_0 f\|$, the mapping $\varphi \to \omega$ is continuous. To prove that $\omega \to \varphi$ is also continuous, let ρ be a unitary matricial representation of G such that $\rho_{i1}(h) = \delta_{i1}$, $1 \leqq i \leqq n = d(\rho)$ for every $h \in H$. Then the functions $f_1, \cdots, f_n$ on M such that $\iota f_k = \rho_{k1}$ belong to $\mathfrak{o}(M)$. Applying $\varphi \in M^c$ to the identity $T_g f_j = \sum_{i=1}^{n} \rho_{i1}(g) f_i$, we get $L_g(\varphi f_j) = \sum_{i=1}^{n} \rho_{i1}(g) \varphi f_i$. The values at e of the both sides of the last identity give the equality

$$(\varphi f_j)(g) = \sum_{i=1}^{n} \rho_{ij}(g)\omega(f_i) \qquad \text{for any} \quad g \in G. \tag{4}$$

Since $|\rho_{ij}(g)| \leqq 1$, the last equality (4) leads to the inequality

$$\|\varphi f_i - \varphi_0 f_i\| \leqq \sum_{i=1}^{n} |\omega(f_i) - \omega_0(f_i)|. \tag{5}$$

As every f in $\mathfrak{o}(M)$ is a finite linear combination of such f_i's, the inequality (5) proves that the mapping $\omega \to \varphi$ is continuous. The lemma is proved.

From now on, we identify $\varphi \in M^c$ with $\omega = \omega_\varphi$ in (3) and M^c with $\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})$. Proposition 2 and Lemma 2 give the structure of an affine algebraic set to M^c. Let $f_1, \cdots, f_n$ be a finite system of the generators of $\mathfrak{o}(M)$. Then the mapping $F: \omega \to (\omega(f_1), \cdots, \omega(f_n))$ is a homeomorphism from $\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C}) = M^c$ onto some algebraic set M^* (a model of M^c) in $\boldsymbol{C}^n$. Since $\mathfrak{o}(M)$ contains $\bar{f}$ together with f, M^c is defined over $\boldsymbol{R}$.

We owe H. Matsumura the proof of the following lemma.

Lemma 3. *Let $\mathfrak{a}$ be a proper (i.e. $\mathfrak{a} \neq \mathfrak{o}(M)$) ideal of the algebra $\mathfrak{o}(M)$ and $\mathfrak{b} = \iota\mathfrak{a}\mathfrak{o}(G)$. Then $\mathfrak{b}$ is a proper ideal of $\mathfrak{o}(G)$.*

Proof. For the sake of simplicity we identify $f \in \mathfrak{o}(M)$ with $\iota f \in \mathfrak{o}(G)$. For every a in $\mathfrak{o}(G)$, we define the mean a^* of a over the subgroup H by

$$a^* = \int_H R_h a dh.$$

Then a^* belongs to $\mathfrak{o}(M)$. Now suppose $\mathfrak{b} = \mathfrak{o}(G)$. Then there exists a finite number of elements $f_1, \cdots, f_n$ in $\mathfrak{a}$ and $a_1, \cdots, a_n$ in $\mathfrak{o}(G)$ such that $\sum_i a_i f_i = 1$. The last identity leads to $\sum_i a_i^* f_i = 1$ which contradicts the fact that $\mathfrak{a}$ is a proper ideal of $\mathfrak{o}(M)$.

Proposition 3. *Let G^c be the associated algebraic group of a compact Lie group G and M^c be the complexification of a homogeneous space $M = G/H$. If we define the operation of G^c on M^c by*

$$\sigma(\varphi) = \sigma \circ \varphi \qquad \textit{for} \quad \sigma \in G^c \textit{ and } \varphi \in M^c,$$

then G^c acts continuously and transitively on M^c. The isotropic subgroup H^* at ι (the canonical injection of $\mathfrak{o}(M)$ into $\mathfrak{o}(G)$) coincides with the associated algebraic group H^c of H.

