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ON DIFFERENTIABILITY AND ANALYTICITY OF SOLUTIONS OF WEIGHTED ELLIPTIC BOUNDARY VALUE PROBLEMS

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The present paper is concerned with the differentiability and analyticity of solutions of weighted elliptic boundary value problems (see [2] for the definition of weighted ellipticity)

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \qquad x\in\Omega, \tag{0.1}$$

$$B_j(x, t, D_x, D_t)u(x, t) = 0, \qquad x\in\partial\Omega, \ j=1, \cdots, m, \tag{0.2}$$

in some cylindrical domain with Ω as its base, where we denote the order type of A by $(2m, l)$. We first investigate such regularity properties of the solution u considered as a function of t with values in $L^2(\Omega)$ or $H_{2m}(\Omega)$ and then the same properties of u as a numerical function of all independent variables (x, t). In [2] S. Agmon and L. Nirenberg proved the differentiability and analyticity in t of the solutions of (0.1)–(0.2) in $L^p(\Omega)$, $1<p<\infty$, under the corresponding hypothesis on f in case in which all the coefficients of A and $\{B_j\}_{j=1}^m$ do not depend on t with the aid of their general results on abstract differential equations

$$\frac{1}{i}\frac{du}{dt}-Au = f(t) \tag{0.3}$$

in a Banach space. Recently in [4] A. Friedman obtained such kind of regularity theorems for the solutions of abstract differential equations

$$\frac{1}{i}\frac{du}{dt}-A(t)u = f(t) \tag{0.4}$$

in a Hilbert space using Fourier transform in t. In his results $A(t)$ may depend on t but is assumed to have a constant domain. In [11] the author showed that A. Friedman's method can be applied to the problem with time-dependent boundary conditions

$$\partial u(x, t)/\partial t+A(x, t, \partial/\partial x)u(x, t) = f(x, t), \qquad x\in\Omega, \tag{0.5}$$

$$B_j(x, t, \partial/\partial x)u(x, t) = 0, \qquad x\in\partial\Omega, \ j=1, \cdots, m, \tag{0.6}$$

where $A(x, t, \partial/\partial x)$ is an elliptic operator of order $2m$, provided that the positive and negative imaginary axes are of minimal growth in the sense of S. Agmon [1] with respect to $(A, \{B_j\}, \Omega)$. In the present paper we shall generalize this result to the higher order problem (0. 1)–(0. 2), the whole contents being based on a slight extention of the inequality (14. 6) in [2]. As in [4] essential use is made of Plancherel theorem, therefore we are obliged to take $L^2(\Omega)$ as the basic Banach space. Roughly speaking our first main result is stated as follows: putting $d=2m/l$ if $A(x, t, D_x, \pm D_y^\alpha)$ is elliptic in $(x, y)\in\bar{\Omega}\times(-\infty, \infty)$ and the Complementing Condition ([3]) is satisfied by $(A(x, t, D_x, \pm D_y^\alpha), \{B_j(x, t, D_x, \pm D_y^\alpha)\})$ in $\bar{\Omega}\times(-\infty, \infty)$ for each fixed t, then the solution u of (0. 1)–(0. 2) is a smooth function of t with values in $L^2(\Omega)$ provided that the coefficients of A and $\{B_j\}$ as well as f are sufficiently smooth and u is analytic in t provided that all the coefficients together with some of their x-diriva-tives and f are analytic in t. With the aid of this result we shall finally prove that the solutions of (0. 1)–(0. 2) are analytic in all variables if the coefficients of A, $\{B_j\}$ and f are all analytic functions of $(x, t)\in\bar{\Omega}\times(-\infty, \infty)$ and the boundary of Ω is an analytic manifold. The proof of this last statement is so lengthy in the general case (0. 1)–(0. 2) that we shall confine ourselves to the special case (0. 5)–(0. 6).

We shall investigate solutions of (0. 1) satisfying the homogenous boundary conditions (0. 2). However, unless the boundary system is equivalent to another one whose coefficients are independent of t, the boundary conditions satisfied by $D_t^k u$ are not necessarily homogenous if u is a solution of (0. 1)–(0. 2). For the the same reason the tangential derivatives of u in the space directions may not satisfy the homogenous boundary conditions (0. 2). Therefore in sections 2 and 3 we shall obtain some estimate for the solutions of the inhomogenous boundary value problems. Based on this estimate we shall prove the differentiability in t of the solutions by means of difference quotient method and their analyticity in t following L. Hörmander's proof of the interior analyticity of the solutions of elliptic differential equations with analytic coefficients ([5], pp. 178–180). In the proof of the analyticity in all independent variables we follow the method of C. B. Morrey and L. Nirenberg [10] and show that the Cauchy data of u are analytic on the boundary so that we may apply Holmgren's theorem to obtain the desired result.

Finally we note that in [13] the analyticity in the abstract sense was proved for the solutions of (0. 5)–(0. 6) in $L^p(\Omega)$, $1<p<\infty$, when $\{B_j\}$ is normal and all the rays $\{re^{i\theta}: 0<r<\infty\}$ with $\pi/2\leqq\theta\leqq3\pi/2$ are of minimal growth with respect to $(A, \{B_j\}, \Omega)$ as an application of a result on the existence problem of abstract differential equations in a

Banach space ([6]). We mention also [12], [9], [7], [8] and [15] for related topics.

