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THE TRACE MAP AND SEPARABLE ALGEBRAS

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K will always denote a commutative ring with 1, A a K algebra with 1 which is finitely generated projective as a K module, and A^* will denote $\mathrm{Hom}_K(A, K)$. Let $x_1, \cdots, x_n$ in A and $f_1 \cdots f_n$ in A^* be a dual basis for A as a K module, then the trace map t given by $t(x)=\Sigma_i f_i(xx_i)$ is an element of A^* and the definition of t is independent of the choice of dual basis. Auslander and Goldman (Prop A. 4 of [1]) proved: If A is a commutative faithful K algebra then A is separable if and only if t generates A^* as a right A module. Here we give the structure of a finitely generated projective K algebra A (not necessarily commutative) with the property that t generates A^* as a right A module. Prop A. 4 of [1] less the hypothesis that A be faithful is an immediate corollary of our result and we thus give an alternate proof of their result. The algebras are the same in the projective case as those studied by T. Kanzaki [6]. As a consequence of our investigation we can give a simple characterization of the algebras he studied. We are led naturally to a study of the descriminent and the ramification theory of orders over domains in separable algebras over the quotient field. Our results in this direction appear in Section 2.

1. As above, A denotes a finitely generated projective K algebra with trace map t.

DEFINITION. A possesses a separable basis in case there is an element $f \in A^*$ ($A^* = \mathrm{Hom}_k(A, K)$) and elements $x_1, \cdots, x_n; y_1, \cdots, y_n \in A$ so that

(a) $\Sigma_i x_i y_i = 1$
(b) $\Sigma_i f(xx_i) y_i = x$ for all $x \in A$
(c) $f(xy) = f(yx)$ for all $x, y \in A$.

One can easily check that t always satisfies (c) and that if A possesses a separable basis, then $f=t$. We will let C denote the center of A and $[A, A]$ denote the C-submodule of A generated by $\{xy-yx \mid$

$x, y \in A\}$.

Theorem 1. *The following statements are equivalent*:

1. *t generates A^* as a right A module*
2. *A possesses a separable basis*
3. *A is a separable K algebra and $A = C \oplus [A, A]$ as C modules*
4. *A is a separable K algebra, $Rank_C(A)$ is defined and is a unit in C.*

Proof. ($1 \to 2$) Let $f_i = t(-x_i)$ and y_i be a dual basis for A as a K module. We show $\Sigma x_i y_i = 1$. For every $z \in A$,

$$\begin{aligned} t[z(1-\Sigma_i x_i y_i)] &= t(z) - t(z\Sigma_i x_i y_i) \\ &= t[z(\Sigma_i y_i x_i - \Sigma_i x_i y_i)] . \end{aligned}$$

Using the fact that $t(xy) = t(yx)$ and that $t(-x_i)$, y_i form a dual basis one shows that $\Sigma_i y_i x_i = \Sigma_i x_i y_i$. Thus $t[z(1-\Sigma x_i y_i)] = 0$ for all $z \in A$. Using the dual basis property of $t(-x_i)$, y_i again, we have $1 = \Sigma_i x_i y_i$.

($2 \to 3$). Let $[x_i, y_i; t]$ form a separable basis and let $\varepsilon \in A \otimes_K A^0$ be given by $\varepsilon = \Sigma_i x_i \otimes y_i$. For any $x \in A$, $\Sigma_i x x_i \otimes y_i = \Sigma_{i,j} t(x x_i x_j) y_j \otimes y_i = \Sigma_j y_j \otimes x_j x$. Thus by Prop. 7.7 of [3] A is separable over K. Define $\pi_\varepsilon \in \mathrm{Hom}_C(A, A)$ by $\pi_\varepsilon(x) = \Sigma_i y_i x x_i$. One can easily check (or see [5]) that π_ε is a projection with kernel $[A, A]$. Moreover, image $(\pi_\varepsilon) = C$ in case $\Sigma_i x_i \otimes y_i = \Sigma_i y_i \otimes x_i$. By the computation immediately above this is the case here so $A = C \oplus [A, A]$ as C modules.