Proof. Clearly the group G^c operates on M^c to the left by the above rule of the operation. Let f_j be the same as in the proof of Lemma 2. Applying σ on the identity (4), we get

$$(\sigma\circ\varphi)f_j = \sum_{i=1}^{n} (\sigma\rho_{ij})\omega(f_i)\,. \tag{6}$$

So we have an inequality

$$\|(\sigma\varphi)f_j - (\sigma_0\varphi_0)f_j\| \leqq \sum_i \|\sigma\rho_{ij} - \sigma_0\rho_{ij}\|(|\varphi_0(f_i)| + |\omega(f_i) - \omega_0(f_i)|) + \sum_i \|\sigma_0\rho_{ij}\|\,|\omega(f_i) - \omega_0(f_i)|$$

which proves that $(\sigma, \varphi)\to\sigma\circ\varphi$ is continuous. Now we shall prove the transitivity of the action of G^c on M^c, that is, for an arbitrary but fixed point φ in M^c we shall find a σ in G^c such that $\sigma\circ\iota=\varphi$. By Lemma 2, $\varphi\in M^c$ corresponds to the unique $\omega=\omega_\varphi\in\mathrm{Hom}\,(\mathfrak{o}(M), \boldsymbol{C})$. Similarly, to each σ in G, there corresponds the unique $\theta=\theta_\sigma$ in $\mathrm{Hom}\,(\mathfrak{o}(G), \boldsymbol{C})$, such that $\theta(f)=(\sigma f)(e)$ for every f in $\mathfrak{o}(G)$. Therefore to prove the existence of $\sigma\in G^c$ such that $\sigma\circ\iota=\varphi$ is equivalent to find a $\theta\in\mathrm{Hom}\,(\mathfrak{o}(G), \boldsymbol{C})$ such that $\theta\circ\iota=\omega$, because we have the equalities

$$[(\sigma\circ\iota)f](g)] = \theta(L_g\iota f) = (\theta\circ\iota)(T_g f) \text{ and}$$
$$\omega(T_g f) = (\varphi f)(g)\,.$$

Let $\mathfrak{a}$ be the kernel of ω. Then by Lemma 3, there exists a maximal ideal $\mathfrak{b}$ of $\mathfrak{o}(G)$ containing $\iota\mathfrak{a}\mathfrak{o}(G)$. Let (ρ, V) be a unitary representation of G whose coefficients generate $\mathfrak{o}(G)$, and $\mathfrak{P}(E)$ be the same as in 2. Let $\mathfrak{b}_0$ be the complete inverse image of the ideal $\mathfrak{b}$ by the homomorphism $\Phi: F\to\hat{F}$ of $\mathfrak{P}(E)$ onto $\mathfrak{o}(G)$. Then $\mathfrak{b}_0$ is a maximal ideal of $\mathfrak{P}(E)$. Since $\mathfrak{P}(E)$ is isomorphic to the polynomial ring $\boldsymbol{C}[t_1, \cdots, t_N]$, the factor algebra $\mathfrak{P}(E)/\mathfrak{b}_0$ is isomorphic to the field $\boldsymbol{C}$. So we have $\mathfrak{o}(G)/\mathfrak{b}\cong\boldsymbol{C}$ and there exists a homomorphism θ of $\mathfrak{o}(G)$ onto $\boldsymbol{C}$ whose kernel coincides with $\mathfrak{b}$. As $\theta\circ\iota$ vanishes on $\mathfrak{a}$, $\theta\circ\iota$ coincides with ω. The transitivity of G^c on M^c is now proved. Let H^* be the subgroup of G^c consisiting of the elements of G^c which keep the canonical injection ι invariant. Let τ be the real automorphism $f\to\bar{f}$ of $\mathfrak{o}(G)$. Then we have