1. Notations and assumptions

We denote by Ω a domain in the n-dimensional Euclidean space E_n and by $\partial\Omega$ its boundary. Let $(x, t)=(x_1, \cdots, x_n, t)$ be the generic point in E_{n+1}. We put $D_x=\left(\frac{1}{(-1)^{1/2}}\frac{\partial}{\partial x_1}, \cdots, \frac{1}{(-1)^{1/2}}\frac{\partial}{\partial x_n}\right)$, $D_t=\frac{1}{(-1)^{1/2}}\frac{\partial}{\partial t}$ and denote by D_x^α, $\alpha=(\alpha_1, \cdots, \alpha_n)$, the x-derivative $D_1^{\alpha_1}\cdots D_n^{\alpha_n}=\left(\frac{1}{(-1)^{1/2}}\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{1}{(-1)^{1/2}}\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. $|\alpha|$ denotes the length of the multi-index $\alpha: |\alpha|=\alpha_1+\cdots+\alpha_n$.

For any integer k we denote by $H_k(\Omega)$ the class of all complex valued measurable functions whose distribution derivatives of order up to k are square integrable in Ω, the norm of $H_k(\Omega)$ being denoted by $\|\ \|_{k,\Omega}$:

$$\|u\|_{k,\Omega}^2=\sum_{|\alpha|\leqq k}\int_\Omega |D_x^\alpha u(x)|^2 dx .$$

Especially $H_0(\Omega)=L^2(\Omega)$.

$H_{k-1/2}(\partial\Omega)$ is to be the class of functions ϕ which are the boundary values of functions v belonging to $H_k(\Omega)$. In this class we introduce the norm

$$\langle\varphi\rangle_{k,\partial\Omega}=\inf\|v\|_{k,\Omega} ,$$

where inf is taken over all functions v in $H_k(\Omega)$ which equal ϕ on the boundary.

Let m and l be positive intergers and let $d=2m/l$. We assume that d is also an integer. $A(x, t, D_x, D_t)$ is a linear differential operator of the form

$$A(x, t, D_x, D_t)=\sum_{k=0}^{l} A_{l-k}(x, t, D_x)D_t^k \tag{1.1}$$

where

$$A_{l-k}(x, t, D_x)=\sum_{|\alpha|\leqq 2m-kd} a_{l-k,\alpha}(x, t)D_x^\alpha , \qquad 0\leqq k<l , \tag{1.2}$$

are differential operators in x of order $2m-kd$ at the most with coefficients defined in $\bar\Omega\times\{t: -\infty<t<\infty\}$ and $A_0(x, t, D_x)\equiv 1$. Let m_j be non negative integers smaller than $2m$ and let $l_j=[m_j/d]$=the integral part of m_j/d. $B_j(x, t, D_x, D_t)$, $j=1, \cdots, m$, are differential operators of the form

$$B_j(x, t, D_x, D_t)=\sum_{k=0}^{l_j} B_{j,l_j-k}(x, t, D_x)D_t^k \tag{1.3}$$

where

$$B_{j,l_j-k}(x, t, D_x) = \sum_{|\beta|\leq m_j-kd} b_{j,l_j-k,\beta}(x, t) D_x^\beta \tag{1.4}$$

are differential operators in x of order m_j-kd at the most with coefficients defined on $\partial\Omega\times\{t: -\infty<t<\infty\}$. In what follows we shall assume without restriction that the coefficients of $B(x, t, D_x, D_t)$, $j=1, \cdots, m$, are defined not only on $\partial\Omega\times\{t: -\infty<t<\infty\}$ but also in $\bar{\Omega}\times\{t: -\infty<t<\infty\}$.

Denote by $A_{l-k}^\#(x, t, D_x)$, $k=0, \cdots, l$, the sum of terms in $A_{l-k}(x, t, D_x)$ which are of precise order $2m-kd$, letting $A_{l-k}^\#=0$ if there are no such terms. Following [2] the weighted principal part $A^\#$ of A is defined as

$$A^\#(x, t, D_x, D_t) = \sum_{k=0}^{l} A_{l-k}^\#(x, t, D_x) D_t^k . \tag{1.5}$$

Similarly we denote by $B_{j,l_j-k}^\#(x, t, D_x)$, $k=0, \cdots, l_j$, $j=1, \cdots, m$, the sum of terms in B_{j,l_j-k} which are of precise order m_j-kd, letting $B_{j,l_j-k}^\#=0$ if there are no such terms. The weighted principal part $B_j^\#$ of B_j is

$$B_j^\#(x, t, D_x, D_t) = \sum_{k=0}^{l_j} B_{j,l_j-k}^\#(x, t, D_x) D_t^k . \tag{1.6}$$

Let y be an auxiliary real variable and we denote by Γ the infinite cyclinder:

$$\Gamma = \{(x, y): x\in\Omega, -\infty<y<\infty\} .$$

For each fixed t, $A(x, t, D_x, \pm D_y^\alpha)$ is a linear differential operator in (x, y) of order $2m$ with coefficients defined in $\bar{\Gamma}$. Similarly for each $i=1, \cdots, m$ and $t\in(-\infty, \infty)$, $B_j(x, t, D_x, \pm D_y^\alpha)$ is a linear differential operator in (x, y) of order m_j at most with coefficients defined in $\bar{\Gamma}$. Clearly the principal part $A'(x, t, D_x, \pm D_y^\alpha)$ of $A(x, t, D_x, \pm D_y^\alpha)$ is

$$A'(x, t, D_x, \pm D_y^\alpha) = A^\#(x, t, D_x, \pm D_y^\alpha) . \tag{1.7}$$

Similarly if the order of $B_j(x, t, D_x, \pm D_y^\alpha)$ is equal to m_j, its principal part is

$$B_j'(x, t, D_x, \pm D_y^\alpha) = B_j^\#(x, t, D_x, \pm D_y^\alpha) . \tag{1.8}$$

ASSUMPTIONS. (I) For each t, $A(x, t, D_x, \pm D_y^\alpha)$ is a uniformly elliptic operator of order $2m$ in $\bar{\Gamma}$; hence $A(x, t, D_x, D_t)$ is a uniformly weighted elliptic operator of order type $(2m, l)$ in the terminology of [2]. For every $(x, t)\in\partial\Omega\times(-\infty, \infty)$ and for every set of real vectors $\xi\in E_n$, $\nu\in E_n$, $\tau\in E_1$ such that $(\xi, \tau)\neq 0$ and $\nu\neq 0$ the polynomial $A^\#(x, t, \xi+s\nu, \tau)$ in s has exactly m roots with a positive imaginary part.

