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# QF-3 AND SEMI-PRIMARY PP-RINGS II

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In the previous paper [5] the author has studied semi-primary left (resp. right) QF-3 rings, which is a ring $A$ with the following property: there exists a faithful, projective, injective left (resp. right) $A$-module. Especially, we have considered, in [5], a semi-primary left QF-3 and partially PP-ring[1]. We have shown in [5], Remark 4 that the basic ring of such a ring is characterized as a special subring of a semi-simple ring.

In § 3 of this short note we shall study a similar problem to the above in a case of a semi-primary left and right QF-3 ring with the following properties: Let $Ae$ is a unique minimal faithful, projective, injective left ideal and $e=\sum_{i=1}^{t}e_i$ a decomposition of $e$ into a sum of mutually orthogonal primitive idempotents $e_i$. 1) The left socle of $Ae$ (the sum of irreducible $A$-module of $Ae$) is $A$-projective 2) $e_iAe_i$ is a division ring for all $i$ and 3) $eAe$ is a direct sum of division rings.

It is clear that 3) implies 2). We shall shown in § 3 that 1) implies 2) and that 3) is equivalent to 1) if $A$ is a left and right QF-3 ring. Furthermore, we shall show that the basic ring of left QF-3 ring is a partially PP-ring if and only if $A$ satisfies condition 1) and a condition that the socle of every primitive left ideal is irreducible.

In § 1 we shall show that if $A$ satisfies left and right minimum conditions, then $A$ is left QF-3 if and only if $A$ is right QF-3. However in § 2 we shall give a semi-primary ring which is left QF-3, but not right QF-3.

## 1. QF-3 rings with minimum conditions.

Let $A$ be a ring with identity element 1 and $N$ the radical of $A$. In this note we always consider a semi-primary ring $A$, namely $A/N$ is a semi-simple ring with minimum conditions and $N$ is nilpotent. We call $A$ a *left QF-3 ring* if there exists a *faithful, projective, injective A-module*. Since $A$ is semi-primary, we obtain a faithful, injective left ideal $Ae$ if $A$ is left QF-3, where $e$ is an idempotent.

1) See [4] or [5].

We shall show in this section that a left QF–3 ring satisfying left and right minimum conditions is a right QF–3 ring. On the other hand in §2 we shall give an example which shows that the above fact is not true for semi-primary left QF–3 rings.

**Theorem 1.** *We assume that $A$ satisfies left and right minimum conditions. Then $A$ is a left QF–3 if and only if $A$ is right QF–3.*

Proof. Let $Q$ be the factor module of the ring of rationals modulo the ring $Z$ of integers. We assume that $A$ is left QF–3 and $L$ a faithful, projective, injective $A$-module. Put $L^* = \mathrm{Hom}_Z(L, Q)$. Since $Q$ is $Z$-injective by [2], p. 134, Proposition 5.1, $L^*$ is a right $A$-faithful module. Furthermore, $L^*$ is $A$-injective by [2], p. 166, Proposition 2.5a. Let $M$ be a finitely generated left $A$-module. The $L^* \underset{A}{\otimes} M \approx \mathrm{Hom}_Z(\mathrm{Hom}_A(M, L), Q)$ by [2], p. 124, Proposition 5,3. Hence, $L^*$ is $A$-flat, since $L$ is $A$-injective and $Q$ is $Z$-injective. Therefore, $L^*$ is a faithful, injective, projective $A$-module by [3]. Hence, $A$ is right QF–3. The converse is similar.

**Corollary 1.** *Let $A$ be as above. Then the left $A$-injective envelope of $A$ is $A$-projective if and only if the right $A$-injective envelope is $A$-projective.*

Proof. It is clear from [8], Theorems 3.2 and 3.1.

## 2. Generalized trianglar matrix rings.

We shall consider a *g.t.a.* matrix ring $T_n(\Delta_i\,;\,M_{i,j})$ over division ringe $\Delta_i$ which is left QF–3, (see [6] for the definition of $T_n(\Delta_i\,;\,M_{i,j})$).