$$\tau H^*\tau = H^*\,. \tag{7}$$

σ in G^c belongs to H^* if and only if $\sigma f=f$ for every f in $\iota(\mathfrak{o}(M))$. Since $\bar{f}$ and f belong to $\iota\mathfrak{o}(M)$ simultaneously, we have $(\tau\sigma\tau)f=\overline{\sigma(\bar{f})}=\bar{\bar{f}}=f$ for σ in H^*. So $\sigma\in H^*$ implies $\tau\sigma\tau\in H^*$ and the equality (7) holds. The identities (1) and (7) prove that $\rho^c(H^*)$ is self-adjoint. On the other hand H^* is an algebraic subgroup of G^c. Moreover, as the functions in $\mathfrak{o}(M)$ separate the points of M (cf. the proof of Theorem 1 below), we have $H^*\cap G=H$. Therefore, by proposition 1, we have $\rho^c(H^*)\cap U(n)=\rho^c(H^*\cap G)=\rho(H)$. So H^* is canonically isomorphic to the associated algebraic group H^c of H. Hence the proposition is proved completely.

As M^c is the quotient manifold of a complex Lie group G^c, M^c has the structure of a complex analytic manifold compatible with the topology of M^c defined earlier. Moreover the operation of G^c on M^c, i.e., the mapping $(\sigma, \varphi)\to\sigma(\varphi)$ is holomorphic. Proposition 1, 2 and 3 lead to the following corollary.

Corollary to Proposition 3. Let *Let $G^*\supset H^*$ be two self-adjoint complex algebraic groups. Then the quotient manifold G^*/H^* has the structure of an affine algebraic set.*

REMARK. G is regarded as a subgroup of G^c by identifying $g\in G$ with the right translation R_g. So the operation of $g\in G$ on M^c is as follows:

$$g\varphi = R_g\ \varphi, \quad \varphi\in M^c.$$

In the realization of M^c as $\mathrm{Hom}(\mathfrak{o}(M), C)$, an element $g\in G$ operates on M^c by the formula

$$(g\omega)(f) = \omega(T_g f), \quad f\in\mathfrak{o}(M), \tag{8}$$

because we have $(g\omega)(f)=(R_g\varphi f)(e)=(\varphi f)(g)=(L_g\varphi f)(e)=\omega(T_g f)$.

5. The duality theorem

Lemma 4. *$\mathfrak{o}(M)$ is dense in $C(M)$ with $\|\ \|$-topology.*

Proof. We identify f in $\mathfrak{o}(M)$ with ιf in $\mathfrak{o}(G)$. By Peter-Weyl's theorem, for any $f\in C(M)$ and $\varepsilon>0$, there exists an $f_0\in\mathfrak{o}(G)$ such that $\|f-f_0\|<\varepsilon$. Let $f_0^*=\int_H R_h f_0 dh$. Then f_0^* belongs to $\mathfrak{o}(M)$. And we have $\|f-f_0\|\leqq\varepsilon$. So the lemma is proved (cf. N. Iwahori [5]).

Theorem 1 (The duality theorem). *Every p in M defines a homomorphism $\omega_p\in\mathrm{Hom}(\mathfrak{o}(M), C)=M^c$ by means of the equality $\omega_p(f)=f(p)$ $(f\in\mathfrak{o}(M))$. Then the mapping $p\to\omega_p$ is a homeomorphism of M onto M_1 $=\{\omega\in M^c;\ \omega\bar{f}=\overline{\omega f}$ for every $f\in\mathfrak{o}(M)\}$. Moreover regarded as G-spaces, M and M_1 are isomorphic to each other, that is,*

$$g\omega_p = \omega_{gp}, \quad \textit{for every} \quad g\in G \quad \textit{and} \quad p\in M. \tag{9}$$

Proof. If p is a point in M, then ω_p belongs to M_1. Let $\omega_p=\omega_{p'}$, for two points p and p' in M. Then we have $f(p)=f(p')$ for all f in $\mathfrak{o}(M)$. These equalities lead to $f(p)=f(p')$ for all f in $C(M)$ by Lemma 4. The last equalities imply that $p=p'$, because M is a normal space.

So the mapping $\Phi: p \rightarrow \omega_p$ is an injection from M into M_1. By the definition of the topology in M^c, the mapping is continuous. Since M is compact, Φ is a homeomorphism of M onto $M_2 = \{\omega_p; p \in M\}$. M is isomorphic to M_2 as the G-space, because we have

$$(g\omega_p)(f) = \omega_p(T_g f) = f(gp) = \omega_{gp}(f)$$

by (8).