(II) For each $j=1, \cdots, m$ and $t\in(-\infty, \infty)$, the order of $B_j(x, t, D_x, \pm D_y^a)$ is equal to m_j.

(III) Let (x, t) be any point on $\partial\Omega\times(-\infty, \infty)$. Let ν be the normal to $\partial\Omega$ and $\xi\in E_n$ be a real vector parallel to $\partial\Omega$ at x. Whenever τ is a real number such that $(\xi, \tau)\neq 0$, the polynomials in $s: B^{\#}(x, t, \xi+s\nu, \tau)$, $j=1, \cdots, m$, are linearly independent modulo the polynomial $\prod_{k=1}^{m}(s-s_k^+(\xi, \tau))$ where $s_k^+(\xi, \tau)$, $k=1, \cdots, m$, are the roots of $A^{\#}(x, t, \xi+s\nu, \tau)$ with positive imaginary part. In other words the Complementing Condition is satisfied by $(A(x, t, D_x, D_y^a), \{B_j(x, t, D_x, D_y^a)\}_{j=1}^m, \Gamma)$ and $(A(x, t, D_x, -D_y^a), \{B_j(x, t, D_x, -D_y^a)\}_{j=1}^m, \Gamma)$.

(IV) For $|\alpha|\leqq 2m-kd$, $k=0, \cdots, l$, $a_{l-k,\alpha}(x, t)$ (recall (1.2)) are continuous in $\bar{\Omega}\times\{t: -\infty<t<\infty\}$. For $|\beta|\leqq m_j-kd$, $k=0, \cdots, l_j$, $|\kappa|\leqq 2m-m_j$, $j=1, \cdots, m$ and $i=0, \cdots, l+1$, $D_x^\kappa b_{j,l_j-k,\beta}(x, t)$ and $D_t^i b_{j,l_j-k\,\beta}(x, t)$ (recall (1.4)) are continuous in $\bar{\Omega}\times\{t: -\infty<t<\infty\}$.

(V) Ω is a bounded domain of class C^{2m}.

We shall first prove some results on differentiability in t of solutions of (0.1)-(0.2) assuming further differentiability of the coefficients. The problem is local, therefore without loss of generality we shall assume

(IV′) All the functions in (IV) are uniformly continuous and bounded in $\bar{\Omega}\times\{t: -\infty<t<\infty\}$.

Throughout this paper it is understood that any solution of (0.1)-(0.2) is a function u such that $D_t^k u(t)$ is a strongly condinuous function of t with values in $H_{2m-kd}(\Omega)$ for $k=0, 1, \cdots, 2m$ and such that (0.1)-(0.2) hold for u. In what follows we denote by $C_1, C_2, \cdots$ constants dependent only on the assumptions (I)$\sim$(V) and (IV′) unless otherwise stated.

2. Estimate in case of time-independent coefficients

As a preparation we obtain some estimates in the special case in which all the coefficients of A and $\{B_j\}_{j=1}^m$ are independent of t:

$$A(x, t, D_x, D_t) = A(x, D_x, D_t)\,,$$
$$B_j(x, t, D_x, D_t) = B_j(x, D_x, D_t)\,, \qquad j = 1, \cdots, m\,.$$

From now on we shall usually write the abbreviated forms $||u||_k$, $\langle\phi\rangle_k$ omitting Ω and $\partial\Omega$ respectively.

Lemma 2.1. *Under the assumptions of section* 1 *for any function* $u\in H_{2m}(\Omega)$ *and real number* λ *we have*

$$\sum_{k=0}^{2m}|\lambda|^{(2m-k)/d}||u||_k \leqq C_1\{||A(x, D_x, \lambda)u||_0 + \sum_{j=1}^{m}|\lambda|^{(2m-m_j)/d}||w_j(\lambda)||_0 + \sum_{j=1}^{m}||w_j(\lambda)||_{2m-m_j} + ||u||_0\}, \tag{2.1}$$

where w_j $(j=1, \cdots, m)$ *is an arbitrary function in* $H_{2m-m_j}(\Omega)$ *which satisfies the boundary condition*

$$B_j(x, D_x, \lambda)u(x) = w_j(x, \lambda), \quad x\in\partial\Omega, \ -\infty<\lambda<\infty, \ j=1, \cdots, m. \tag{2.2}$$

Proof. We follow the proof of Theorem 5.2 of [2]. Let $\zeta(y)$ be an infinitely differentiable function on the real line such that $\zeta=1$ for $|y|\leqq 1$, $\zeta=0$ for $|y|\geqq 2$. We introduce the function

$$v(x, y) = \zeta(y)e^{(-1)^{1/2}\mu y}u(x)$$

where μ is a real number and $u\in H_{2m}(\Omega)$. By the assumptions and the remark in the proof of Theorem 5.2 of [2], we have

$$||v||_{2m,\Gamma}\leqq C_2\{||A(x, D_x, D_y^q)v||_{0,\Gamma} + \sum_{j=1}^{m}\langle B_j(x, D_x, D_y^q)v\rangle_{2m-m_j,\partial\Gamma} + ||v||_{0,\Gamma}\}.$$