**Proposition 1.** *Let $A$ be a g.t.a. matrix ring $T_n(\Delta_i\,;\,M_{i,j})$ over division rings. We assume $Ae_i$ is $A$-injective and $t$ is the maximal index among $j$ such that $M_{j,i} \neq (0)$. Then $Ae_i \approx \mathrm{Hom}_{\Delta_t}(e_tA, \Delta_t)$, and this isomorphism is given by the multiplication of elements in $Ae_i$ from the right side, where $e_i = T_n(o, \overset{\overset{i}{\vee}}{1}, o\,;\,o)$.*

Proof. First, we show that $M_{j,i} \approx \mathrm{Hom}_{\Delta_t}(M_{t,j}, M_{t,i})$ by the multiplication of elements in $M_{j,i}$. Since $M_{k,i} = (0)$ for $k < t$ and $Ae_i$ is an indecomposable injective ideal, $M_{t,i}$ is a unique minimal left ideal in $Ae_i$. Hence, $[M_{t,i} : \Delta_t] = 1$ and $M_{t,i} \approx \Delta_t$ as a left $\Delta_t$-module. Let $X = \mathrm{Hom}_{\Delta_t}(M_{t,j}, M_{t,i})$ and $f \in X$. We define $\bar{f} \in \mathrm{Hom}_A(\sum_{k=t}^{n} \oplus M_{k,j}, M_{t,i})$ by setting $\bar{f}(M_{k,j}) = (0)$ for $k > t$ and $\bar{f} | M_{t,j} = f$. Since $M_{t,i} \subseteq Ae_i$, there exists an element $m_j \in M_{j,i}$ such that $f(m) = mm_j$ for any $m \in M_{t,j}$. Therefore, $X$

coincides with the set of right multiplication of elements in $M_{j,i}$. Furthermore, $Ae_i \supseteq Am_j \cap M_{t,i} = M_{t,j}m_j$. Hence, $M_{t,j}m_j \neq (0)$ whenever $m_j \neq 0$, since $M_{t,i}$ is the socle of $Ae_i$. Therefore, $X \approx M_{j,i}$. It is clear from this fact that $Ae_i \approx \mathrm{Hom}_{\Delta_t}(e_t A, M_{t,i}) \approx \mathrm{Hom}_{\Delta_t}(e_t A, \Delta_t)$ as a left $A$-module.

**Corollary 2.** *Let $A$ be as above. We assume that $A$ is a left and right QF-3 ring. Then $[Ae_1 : \Delta_1] < \infty$.*

Proof. Since $A$ is left QF-3, $Ae_1$ must be $A$-injective. We assum $M_{t,1} \neq (0)$ and $M_{k,1} = (0)$ for $k > t$. Then $e_k A M_{t,1} \subseteq M_{k,t} M_{k,1} = M_{k,1} = (0)$ for $k \neq t$. Therefore, if $A$ is right QF-3, $e_t A$ must be $A$-injective. Furthermore, $Ae_1 \approx \mathrm{Hom}_{\Delta_t}(e_t A, M_{t,1})$, $e_t A \approx \mathrm{Hom}_{\Delta_1}(Ae_1, M_{t,1})$ and $M_{t,1} = \Delta_t x = x\Delta_1$ for some $x \in M_{t,1}$ by Poroposition 1. Therefore, $[Ae_1 : \Delta_1] < \infty$ by [7], p. 68, Theorem 1.

EXAMPLE. Let $\Delta = \Delta_1 = \Delta_3$ and $\Delta_2$ be division rings and $M_{3,2}$ a $\Delta_3 - \Delta_2$ module such that $[M_{3,2} : \Delta_3] = \infty$. Put $M_{3,1} = \Delta$ and $M_{2,1} = \mathrm{Hom}_{\Delta}(M_{3,2}, M_{3,1})$. Let

$$A = \begin{pmatrix} \Delta_1 & 0 & 0 \\ M_{2,1} & \Delta_2 & 0 \\ M_{3,1} & M_{3,2} & \Delta_3 \end{pmatrix}.$$

Then $Ae_1$ is $A$-faithful. Furthermore, $Ae_1 \approx \mathrm{Hom}_{\Delta_3}(e_3 A, \Delta_3)$ as an $A$-module. Therefore, $A$ is left QF-3. However, $A$ is not right QF-3 from Corollary 2.

We obtain immediately Lemma 5 in [5] from Poroposition 1.

### 3. PP-rings.

Let $e$ be a primitive idempotent of a semi-primary ring $A$. In this section we shall study the ring $A$ such that $Ae$ is injective and its socle is $A$-projective. If $A$ is a partially PP-ring[2] and QF-3 ring, then $A$ satisfies the above condition, (cf. Theorem 2 below).