Clearly M_2 is contained in M_1. Now we shall prove the converse: $M_1 \subset M_2$. Let Q be the involutive transformation of $M^c = \mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})$ defined by

$$(Q\omega)(f) = \overline{\omega(\bar{f})} \qquad f \in \mathfrak{o}(M), \qquad \omega \in M^c,$$

and S the involutive automorphism of $G^c = \mathrm{Hom}(\mathfrak{o}(G), \boldsymbol{C})$ defined by

$$(S\theta)(f) = \overline{\theta(\bar{f})}, \qquad f \in \mathfrak{o}(G), \qquad \theta \in G^c.$$

Let π be the canonical projection of G^c onto $M^c = G^c/H^c$. If σ belongs to $G^c = \mathrm{Hom}(\mathfrak{o}(G), \boldsymbol{C})$, then the element $\pi(\sigma) \in M^c$, regarded as an element of $\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})$, is equal to $\sigma \circ \iota$. So we have

$$Q \circ \pi = \pi \circ S. \tag{10}$$

Since G^c acts on M^c transitively, every ω in M^c can be written as $\omega = \pi(\sigma)$ for some σ in G^c. ω belongs to M_1 if and only if $Q\omega = \omega$ which is equivalent to $S(\sigma)H^c = \sigma H^c$ by the identity (10). Let $\omega = \pi(\sigma)$ be an arbitrary element of M_1. σ is expressed uniquely as

$$\sigma = ga, \quad g \in G \quad \text{and} \quad a = \exp\sqrt{-1}\,X, \ X \in \mathfrak{g} \tag{11}$$

where $\mathfrak{g}$ is the Lie algebra of G. Then, by the identity (11), $S(\sigma)$ is decomposed as

$$S(\sigma) = ga^{-1}$$

As $S(\sigma)$ is congruent to σ modulo H^c, we can find an element η of H^c such that $\sigma = S(\sigma)\eta$. η is expressed uniquely as

$$\eta = hb, \quad h \in H \quad \text{and} \quad b = \exp\sqrt{-1}\,Y, \quad Y \in \mathfrak{h},$$

where $\mathfrak{h}$ is the Lie algebra of H. The identity $\sigma = S(\sigma)\eta$ leads to $ga^{-1}\eta = ga$ and $a^2 = \eta = hb$. So we have by the uniqueness of the expression (11)

$$h = e, \qquad X = 2^{-1}Y \in \mathfrak{h}$$

and $\sigma = ga = g\exp 2^{-1}\sqrt{-1}\,Y \in gH^c$ where g lies in the original compact

Lie group G. Therefore we have proved $\omega=\pi(\sigma)=\pi(g)=\omega_{gp_0}\in M_2$ and $M_1=M_2$. The theorem is now proved completely.

DEFINITION. Let ρ be a matricial representation of a group G and n be the degree of ρ. The continuous mapping F of $M=G/H$ into $\boldsymbol{C}^n$ satisfying

$$F(gp) = \rho(g)F(p) \text{ for every } g\in G \text{ and } p\in M, \tag{12}$$

is called a (linear) representation of M associated with ρ. n is called the degree of the representation F and is denoted by $d(F)$.

Two representations $(F_1, \rho_1, \boldsymbol{C}^{n_1})$ and $(F_2, \rho_2, \boldsymbol{C}^{n_2})$ of M are called equivalent if $n_1=n_2$ and there exists a non singular matrix γ of degree n_1 satisfying $\gamma F_1=F_2$ and $\gamma\circ\rho_1(g)=\rho_2(g)\circ\gamma$ for every $G\in G$.