($3 \to 4$). We show $\mathrm{Rank}_C(A)$ is defined and a unit in C. Let t_C be the trace from A to C whose existence is a consequence of Theorem 2.3 of [1]. It suffices to show $t(1)$ is a unit in C. Since the trace on an algebra over a field is just the usual matrix of the matrices afforded by the regular representation of the algebra, for each maximal ideal $\mathfrak{a}$ of C, the characteristic of $C/\mathfrak{a}$ divides $\mathrm{Rank}_{C/\mathfrak{a}}$ $(C/\mathfrak{a} \otimes_C A)$ if and only if $1 \otimes t_C(1 \otimes 1) = 0$.

By Corollary 1.6 of [1], $C/\mathfrak{a} \otimes_C A$ is a central separable over the field $C/\mathfrak{a}$. Now if S is a central separable algebra over the field F, it is not hard to show (see Lemma 2 of [6]) that $1 \in [S, S]$ if the characteristic of F divides $\mathrm{Rank}_F(S)$. But by 3, $A \simeq C \oplus [A, A]$ so $C/\mathfrak{a} \otimes_C A = C/\mathfrak{a} \oplus [C/\mathfrak{a} \otimes_C A, C/\mathfrak{a} \otimes_C A]$. Thus the characteristic of $C/\mathfrak{a}$ does not divide $\mathrm{Rank}_{C/\mathfrak{a}}(C/\mathfrak{a} \otimes_C A)$ so $t_C(1)$ is a unit in C.

($4 \to 1$). Let $\mathfrak{a}$ be a maximal ideal in K containing the annihilator of A in K. $K/\mathfrak{a} \otimes_K A$ is a separable algebra over the field $K/\mathfrak{a}$. By Theorem 2.3 of [1] the center $(K/\mathfrak{a} \otimes_K C)\varepsilon$ of a simple component of $K/\mathfrak{a} \otimes_K A$ generated by the minimal central idempotent ε is a separable field extension of $K/\mathfrak{a}$. The trace map is transitive so if t_C denotes the trace from A to C and t_K denotes the trace from C to K, then $t=t_K t_C$. By the Corollary at the top of page 96 of [7], there is an element $y\in(K/\mathfrak{a}\otimes_K C)$ so that $1\otimes t_K(y)\neq 0$. Since the characteristic of $K/\mathfrak{a}$ is relatively prime to the rank of the simple components of $K/\mathfrak{a}\otimes_K A$ over their centers, there is an element $z\in(K/\mathfrak{a}\otimes_K C)\varepsilon$ so that $1\otimes t_K(z)=y$. By simplicity of $(K/\mathfrak{a}\otimes_K A)\varepsilon$, for any $b\in(K/\mathfrak{a}\otimes_K A)\varepsilon$ there is v_i, $u_i\in(K/\mathfrak{a}\otimes_K A)$ so that $y=\Sigma_i v_i b y_i$. Thus $1\otimes t(\Sigma_i u_i v_i\cdot b)=1\otimes t(y)\neq 0$ so the map $x\to t(-x)$ from $K/\mathfrak{a}\otimes_K A$ to $\mathrm{Hom}_{K/\mathfrak{a}}(K/\mathfrak{a}\otimes A, K/\mathfrak{a})$ is a monomorphism. By a dimension argument, this map is an isomorphism, and $1\otimes t$ generates $\mathrm{Hom}_{K/\mathfrak{a}}(K/\mathfrak{a}\otimes_K A, K/\mathfrak{a})$ in case the maximal ideal $\mathfrak{a}$ contains the annihilator of A in K. If a maximal ideal $\mathfrak{a}$ of K does not contain the annihilator of A, then $K/\mathfrak{a}\otimes_K A^*=0$. Thus for every maximal ideal of K, $K/\mathfrak{a}\otimes_K A^*/t(-A)=0$. A^* is finitely generated as a K module and thus A^* is generated as a right A module by t. This completes the proof of Theorem 1.

REMARK 1. The equivalence of (1) and (2) above in the special case where A is a faithful commutative K algebra was proved in a seminar at the University of Oregon by D. K. Harrison.