Let $A^*$ be a basic ring[3] of $A$. Then $A$ is isomorphic to the endomorphism rings of a finitely generated projective right $A^*$-module (see [6]). We note from this fact that primitive left ideals in $A$ and $A^*$ enjoy many similar properties.

**Lemma.** *Let $e$, $f$ be primitive idempotents. We assume that the left socle of $Ae$ is $A$-irreducible and $A$-projective. If $fAe \neq (0)$, then either*

2) See [4].
3) See [5].

*Ae contains an isomorphic image of Af or the left socle of Af is not irreducible.*

Proof. If $x \neq 0 \in fAe$. Then $Afx = Ax \neq (0)$ in $Ae$. Let $\varphi$ be an $A$-homomorphism of $Af$ to $Ax$ by setting $\varphi(yf) = yfx$; $y \in A$. Since $Ax \neq (0)$, $Ax$ contains the left socles $S$ of $Ae$. Then $o \to \varphi^{-1}(o) \to \varphi^{-1}(S) \to S \to o$ is exact. Hence, $\varphi^{-1}(S) \approx S \oplus \varphi^{-1}(o)$. If the left socle of $Af$ is irreducible, then $\varphi^{-1}(o) = (0)$. Therefore, $\varphi$ is isomorphic.

**Proposition 2.** 1) *Let A be semi-primary and e a primitive idempotent in A. If Ae is A-injective and its left socle is A-projective, then* $Ae \approx \mathrm{Hom}_{fAf}(fA, fAe)$ *as a left A-module and* $eAe \approx fAf$ *is a division ring.* 2) *Furthermore, we assume that A is a left QF-3 ring with faithful injective ideal AE. If the left socle of AE is A-projective, then EAE is a semi-simple ring, where f is a prinitive idempotent.*

Proof. We may assume that $A$ coincides with its basic ring. Let $S$ be the socle of $Ae$. Since $S$ is $A$-projective, $S \approx Af$ and $Nf = (0)$ for some primitive idempotent $f$. Let $1 = \sum_{j=1}^{n} e_j$ a decomposition of 1 into a sum of mutually orthogonal primitive idempotents $e_j$ (assume $e = e_i$, $f = e_k$). Then $e_jAe_k = e_jNe_k = (0)$ for $j \neq k$. It is clear that $\Delta \equiv e_kAe_k = e_kAe_k/e_kNe_k$ is a division ring. Since $e_lAe_kAe_j = (0) = e_lAS$ for $l \neq k$, $\mathrm{Hom}_\Delta(e_kA, S) = \sum_{j=1}^{n} \mathrm{Hom}_A(e_kAe_j, S)$. Furthermore, $S = e_kAe_i$, since $e_kAe_i$ is a left ideal in $Ae_i$. Then we can prove similarly to Proposition 1 that $\mathrm{Hom}_\Delta(e_kA, S) \approx Ae_i$ as a left $A$-module. We have the last part of 1) from this isomorphism (cf. the proof of Theorem 3 below).
2) Let $E = \sum_{i=1}^{t} e_i$ be the usual decomposition. From the assumption we know that the left socle of $Ae_i$ is $A$-projective. Then we obtain $e_iAe_j = (0)$ for $i \neq j$ by Lemma. Therefore, we have proved 2) from 1).

**Corollary 3.** *Let A, Ae and fA be as above. Then the following facts are equivalent.*
1) $eN = (0)$, 2) *The right socle of fA is A-projective.*

Proof. We assume that $A$ is basic. Let $T$ be the right socle of $fA$, then $\Delta T = T$, where $\Delta = fAf$. Since $Ae \approx \mathrm{Hom}_\Delta(fA, \Delta)$, $\Delta \approx eAe \approx Ae/Ne \approx \mathrm{Hom}_\Delta(T, \Delta)$. Hence, $T \approx eA/eN$. Therefore, 1) and 2) are equivalent.

We note that if $A$ is a semi-primary left QF-3 ring with a faithful injective $Ae$, then every irreducible left ideal in a primitive left ideal is isomorphic to one of $Ae$, (cf. [8]).