The direct sum and the tensor product of two representations (F_1, ρ_1, C^{n_1}) and (F_2, ρ_2, C^{n_2}) can be defined naturally and are denoted by

$$\begin{array}{ll} (F_1\oplus F_2,\ \rho_1\oplus\rho_2,\ C^{n_1+n_2}) & \text{and} \\ (F_1\otimes F_2,\ \rho_1\otimes\rho_2,\ C^{n_1n_2}) & \text{respectively.} \end{array}$$

The complex conjugate representation $(\bar{F}, \bar{\rho}, \boldsymbol{C}^n)$ of $(F, \rho, \boldsymbol{C}^n)$ is also defined in a natural way.

DEFINITION. Let $\mathfrak{R}$ be the set of all representations of $M=G/H$. By a representation of $\mathfrak{R}$, we understand a mapping ζ which assigns to every $F\in\mathfrak{R}$ a vector $\zeta(F)$ in $\boldsymbol{C}^n$ ($n=d(F)$), in such a way that the equalities

1) $\eta(\gamma\circ F) = \gamma\zeta(F)$
2) $\zeta(F_1\oplus F_2) = \zeta(F_1)\oplus\zeta(F_2)$
3) $\zeta(F_1\otimes F_2) = \zeta(F_1)\otimes\zeta(F_2)$
4) $\zeta(E) = 1$

hold for any representations F_1, F_2, F of M and any regular matrix γ of degree $d(F)$ and the (1-dimensional) unit representation $E: p\to 1$.

Let $F\in\mathfrak{R}$ and $F(p)={}^t(f_1(p), \cdots, f_n(p))$. Then each f_i belongs to $\mathfrak{o}(M)$ and $\mathfrak{o}(M)$ is spanned by such f_i's. We denote f_i by $f(i, F)$ and introduce the indeterminates $u(i, F)$ ($1\leqq i\leqq d(F)$) for every $F\in\mathfrak{R}$. Let $\mathfrak{U}$ be the ring of polynomials in the variables $u(i, F)$'s with the coefficients in $\boldsymbol{C}$. Then there exists a homomorphism Ψ of $\mathfrak{U}$ onto $\mathfrak{o}(M)$ which maps each $u(i, F)$ upon the corresponding $f(i, F)$. Let $\mathfrak{d}$ be the kernel of this homomorphism.

To every representation $F\in\mathfrak{R}$, let us assign the vector $U(F)={}^t(u(1,$

$F), \cdots, u(d(F), F))$. Among the polynomials belonging to the ideal $\mathfrak{d}=$ ker Ψ, we find in particular the following ones:

1) The components of the vectors $U(\gamma \circ F)-\gamma U(F)$, where γ is any regular matrix of degree $d(F)$.
2) the components of the vectors $U(F_1 \oplus F_2)-(U(F_1)\oplus U(F_2))$
3) the components of the vectors $U(F_1 \otimes F_2)-(U(F_1)\otimes U(F_2))$
4) $u(1, E)-1$.

Lemma 5. *Let $M=G/H$ be a homogeneous space of a compact Lie group G. Then the polynomials listed above under the headings* 1), 2), 3), *and* 4) *form a set of generators of the ideal $\mathfrak{d}=ker\,\Psi$.*

The proof of this Lemma is same as the proof of the similar proposition for the algebra $\mathfrak{o}(G)$ (Chevalley [2], Ch. VI § VIII, Proposition 1).

G^c operates on the set $\mathfrak{M}$ of all representations of $\mathfrak{R}$ by the formula

$$(\sigma\zeta)(F) = \rho^c(\sigma)\zeta(F), \ \sigma\in G^c, \ \zeta\in\mathfrak{M}, \ F\in\mathfrak{R}. \tag{13}$$

After these preparations, Theorem 1 can be interpreted to a duality theorem of usual Tannaka-Chevalley type.

Theorem 2. *Let $\mathfrak{M}$ be the set of all representations of $\mathfrak{R}$. Then every $\omega\in\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})=M^c$ corresponds to a representation ζ of $\mathfrak{R}$ defined by*

$$\zeta_\omega(F) = {}^t(\omega(f(1, F)), \cdots, \omega(f(d(F), F))). \tag{14}$$

The mapping $\omega\to\zeta_\omega$ is a G^c-space isomorphism from M^c onto $\mathfrak{M}$.