REMARK 2. The algebras discussed by T. Kanzaki in [6] reduce to those in the finitely generated projective case. If S is a separable K algebra, then using the ideas in the proof that $2\to 3$ one can show that $S=C\oplus[S, S]$ as C modules if and only if there is an idempotent $\varepsilon=\Sigma_i x_i\otimes y_i\in S\otimes_K S^0$ so that $(1\otimes x-x\otimes 1)\varepsilon=0$ for all $x\in S$, $\Sigma x_i y_i=1$, and $\Sigma_i x_i\otimes y_i=\Sigma_i y_i\otimes x_i$. This gives a simple characterization of the algebras studied in [6].

2. The algebras discussed in Section 1 lend themselves to a discussion of the classical descriminent. Henceforth, A will denote a faithful finitely generated projective K algebra with trace map t. If $x_i, \cdots, x_n$ is a generating set for A as a K module we let $[t(x_i x_j)]$ be the matrix whose t, j^{th} entry is $t(x_i x_j)$.

Lemma 2. *There is a generating set $x_1, \cdots, x_n$ of A as a K module so that $det[t(x_i x_j)]$ is a unit in K if and only if t generates A^* as a right A module and A is free as a K module.*

Proof. $(\to)$ There exists $x_1, \cdots, x_n\in A$ so that $\det[t(x_i x_j)]$ is a unit in K, thus the system

$$x_j = \Sigma_i t(x_j x_i) Y_i \qquad j = 1, \cdots, n$$

has a unique solution $y_1, \cdots, y_n \in A$. The elements $t(-x_i)$, y_i form a dual basis so the elements $t(-x_i)$ generate A^*. Moreover,

$$[t(x_ix_j)][t(y_iy_j)]=[t(x_iy_i)]$$

and using the dual basis property we have that $[t(x_iy_j)]^2=[t(x_iy_j)]$ and $[t(x_iy_j)][t(x_ix_j)]=[t(x_ix_j)]$. Thus $[t(x_iy_j)]$ is an idempotent invertible matrix so $t(x_iy_j)=\begin{cases}0 & i\neq j\\ 1 & i=j.\end{cases}$ Thus $x_1, \cdots, x_n$ are a free K basis for A.

($\leftarrow$) Let $y_1, \cdots, y_n$ be a free K basis with $\{t(-x_i), y_i\}$ a dual basis. By (2) of Theorem 1, $[t(x_iy_j)]=I_n$. since $[t(x_ix_j)][t(y_iy_j)]=[t(x_iy_j)]$, $\det[t(y_iy_j)]$ is a unit in K which proves the lemma.

Now let R be a commutative integral domain with quotient field K, let A be a K algebra and assume t generates A^* as a right A module. Let Γ be an R order in A, projective over R. The descriminent ideal $\mathrm{Des}(\Gamma/R)$ is the ideal in R generated by the elements $\det[t(x_ix_j)]$ as $\{x_1, \cdots, x_n\}$ ranges over K basis for A in Γ. We let $\Gamma^*=\mathrm{Hom}_R(\Gamma, R)$.

Theorem 3. *Γ^* is generated as a right Γ module by t if and only if $Des(\Gamma/R)=R$.*

Proof. Let R_ρ be the localization of R at the prime ideal ρ. It is easy to see that t_R generates Γ^* if and only if t_{R_ρ} generates $(R_\rho\otimes_R\Gamma)$ $=\mathrm{Hom}_{R_\rho}(R_\rho\otimes_R\Gamma, R_\rho)$ for all prime ideals ρ of R.

If t_R generates Γ^*, then $1\otimes t_R=t_{R_\rho}$ generates $(R_\rho\otimes_R\Gamma)^*$ and $R_\rho\otimes_R\Gamma$ is a free R_ρ module. By Lemma 2 there is a basis $x_1, \cdots, x_n$ of A in $R_\rho\otimes_R\Gamma$ so that $\det[t_{R_\rho}(x_ix_j)]$ is a unit in R_ρ. By multipling the x_i by elements not in ρ, we obtain a basis $x_1', \cdots, x_n'$ of A in Γ so that det $[t(x_i'x_j')]\notin\rho$ ann then $\mathrm{Des}(\Gamma/R)=R$.