**Theorem 2.** *Let $A$ be a semi-primary left $QF$-3 ring with faithful injective left ideal $Ae$. Then the basic ring of $A$ is a partially PP-ring*[1] *if and only if the left socle of $Ae$ is $A$-projective and the socle of any primitive left ideal is irreducible. In this case $A$ is also right $QF$-3.*

Proof. Let $A$ be a basic and partially PP-ring. Then we may assume by [5], Theorem 3 that $A=T_n(\Delta_i\,;\,M_{i,j})$ and $[M_{n,i}:\Delta_n]=1$ for all $i$, where the $\Delta_i$'s are division rings. Let $e'=T_n(1, o, \cdots, o\,;\,o)$, then $Ae'$ is faithful and injective. Therefore, the irreducible left ideal in a primitive left ideal $Af$ is isomorphic to the socle $T_n(o\,;M_{n,1}, o, \cdots, o)$, which is $A$-projective. Since $[M_{n,i}:\Delta_n]=1$, we obtain that the socle of $Af$ is irreducible. Conversely, we assume that $A$ is basic and $Ae$ is a faithful injective and its socle is $A$-projective, and that the socle of any primitive left ideal is irreducible. Let $1=\sum_{i=1,j=1}^{t,\rho(i)} e_{i,j}$ be a decomposition of 1 into a sum of mutually orthogonal primitive idempotent $e_{i,j}$ such that $e=\sum_{i=1}^{t} e_{i,1}$ and the left socle of $Ae_{i,j}$ is isomorphic to one of $Ae_{i,1}$ and not isomorphic to one of $Ae_{k\,1}$ for $k\neq i$. Then $e_{i,j}Ae_{k,l}=(0)$ for $i\neq k$ by Lemma. Therefore, $A=\sum_i\oplus E_iAE_i$ as a ring, where $E_i=\sum_{j=1}^{\rho(i)} e_{i,j}$. Hence, we may assume $t=1$, $e=e_1$, and $1=e_1+\cdots+e_s$. By $n(e_i)$[4] we shall mean the largest integer $p$ such that $N^pe_i\neq(0)$. If $e_iAe_j\neq(0)$ for $i\neq j$, then there exists an isomorphism $\varphi$ of $Ae_i$ into $Ae_j$. Since $Ne_j$ is a unique maximal left ideal in $Ae_j$, $\varphi(Ae_i)\subseteq Ne_j$. Hence, $n(e_j)>n(e_i)$. After rearranging $\{e_k\}$, we may assume that $1=f_1+\cdots+f_s$, $n(f_i)\geqslant n(f_j)$ if $i\leqslant j$ and $\{f_i, \cdots, f_s\}\equiv\{e_1, \cdots, e_s\}$. Since $Ae_1$ is faithful, $e_1=f_1$. Furthermore, $f_iAf_j=(0)$ if $i<j$ from the above. Hence, $A=T_s(f_iAf_i\,;\,f_kAf_p)$[5]. It is clear that the $f_iAf_i$ are division rings[6]. Furthermore, $Af_1$ is injective and $f_kAf_1\neq(0)$ for all $k$ and hence, $Af_1\approx\mathrm{Hom}_{f_sAf_s}(f_sA, f_sAf_1)$ by Proposition 1. Therefore, $f_sAf_i\neq(0)$ for all $i$. Since $Af_i$ has the irreducible socle, $[f_sAf_i:f_sAf_s]=1$. Hence $[f_iAf_1:f_1Af_1]=1$. Therefore, $A$ is a partially PP-ring by [5], Proposition 5.

REMARK. Using the similar argument as above and Corollary 3 we can prove directly [5], Theorem 1.

**Theorem 3.** *Let $A$ be a left and right $QF$-3 ring and semi-primary.*

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4) cf. [6].

5) See [6].

6) Since $f_sAf_1\neq(0)$ is left ideal in $Af_1$, the socle $S$ of $Af_1$ is contained in $f_sAf_1$. Hence $S\approx Af_k$. We know $f_sAf_s\equiv\Delta_s$ is a division ring by Prop. 2. In the proof. of Prop. 1 we have used a fact that $\Delta_s$ is a division ring, and hence we obtain $[f_iAf_1:f_1Af_1]=1$ for all $i$ as in the proof. Therefore, $f_iNf_iAf_i=(0)$ and hence $f_iNf_i=(0)$