From the obvious relation

$$A(x, D_x, D_y^q)v(x, y) = \zeta(y)e^{(-1)^{1/2}\mu y}A(x, D_x, \mu^d)u(x) + \sum_{k=1}^{l}A_{l-k}(x, D_x)u(x)\sum_{p=0}^{kd-1}\binom{kd}{p}D_y^{kd-p}\zeta(y)\mu^p e^{(-1)^{1/2}\mu y},$$

it follows that

$$||A(x, D_x, D_y^q)v||_{0\,\Gamma} \leqq C_3\{||A(x, D_x, \mu^d)u||_0 + \sum_{k=1}^{l}(1+|\mu|^{kd-1})||u||_{2m-kd}\}. \tag{2.3}$$

Recalling the definition of the boundary norm $\langle\ \rangle_k$ and noting that $B_j(x, D_x, \mu^d)u(x)=w_j(x, \mu^d)$ on $\partial\Omega$, we get

$$\begin{aligned}\langle B_j(x, D_x, D_y^q)v\rangle_{2m-m_j,\partial\Gamma} &\leqq ||\zeta e^{(-1)^{1/2}\mu y}w_j(\mu^d)||_{2m-m_j,\Gamma} \\ &+ ||\sum_{k=1}^{l_j}B_{l_j-k}(x, D_x)u\sum_{p=0}^{kd-1}\binom{kd}{p}D_y^{kd-p}\zeta\cdot\mu^p e^{(-1)^{1/2}\mu y}||_{2m-m_j,\Gamma} \\ &\leqq C_4\{\sum_{q=0}^{2m-m_j}(1+|\mu|)^{2m-m_j-q}||w_j(\mu^d)||_q \\ &+ \sum_{k=1}^{l_j}\sum_{q=0}^{2m-m_j}(1+|\mu|)^{kd-1+q}||u||_{2m-q-kd}\}.\end{aligned} \tag{2.4}$$

Noting that $v(x, y)=e^{(-1)^{1/2}\mu y}u(x)$ for $|y|\leqq 1$, we obtain

$$||v||_{2m,\Gamma}^2\geqq||ue^{(-1)^{1/2}\mu y}||_{2m,\Gamma}^2\geqq\sum_{k=0}^{2m}||u||_k^2|\mu|^{2m-k}. \tag{2.5}$$

Thus if $|\mu|$ is sufficiently large we find

$$\sum_{k=0}^{2m}|\mu|^{2m-k}\|u\|_k \leqq C_5\{\|A(x, D_x, \mu^d)u\|_0 + \sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\mu|^{2m-m_j-k}\|w_j(\mu^d)\|_k + |\mu|^{-1}\sum_{k=0}^{2m-1}|\mu|^{2m-k}\|u\|_k\}. \quad (2.6)$$

Replacing $A(x, D_x, D_y^d)$ and $\{B_j(x, D_x, D_y^d)\}_{j=1}^m$ by $A(x, D_x, -D_y^d)$ and $\{B_j(x, D_x, -D_y^d)\}_{j=1}^m$ respectively, we obtain an estimate similar to (2.6). Putting $\lambda=\mu^d$ or $\lambda=-\mu^d$, we can show that if λ is a real number with a sufficiently large absolute value

$$\sum_{k=0}^{2m}|\lambda|^{(2m-k)/d}\|u\|_k \leqq C_6\{\|A(x, D_x, \lambda)u\|_0 + \sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda|^{(2m-m_j-k)/d}\|w_j(\lambda)\|_k\}.$$

The proof of the lemma will be easily completed noting that for $0<k<2m-m_j$

$$|\lambda|^{(2m-m_j-k)/d}\|w_j\|_k \leqq c_0\|w_j\|_{2m-m_j}^{k/(2m-m_j)}\|w_j\|_0^{(2m-m_j-k)/(2m-m_j)}|\lambda|^{(2m-m_j-k)/d}$$
$$\leqq c_0(\|w_j\|_{2m-m_j} + |\lambda|^{(2m-m_j)/d}\|w_j\|_0).$$

If $u(x, t)$ is a function defined for $x\in\Omega$, $-\infty<t<\infty$, we denote by $\hat{u}(x, \lambda)$ the Fourier transform of u with respect to t:

$$\hat{u}(x, \lambda) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-(-1)^{1/2}\lambda t}u(x, t)\,dt. \quad (2.7)$$

Lemma 2.2. *Let u be a function of t with values in $H_{2m}(\Omega)$ satisfying*

$$A(x, D_x, D_t)u(x, t) = f(x, t), \quad x\in\Omega, \quad -\infty<t<\infty, \quad (2.8)$$
$$B_j(x, D_x, D_t)u(x, t) = g_j(x, t), \quad x\in\partial\Omega, \quad -\infty<t<\infty, \quad j=1, \cdots, m, \quad (2.9)$$

where f and g_j, $j=1, \cdots, m$, are functions of t with values in $L^2(\Omega)$ and $H_{2m-m_j}(\Omega)$ respectively. Then we have

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{(2m-k)/d}\|\hat{u}(\lambda)\|_k)^2 d\lambda \leqq C_7\Big\{\int_{-\infty}^{\infty}\|f(t)\|_0^2 dt + \sum_{j=1}^{m}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\hat{g}_j(\lambda)\|_0)^2 d\lambda + \sum_{j=1}^{m}\int_{-\infty}^{\infty}\|g_j(t)\|_{2m-m_j}^2 dt + \int_{-\infty}^{\infty}\|u(t)\|_0^2 dt\Big\} \quad (2.10)$$

if all the terms of the above inequality are finite.

Note that (2.9) is assumed to hold only on the boundary although the functions on both sides are defined also in the interior.