Moreover let $\mathfrak{M}_1=\{\zeta\in\mathfrak{M}\,;\,\zeta(\bar{F})=\overline{\zeta(F)}$ for every $F\in\mathfrak{R}\}$. Then the mapping $p\to\zeta_{\omega_p}$ is a G-space isomorphism of M onto $\mathfrak{M}_1$.

Proof. ζ_ω belongs to $\mathfrak{M}$, because ω is a homomorphism of $\mathfrak{o}(M)$ onto $\boldsymbol{C}$. If ω_1 and ω_2 are two distinct elements in M^c, then there exists a function $f(i, F)$, $F\in\mathfrak{R}$ such that $\omega_1(f(i, F))\neq\omega_2(f(i, F))$. It follows $\zeta_{\omega_1}(F)\neq\zeta_{\omega_2}(F)$. And the mapping $\omega\to\zeta_\omega$ is an injection. Let ζ be an arbitrary element in $\mathfrak{M}$. Then ζ defines consistently an $\omega\in\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})$ satisfying (14) for $\zeta_\omega=\zeta$ by virtue of Lemma 5. So the first half of the theorem is proved. The second half of the theorem is a direct consequence of Theorem 1 because ω belongs to the set M_1 in Theorem 1 if and only if ζ_ω belongs to $\mathfrak{M}_1$. Now the theorem is proved.

Every representation F of $M=G/H$ can be extended to a representation F^c of $M^c=G^c/H^c$ identified with $\mathfrak{M}$ by defining

$$F^c(\omega) = \zeta_\omega(F). \tag{15}$$

F^c is an everywhere defined rational mapping of M^c into $\boldsymbol{C}^{d(F)}$.

Lemma 6. *Let F be a representation of M whose components form a system of generators of $\mathfrak{o}(M)$. Then the extension F^c of F defined by* (15) *is a faithful rational representation of M^c in $\boldsymbol{C}^n$* ($n=d(F)$). *$F^c(M^c)$ is the smallest algebraic set in $\boldsymbol{C}^n$ containing $F(M)$.*

Proof. The first half of the Lemma is clear. Let $\mathfrak{a}$ be the ideal of $\boldsymbol{C}[t_1, \cdots, t_n]$ consisting of all polynomials P satisfying $P(f(1, F), \cdots, f(n, F))=0$ and N be the algebraic set in $\boldsymbol{C}^n$ defined by $\mathfrak{a}$. Let ω be an element of $\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})=M^c$ and P in $\mathfrak{a}$. Then $P(\omega(f(1, F)), \cdots, (f(n, F)))=0$. So $F^c(M^c)\subset N$. Conversely, let $z=(z_1, \cdots, z_n)$ be any point in N. Then there exists the unique $\omega\in\mathrm{Hom}(\mathfrak{o}(M), \boldsymbol{C})=M^c$ such that $\omega(f(i, F))=z_i$ $(1\leqq i\leqq n)$. So we have proved $F^c(M^c)=N$. The Lemma is proved.

Let (F, ρ, C^n) be a representation of $M=G/H$ such that the components of F generate $\mathfrak{o}(M)$ and $\rho(G)$ is contained in $GL(n, \boldsymbol{R})$. We identify $\omega\in M^c$ with $F^c(\omega)$ and M^c with its model $F^c(M^c)$. Then we have the following theorem.

Theorem 3. 1) $M=M^c\cap\boldsymbol{R}^n$. 2) *$M$ is a real affine algebraic set.* 3) *M^c is defined over $\boldsymbol{R}$.* 4) *$M=G/H$ has a faithful representation.* 5) *The operation of G^c on M^c is rational, that is, the mapping $(\sigma, \omega)\to\sigma(\omega)$ is an everywhere defined rational mapping of $G^c\times M^c$ onto M^c.*

Proof. 1) is clear from Theorem 2. 2) and 3) are obtained from 1) and Lemma 6. 4) is clear since F^c is faithful. Since ρ^c is a rational representation, 5) follows directly from the formula (13).

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