*We assume $Ae$ (resp. $fA$) is a unique minimal faithful, projective, injective left (resp. right) $A$-module, and $e=e_1+\cdots+e_t$ ($f=f_1+\cdots+f_s$) is a decomposition of $e$ (resp. $f$) into a sum of mutually orthogonal primitive idempotents $e_i$ (resp. $f_i$). If* 1) *$e_iAe_i=\Delta_i$ is a division ring for all $i$, then* 2) *$s=t$ and $e_iAe_i\approx f_{\pi(i)}Af_{\pi(i)}$. And furthermore, $A$ is contained in a semi-simple ring $B=\sum_{i=1}^{t}\oplus(\Delta_i)_{n_i}$ such that* 3) *$Ae_i$ (resp. $f_{\pi(i)}A$) is isomorphic to an irreducible left $B$-ideals in $(\Delta_i)_{n_i}$ as $A-\Delta_i$ (resp. $\Delta_i-A$) module for $i=1,\cdots,t$. Conversely, if $A$ is a subring in a semi-simple ring $B$ satisfying* 1) 2) *and* 3), *then $A$ is a left and right QF-3 ring, (cf.* [5], *Remark* 4).

Proof. If A satisfies 1), 2) and 3), then $Ae_i\approx Be_{ii}\approx\mathrm{Hom}_{\Delta_i}(e_{ii}B,\Delta_i)\approx\mathrm{Hom}_{\Delta_i}(f_{\pi(i)}A,\Delta_i)$, where $e_{ii}$ is a matrix unit in $(\Delta_i)_{n_i}$. Hence, $Ae_i$ is $A$-injective by [2], p. 166, Proposition 2.5a. Similarly we obtain that $f_iA$ is $A$-injective. Since $\sum_{i=1}^{t}Be_{ii}$ is $B$-faithful, $\sum Ae_i$ is $A$-faithful. We assume that $A$ is semi-primary left and right QF-3 ring satisfying 1). First we assume that $A$ is basic. We put $e_i=g$, $\Delta_i=e_iAe_i=\Delta$ and $\mathrm{Hom}_\Delta(Ag,\Delta)=(Ag)^*$. $T=\{t\in(Ag)^*\mid, t(Ng)=(0)\}$ is a right $A$-submodule of $(Ag)^*$ and $TN=(0)$. Conversely, if $y\in(Ag)^*$ satisfies $yN=(0)$, $(0)=(yN)(Ag)$. Hence, $y\in T$. Therefore, $T$ coincides with right socle of $(Ag)^*$. It is clear that $T\approx\mathrm{Hom}_\Delta(Ag/Ng,\Delta)$ as an $A$-module and since $Ag/Ng\approx\Delta$ ($A$ is basic), $T$ is an irreducible right $A$-submodule of $(Ag)^*$. Hence, $(Ag)^*$ is an indecomposable $A$-module ($A$ is semi-primary). Put $M=\sum\oplus(Ae_i)^*$. Since $\sum\oplus Ae_i$ is $A$-faithful, $M$ is $A$-faithful as right $A$-module. Hence, $M$ contains $fA$ as a direct summand. Since $(Ae)^*$ is an indecomposable injective $A$-module, $f_iA\approx(Ae_{\pi(t)})^*$ for $i=1,\cdots,s$ by the generalized Krull-Schmidt's Theorem in [1], where $\pi$ is a one-to-one mapping of $\{1,\cdots,s\}$ to $\{1,\cdots,t\}$. By $\varphi$ we denote the isomprphism $f_{\pi^{-1}(t)}A\approx\mathrm{Hom}_{\Delta_i}(Ae_i,\Delta_i)$. The $\varphi(T)\approx\mathrm{Hom}_{\Delta_i}(Ae_i/Ne_i,\Delta_i)$. Since $e_i(Ae_i/Ne_i)=Ae_i/Ne_i$, $T=f_kTe_i$, $(k=\pi^{-1}(\mathrm{t}))$. $\varphi(f_kAe_i)=\mathrm{Hom}_{\Delta_i}(\Delta_i,\Delta_i)\approx\Delta_i$. Therefore, $T=f_kTe_i=f_kAe_i$. Let $S$ be the left socle $Ae_i$. Then $S=S\Delta_i$, since $Ae_i$ is $A$-injective. Hence, we obtain $\varphi(f_kA/f_kN)=\mathrm{Hom}_{\Delta_i}(S,\Delta_i)$. Therefore, $f_kS=S=f_kSe_i=f_kAe_i$. Furthermore, $\varphi(f_kNf_k)(Ae_i)=\varphi(f_k)(NS)=(0)$, $f_kAf_k/f_kNf_k=f_kAf_k$ is a division ring. Therefore, if we exchange "left" and "right" in the above argument, we obtain $(f_kA)^*\approx Ae_i$ and $s=t$. Similarly we obtain $[f_kAe_i:\Delta_k]=1$. Hence, $f_kAe_i=\Delta_kx'=x'\Delta_i$ for some $x'\neq0$. Therefore, we have an isomorphism $\psi$ of $\Delta_i$ to $\Delta_k$ such that $x'\delta=\psi(\delta)x'$ for $\delta\in\Delta_i$. Let $\varphi(f_k)=g$, then $gf_k=g$ and $g(ye_i)=g(f_kye_i)=g(x')\delta$, where $f_kye_i=x\delta$ and $y\in A$. We may assume $g(x)=1\in\Delta_i$ for some $x\in f_kAe_i$. Then we can easily check that $\varphi$ is given by a multiplication of element of $f_kA$ from the left side and $\varphi$ is a right $A$ and left $(\Delta_k, \psi)$-semilinear mapping. Therefore, the facts $(Ae_i)^* \approx f_k A$ and $(f_k A)^* \approx Ae_i$ imply that $[Ae_i : \Delta_i] = [f_k A : \Delta_k] = n_i < \infty$ by [7], p. 68, Theorm 1. Let $C = \sum \oplus \Delta_i$ and $B = \mathrm{End}_C(Ae) = \sum \oplus \mathrm{End}_{\Delta_i}(Ae_i)$. Since $Ae$ is $A$-faithful, we may regard $A$ as a subring of $B$. Then it is clear from the fact $(Ae_i)^* \approx f_k A$ that $f_k A$ is isomorphic to an irreducible right ideal in $B$ as $\Delta_i - A$ module. Furthermore, $(Ae_i)(Ae_i) = Ae_i$, hence $Ae_i$ is isomorphic to an irreducible left ideal in $B$ as an $A - \Delta_i$ module. If $A$ is not basic, then we can use the same argument as the above after enlarging the degree $n_i$ of the simple rings $B_i = \mathrm{End}_{\Delta_i}(Ae_i)$.