Proof. $\hat{u}(x, \lambda)$ satisfies

$$A(x, D_x, \lambda)\hat{u}(x, \lambda) = \hat{f}(x, \lambda), \quad x\in\Omega, \quad -\infty<\lambda<\infty,$$
$$B_j(x, D_x, \lambda)\hat{u}(x, \lambda) = \hat{g}_j(x, \lambda), \quad x\in\partial\Omega, \quad -\infty<\lambda<\infty.$$

If we apply Lemma 2.1 to $\hat{u}(x, \lambda)$ taking $\hat{g}_j(x, \lambda)$ as $w_j(x, \lambda)$, we get

$$\sum_{k=0}^{2m}(|\lambda|^{(2m-k)/d}\|\hat{u}(\lambda)\|_k)^2\leq C_8\{\|\hat{f}(\lambda)\|_0^2+\sum_{j=1}^{m}(|\lambda|^{(2m-m_j)/d}\|\hat{g}_j(\lambda)\|_0)^2 + \sum_{j=1}^{m}\|\hat{g}_j(\lambda)\|_{2m-m_j}^2+\|\hat{u}(\lambda)\|_0^2\}.$$

Integrating the above relation over $-\infty<\lambda<\infty$ and applying Plancherel theorem we complete the proof.

3. Estimates in general case

In this section we obtain some estimates in the general case of time-dependent coefficients.

Lemma 3.1. *Let v be a solution of*

$$A(x, t, D_x, D_t)v(x, t) = f(x, t), \quad x\in\Omega, \quad -\infty<t<\infty, \tag{3.1}$$
$$B_j(x, t, D_x, D_t)v(x, t) = g_j(x, t), \quad x\in\partial\Omega, \quad -\infty<t<\infty, \quad j=1, \cdots, m. \tag{3.2}$$

If the support of v as a function of t is sufficiently small, then the same estimate as (2.10) *holds for v replacing C_7 by another constant if necessary.*

Proof. Let s be an arbitrary real number and suppose

$$v(x, t) = 0 \quad \text{if} \quad |t-s|>\delta \tag{3.3}$$

where δ is some small positive number. Let $\varphi(t)$ be a smooth real valued function such that $\varphi(t)=1$ for $|t|\leq 1$ and $\varphi(t)=0$ for $|t|>2$. If we write $\psi(t)=\varphi((t-s)/\delta)$, then $\psi(t)=1$ on the support of v. Clearly v satisfies

$$A(x, s, D_x, D_t)v(x, t) = F(x, t), \quad x\in\Omega, \quad -\infty<t<\infty, \tag{3.4}$$
$$B_j(x, s, D_x, D_t)v(x, t) = G_j(x, t), \quad x\in\partial\Omega, \quad -\infty<t<\infty, \quad j=1, \cdots, m, \tag{3.5}$$

where

$$F = f+\sum_{k=0}^{l}\sum_{|\alpha|\leq 2m-kd}\psi(t)(a_{l-k,\alpha}(x, s)-a_{l-k,\alpha}(x, t))D_x^\alpha D_t^k v, \tag{3.6}$$

$$G_j = g_j+\sum_{k=0}^{l_j}\sum_{|\beta|\leq m_j-kd}\gamma_{j,l_j-k,\beta}(x, s, t)D_x^\beta D_t^k v, \tag{3.7}$$

with the notation

$$\gamma_{j,l_j-k,\beta}(x, s, t) = \psi(t)(b_{j,l_j-k,\beta}(x, s)-b_{j,l_j-k,\beta}(x, t)) . \tag{3.8}$$

As is easily seen

$$|\gamma_{j,l_j-k,\beta}(x, s, t)| \leqq C_{10}\delta , \tag{3.9}$$

$$|D_t^{l+1}\gamma_{j,l_j-k,\beta}(x, s, t)| \leqq C_{11}\delta^{-l} , \tag{3.10}$$

and hence

$$|\hat{\gamma}_{j,l_j-k,\beta}(x, s, \lambda)| \leqq C_{12}\delta^2 , \tag{3.11}$$

$$|\lambda^{l+1}\hat{\gamma}_{j,l_j-k,\beta}(x, s, \lambda)| \leqq C_{13}\delta^{1-l} . \tag{3.12}$$

It follows from the above two inequalities that

$$\int_{-\infty}^{\infty} |\hat{\gamma}_{j\, l_j-k,\beta}(x, s, \lambda)|\, d\lambda \leqq C_{14}\delta , \tag{3.13}$$

$$\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/d} |\hat{\gamma}_{,l_jj-k,\beta}(x, s, \lambda)|\, d\lambda \leqq C_{15}\delta^{1-(2m-m_j)/d} \quad \text{if } m_j>0 . \tag{3.14}$$

The Fourier transform of $G_j(x, t)$ is

$$\hat{G}_j(x, \lambda) = \hat{g}_j(x, \lambda) + \sum_{k=0}^{l_j} \sum_{|\beta|\leqq m_j-kd} \hat{\gamma}_{j,l_j-k,\beta}(x, s, \cdot)*(D_x^\beta D_t^k v)^\wedge(x, \cdot) . \tag{3.15}$$

If we notice the following lemma:

Lemma 3.2. *If f and g are complex valued functions on the real line $-\infty<\lambda<\infty$, we have for $\gamma>0$*

$$\begin{aligned}\left(\int_{-\infty}^{\infty} (|\lambda|^\gamma |(f*g)(\lambda)|)^2 d\lambda\right)^{1/2} &\leqq 2^\gamma \int_{-\infty}^{\infty} |f(\lambda)|\, d\lambda \left(\int_{-\infty}^{\infty} (|\lambda|^\gamma |g(\lambda)|)^2 d\lambda\right)^{1/2} \\ &\quad + 2^\gamma \int_{-\infty}^{\infty} |\lambda|^\gamma |f(\lambda)|\, d\lambda \left(\int_{-\infty}^{\infty} |g(\lambda)|^2 d\lambda\right)^{1/2}\end{aligned} \tag{3.16}$$

provided that all the terms of the inequality are finite;

we get from (3.11), (3.9), (3.10)

$$\begin{aligned}\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} |\hat{G}_j(x, \lambda)|)^2 d\lambda \leqq C_{16} \Big\{ &\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} |\hat{g}_j(x, \lambda)|)^2 d\lambda \\ &+ \delta^2 \sum_{k=0}^{l_j} \sum_{|\beta|\leqq m_j-kd} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} |\lambda^k D_x^\beta \hat{v}(x, \lambda)|)^2 d\lambda \\ &+ \sum_{k=0}^{l_j} \sum_{|\beta|\leqq m_j-kd} \delta^{2(1-(2m-m_j)/d)} \int_{-\infty}^{\infty} |\lambda^k D_x^\beta \hat{v}(x, \lambda)|^2 d\lambda .\end{aligned}$$

Integrating over $x\in\Omega$ we get

$$\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{G}_j(\lambda)\|_0)^2 d\lambda \leqq C_{17} \Big\{ \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/d} \|\hat{g}_j(\lambda)\|_0)^2 d\lambda$$

$$+\delta^2\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d+k}||\hat{v}(\lambda)||_{m_j-kd})^2d\lambda \tag{3.17}$$
$$+\sum_{k=0}^{l_j}\delta^{2(1-(2m-m_j)/d)}\int_{-\infty}^{\infty}(|\lambda|^k||\hat{v}(\lambda)||_{m_j-kd})^2d\lambda\Big\}.$$

Since

$$|\lambda|^k=(\delta^{(2m-m_j)/d}|\lambda|^{(2m-m_j+kd)/d})^{kd/(2m-m_j+kd)}\delta^{-(2m-m_j)k/(2m-m_j+kd)}$$
$$\leqq\delta^{(2m-m_j)/d}|\lambda|^{(2m-m_j+kd)/d}+\delta^{-k},$$

we have

$$\delta^{2(1-(2m-m_j)/d)}\int_{-\infty}^{\infty}(|\lambda|^k||\hat{v}(\lambda)||_{m_j-kd})^2d\lambda$$
$$\leqq 2\delta^2\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d+k}||\hat{v}(\lambda)||_{m_j-kd})^2d\lambda \tag{3.18}$$
$$+2\delta^{2(1-(2m-m_j)/d-k)}\int_{-\infty}^{\infty}||\hat{v}(\lambda)||_{m_j-kd}^2\,d\lambda\,.$$

Noting

$$||\hat{v}(\lambda)||_{m_j-kd}\leqq c_0(\delta^{(2m-m_j)/d+k}||\hat{v}(\lambda)||_{2m})^{(m_j-kd)/2m}(\delta^{-(m_j-kd)/d}||\hat{v}(\lambda)||_0)^{(2m-m_j+kd)/2m}$$
$$\leqq c_0(\delta^{(2m-m_j)/d+k}||\hat{v}(\lambda)||_{2m}+\delta^{-(m_j-kd)/d}||\hat{v}(\lambda)||_0),$$

we have also

$$\delta^{2(1-(2m-m_j)/d-k)}\int_{-\infty}^{\infty}||\hat{v}(\lambda)||_{m_j-kd}^2d\lambda$$
$$\leqq 2c_0\delta^2\int_{-\infty}^{\infty}||\hat{v}(\lambda)||_{2m}^2d\lambda+2c_0^2\delta^{2(1-l)}\int_{-\infty}^{\infty}||\hat{v}(\lambda)||_0^2\,d\lambda\,. \tag{3.19}$$

It should be noted here that (3.19) is true for $k=0$ and also for $k=l_j$ if $l_j=m_j/d$. Summing up we get

$$\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}||\hat{G}_j(\lambda)||_0)^2d\lambda\leqq C_{18}\Big\{\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}||\hat{g}_j(\lambda)||_0)^2d\lambda$$
$$+\delta^2\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j+kd)/d}||\hat{v}(\lambda)||_{m_j-kd})^2d\lambda \tag{3.20}$$
$$+\delta^2\int_{-\infty}^{\infty}||\hat{v}(\lambda)||_{2m}^2d\lambda+\delta^{2(1-l)}\int_{-\infty}^{\infty}||\hat{v}(\lambda)||_0^2\,d\lambda\Big\}.$$

The following two inequalities are easily proved

$$\int_{-\infty}^{\infty}||G_j(t)||_{2m-m_j}^2dt\leqq C_{19}\Big\{\int_{-\infty}^{\infty}||g_j(t)||_{2m-m_j}^2dt$$
$$+\varepsilon(\delta)\sum_{k=0}^{l_j}\int_{-\infty}^{\infty}(|\lambda|^k||\hat{v}(\lambda)||_{2m-kd})^2d\lambda\Big\}, \tag{3.21}$$

$$\int_{-\infty}^{\infty}||F(t)||_0^2\,dt\leqq C_{20}\Big\{\int_{-\infty}^{\infty}||f(t)||_0^2\,dt+\varepsilon(\delta)\sum_{k=0}^{l}\int_{-\infty}^{\infty}||D_t^kv(t)||_{2m-kd}^2dt\Big\}, \tag{3.22}$$

where $\varepsilon(\delta)$ is a function such that $\varepsilon(\delta)\to 0$ as $\delta\to 0$. Making use of (3.20), (3.21), (3.22) in the application of Lemma 2.2 to v, we get

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{(2m-k)/d}\|\hat{v}(\lambda)\|_k)^2 d\lambda \leqq C_{21}\Big\{\int_{-\infty}^{\infty}\|f(t)\|_0^2 dt$$
$$+\sum_{j=1}^{m}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\hat{g}_j(\lambda)\|_0)^2 d\lambda+\sum_{j=1}^{m}\int_{-\infty}^{\infty}\|g_j(t)\|_{2m-m_j}^2 dt \tag{3.23}$$
$$+(\delta^{2(1-l)}+1)\int_{-\infty}^{\infty}\|v(t)\|_0^2 dt+\varepsilon(\delta)\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{(2m-k)/d}\|\hat{v}(\lambda)\|_k)^2 d\lambda\Big\};$$

the desired inequality for v follows from (3.23) when $0<\delta\leqq\delta_0$, δ_0 being some small positive number independent of s and u.