Conversely, if $\mathrm{Des}(\Gamma/R)=R$ then for each prime ideal ρ of R there is a basis $x_1, \cdots, x_n$ of A in Γ so that $\det[t(x_ix_j)]\notin\rho$. Then det $[t(x_ix_j)]$ is a unit in R_ρ so by Lemma 2 t_{R_ρ} generates $(R_\rho\otimes_R\Gamma)^*$ and this proves the theorem.

Assume now that R is an integrally closed domain in its quotient field K and that Σ is a central separable K algebra. If Γ is a projective R order in Σ, the reduced trace on Γ provides a criterion that Γ be separable. The reduced trace "trd" is the trace map afforded by a representation of $\bar{K}\otimes_K\Sigma$ as matrices over $\bar{K}$, where $\bar{K}$ is the algebraic closure of K. It turns out that "trd" is a well defined element of $\mathrm{Hom}_K(\Sigma, K)$ and that trd restricted to Γ maps Γ to R. (For an exposition of these facts we refer the reader to [2], pp 142–5.

Theorem 4. *With R, K, Σ, Γ as above; Γ is a separable R algebra* *if and only if trd restricted to* Γ *generates* $Hom_R(\Gamma, R)$ *as a right* Γ *module.*

Proof. The results in [4] apply here so we adopt the notation of [4] and begin by collecting some of the pertinent facts which appear there. Let $q: \Sigma \rightarrow \mathrm{Hom}_K(\Sigma, K)$ by $q(x)=\mathrm{trd}(-x)$, q is an isomorphism. Let $C(\Gamma/R)=\{x\in\Sigma \mid \mathrm{trd}(x\Gamma)\subseteq R\}$ and $D(\Gamma/R)=\{x\in\Sigma \mid C(\Gamma/R)x\subseteq\Gamma\}$. $D(\Gamma/R)$ is called the Dedekind different. If $\sigma: \Gamma\otimes_R\Gamma^0\rightarrow\Gamma$ by $\sigma(x\otimes y)=xy$, if J denotes ker σ, and if $(0:J)$ denotes $\{u\in\Gamma\otimes_R\Gamma^0 \mid Ju=0\}$; then is separable over R and if and only if $\sigma(0:J)=R$ (Prop 1 of [1]). Let $\sigma(0:J)=N(\Gamma/R)$, the homological or Noetherian different. Define η: $\Gamma\otimes_R\Gamma^0\rightarrow\mathrm{Hom}_R(\Gamma, \Gamma)$ by $\eta(x\otimes y)[a]=xay$. By [4], $q: D(\Gamma/R)\rightarrow(0:J)$ is an isomorphism and $\eta(0:J)\subseteq\mathrm{Hom}_R(\Gamma, R)$. Moreover $q: C(\Gamma/K)\rightarrow\mathrm{Hom}_R(\Gamma, R)$ is an isomorphism.

Now trd generates $\mathrm{Hom}_R(\Gamma, R)$ if and only if $D(\Gamma/R)=\Gamma$. Since $q: D(\Gamma/R)\rightarrow\eta(0:J)$ is an isomorphism, $D(\Gamma/R)=\Gamma$ if and only if $\eta(0:J)=\mathrm{Hom}_R(\Gamma, R)$ since $C(\Gamma/R)=D(\Gamma/R)=\Gamma$. By the discussion at the top of page 370 of [1] we infer that $\eta(0:J)=\mathrm{Hom}_R(\Gamma, R)$ if and only if $\sigma(0:J)=R$ and this chain of implications proves the result.

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Bibliography

[1] M. Auslander and O. Goldman : *The Brauer group of a commutative ring.* Trans. Amer. Math. Soc. **97** (1960), 367–409.

[2] N. Bourbaki : Algebre, chapitre 8, Modules et Anneaux Semi-Simples, Hermann, Paris, 1958.

[3] H. Cartan and S. Eilenberg : Homological Algebra, Princeton, 1956.

[4] R. M. Fossum : *The Noetherian different of projective orders,* Illinois J. Math. (to appear).

[5] A. Hattori : *Rank element of a projective module,* Nagoya Math. J. **25** (1965), 113–130.

[6] T. Kanzaki : *Special type of separable algebra over a commutative ring,* Proc. Japan Acad. **40** (1964), 781–786.

[7] O. Zariski and P. Samuel : Commutative Algebra, vol. 1., Van Nostrand 1958.