We shall consider the converse of Proposition 2.2)

**Theorem 4.** *In Theorem* 3 *we assume furthermore that* $e_i A e_j = (0)$ *for* $i \neq j$. *Then* $Ae_i$ *and* $f_i A$ *coincide with irreducible left ideal and right ideals in* $B$, *respectively, and the socle of* $Ae_i$ *and* $f_i A$ *ard projective.*

Proof. We may assume $(Ae_i)^* \approx f_i A$ and $(f_i A)^* \approx Ae_i$ and $A$ is basic. Since $e_i A e_j = (0)$ for $i \neq j$, $Ae_i \subseteq \mathrm{Hom}_{\Delta_i}(Ae_i, Ae_i) = B_i$ coincides to an irreducible left ideal in $B_i$. Since $f_i A \approx \mathrm{Hom}_{\Delta_i}(Ae_i, \Delta_i)$, $f_i A e_j \approx \mathrm{Hom}_{\Delta_i}(e_j A e_i, \Delta_i) = (0)$ for $i \neq j$. Hence, $f_i A = \mathrm{Hom}_{\Delta_i}(Ae_i, \Delta_i) \subseteq B_i$. Hence, we can assume that $e_i = e_{ii}^{(i)}$, $f_i = e_{mm}^{(i)}$, where the $e_{jj}^{(j)}$ are matrix units in $B_i$. Let $E_i$ be the identity element of $B_i$. If $E_i A$ contains $x = \sum_j \delta_j e_j m$ such that $\delta_j \neq 0$ for some $j \neq m$, then $x f_i A e_i \not\subset f_i A e_i$. However, $f_i A e_i$ is the left socle of $Ae_i$ and $E_i A e_i = Ae_i$. Therefore, we obtain $E_i A \cap B_i f_i = \Delta_i f_i$. Hence $Af_i = f_i A f_i$ is a $A$-projective. Since $f_i A e_i \approx f_i A f_i$ as a left $A$-module, $f_i A e_i$ is $A$-projective.

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