Let a and b be any real numbers satisfying $0<b-a\leqq 2\delta_0$ where δ_0 is a positive number such that (3.3) holds whenever the length of the support of v is not larger than $2\delta_0$ (cf. the end of the proof of Lemma 3.1). Let δ and δ' be any pair of positive numbers satisfying $\delta'<\delta<\delta_0$ and $\varphi(t)$ be a smooth real valued function such that $\varphi(t)=0$ if $t<a+\delta'$ or $t>b+\delta'$ and $\varphi(t)=1$ if $a+\delta\leqq t\leqq b-\delta$ and

$$|D_t^k\varphi(t)|\leqq K(\delta-\delta')^{-k},\qquad k=1,\cdots,l, \tag{3.24}$$

with some constant K independent of δ and δ'. η is to be a smooth function having a compact support in the real line and satisfying $\eta(t)=1$ in some open set containing the closed interval $[a, b]$.

Lemma 3.3. *Using the above notations for any solution u of*

$$A(x,t,D_x,D_t)u(x,t)=f(x,t),\qquad x\in\Omega,\ a<t<b, \tag{3.25}$$
$$B_j(x,t,D_x,D_t)u(x,t)=g_j(x,t),\qquad x\in\partial\Omega,\ a<t<b,\quad j=1,\cdots,m, \tag{3.26}$$

we have

$$\sum_{k=0}^{l}\int_{a+\delta}^{b-\delta}\|D_t^{l-k}u(t)\|_{ka}^2 dt\leqq C_{22}\Big\{\int_{a+\delta'}^{b-\delta'}\|f(t)\|_0^2 dt$$
$$+\sum_{j=1}^{m}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\widehat{\varphi g_j}(\lambda)\|_0)^2 d\lambda+\sum_{j=1}^{m}\int_{a+\delta'}^{b-\delta'}\|g_j(t)\|_{2m-m_j}^2 dt \tag{3.27}$$
$$+(M_0K)^2\sum_{\substack{i+k\leqq l\\ i\geqq 1}}\frac{1}{(\delta-\delta')^{2i}}\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k-i}u(t)\|_{ka}^2 dt+\int_{a+\delta'}^{b-\delta'}\|u(t)\|_0^2 dt\Big\},$$

where M_0 is a constant such that

$$|a_{l-k,\alpha}(x,t)|\leqq M_0,\quad x\in\Omega,\ -\infty<t<\infty,\ k=0,\cdots,l,\ |\alpha|\leqq 2m,$$
$$|D_x^\kappa b_{j,l_j-k,\beta}(x,t)|\leqq M_0,\ \int_{-\infty}^{\infty}|\widehat{\eta b}_{j,l_j-k,\beta}(x,\lambda)|d\lambda\leqq M_0,\quad x\in\Omega,\ -\infty<t<\infty,$$
$$k=0,\cdots,l_j,\ |\beta|\leqq m_j-kd,\ |\kappa|\leqq 2m-m_j,\ j=1,\cdots,m.$$

Proof. If we write

$$v(x, t) = \varphi(t)u(x, t),$$

then the length of the support of v is not larger than $2\delta_0$ and v satisfies

$$\begin{aligned} A(x, t, D_x, D_t)v(x, t) &= F(x, t), & x\in\Omega,\ -\infty<t<\infty, \\ B_j(x, t, D_x, D_t)v(x, t) &= G_j(x, t), & x\in\partial\Omega,\ -\infty<t<\infty, \\ & & j=1, \cdots, m, \end{aligned}$$

where

$$F = \varphi f+\sum_{k=1}^{l} A_{l-k}(x, t, D_x)\sum_{p=0}^{k-1}\binom{k}{p}D_t^{k-p}\varphi\cdot D_t^p u,$$

$$\begin{aligned} G_j &= \varphi g_j+\sum_{k=1}^{l_j} B_{j,l_j-k}(x, t, D_x)\sum_{p=0}^{k-1}\binom{k}{p}D_t^{k-p}\varphi\cdot D_t^p u \\ &= \varphi g_j+\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\binom{k}{p}\sum_{|\beta|\leq m_j-kd}\eta b_{j,l_j-k,\beta}D_x^\beta D_t^{k-p}\varphi\cdot D_t^p u. \end{aligned}$$

As in the proof of Lemma 3.1 we get

$$\begin{aligned} \int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\hat{G}_j(\lambda)\|_0)^2 d\lambda \leq C_{23}\Big\{&\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\widehat{\varphi g}_j(\lambda)\|_0)^2 d\lambda \\ &+M_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)\|_{m_j-kd})^2 d\lambda \\ &+M_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}\|(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)\|_{m_j-kd}^2 d\lambda\Big\}. \end{aligned}$$

If $l_j=m_j/d$, $(2m-m_j)/d$ is an integer and we can immediately apply Parseval theorem to all the summands in the second sum on the right. If m_j/d is not an integer, we shall replace these summands by the sums of other integrals in which only integers occur as the exponents of $|\lambda|$. Hence suppose m_j/d is not an integer. We notice

$$\frac{2m-m_j}{d} = (l-l_j-1)\frac{m_j-kd}{(l_j+1-k)d}+(l-k)\frac{(l_j+1)d-m_j}{(l_j+1-k)d}, \quad k=1, \cdots, l_j. \tag{3.29}$$

By (3.29) we get for any function $w\in H_{m_j-kd}(\Omega)$

$$\begin{aligned} &|\lambda|^{(2m-m_j)/d}\|w\|_{m_j-kd} \\ &\leq c_0(|\lambda|^{l-l_j-1}\|w\|_{(l_j+1-k)d})^{(m_j-kd)/(l_j+1-k)d}(|\lambda|^{l-k}\|w\|_0)^{((l_j+1)d-m_j)/(l_j+1-k)d} \\ &\leq c_0(|\lambda|^{l-l_j-1}\|w\|_{(l_j+1-k)d}+|\lambda|^{l-k}\|w\|_0). \end{aligned} \tag{3.30}$$

By (3.30) with $(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)$ in place of w we get

$$\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)\|_{m_j-kd})^2 d\lambda$$

$$\leqq c_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}(|\lambda|^{l-l_j-1}\|(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)\|_{(l_j+1-k)d})^2 d\lambda$$

$$+c_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}(|\lambda|^{l-k}\|(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)\|_0)^2 d\lambda$$

$$\leqq c_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}\|D_t^{l-l_j-1}(D_t^{k-p}\varphi(t)\cdot D_t^p u(t))\|^2_{(l_j+1-k)d}dt$$

$$+c_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\int_{-\infty}^{\infty}\|D_t^{l-k}(D_t^{k-p}\varphi(t)\cdot D_t^p u(t))\|^2_0 dt$$

$$\leqq c_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\sum_{i=0}^{l-l_j-1}\binom{l-l_j-1}{i}^2\left\{\frac{K}{(\delta-\delta')^{l-l_j-1-i+k-p}}\right\}^2\int_{a+\delta'}^{b-\delta'}\|D_t^{p+i}u(t)\|_{(l_j+1-k)d}dt$$

$$+c_0^2\sum_{k=1}^{l_j}\sum_{p=0}^{k-1}\sum_{i=0}^{l-k}\binom{l-k}{i}^2\left\{\frac{K}{(\delta-\delta')^{l-k-i+k+p}}\right\}^2\int_{a+\delta'}^{b-\delta'}\|D_t^{p+i}u(t)\|_0^2 dt$$

$$\leqq (c_0C_{24}K)^2\sum_{\substack{i+k\leqq l\\ 1\leqq i}}\frac{1}{(\delta-\delta')^{2i}}\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k-i}u(t)\|^2_{kd}dt\,.$$

Clearly

$$\int_{-\infty}^{\infty}\|(D_t^{k-p}\varphi\cdot D_t^p u)^\wedge(\lambda)\|^2_{m_j-kd}d\lambda\leqq\int_{-\infty}^{\infty}\|D_t^{k-p}\varphi(t)\cdot D_t^p u(t)\|^2_{(l-k)d}dt$$

$$\leqq\frac{K^2}{(\delta-\delta')^{2(k-p)}}\int_{a+\delta'}^{b-\delta'}\|D_t^p u(t)\|^2_{(l-k)d}dt\,.$$

Summing up we obtain

$$\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\hat{G}_j(\lambda)\|_0)^2 d\lambda\leqq C_{25}\left\{\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\widehat{\varphi g}_j(\lambda)\|_0)^2 d\lambda\right.$$

$$\left.+(M_0c_0K)^2\sum_{\substack{i+k\leqq l\\ 1\leqq i}}\frac{1}{(\delta-\delta')^{2i}}\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k-i}u(t)\|^2_{kd}dt\,.\right.$$

As is easily seen

$$\int_{-\infty}^{\infty}\|F(t)\|_0^2 dt\leqq C_{26}\left\{\int_{a+\delta'}^{b-\delta'}\|f(t)\|_0^2 dt\right.$$

$$\left.+(M_0K)^2\sum_{\substack{i+k\leqq l\\ 1\leqq i}}\frac{1}{(\delta-\delta')^{2i}}\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k-i}u(t)\|^2_{kd}dt\right\},$$

$$\int_{-\infty}^{\infty}\|G_j(t)\|^2_{2m-m_j}dt\leqq C_{27}\left\{\int_{a+\delta'}^{b-\delta'}\|g_j(t)\|^2_{2m-m_j}dt\right.$$

$$\left.+(M_0K)^2\sum_{\substack{i+k\leqq l\\ 1\leqq i}}\frac{1}{(\delta-\delta')^{2i}}\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k-i}u(t)\|^2_{kd}dt\right\}.$$

Hence with the aid of Lemma 3.1 we get (3.27).

4. Differentiability in t

From now now on we shall write $f^{(p)}(x, t)=D_t^p f(x, t)$ for any function f of (x, t). By q we denote a natural number. In addition to the assumptions of section 1 here we make the following

ASSUMPTIONS. (VI$_q$) $a_{l-k,\alpha}(x, t)$, $|\alpha|\leqq 2m-kd$, $k=0, \cdots, l-1$, are q times continuously differentiable in t in $\bar{\Omega}\times(-\infty, \infty)$;

(VII$_q$) $D_x^\kappa b_{j,l_j-k,\beta}(x, t)$ and $b_{j,l_j-k,\beta}(x, t)$, $|\beta|\leqq m_j-kd$, $k=0, \cdots, l_j$, $|\kappa|\leqq 2m-m_j$, $j=1, \cdots, m$, are q times and $q+l+1$ times continuously differentiable in t respectively in $\bar{\Omega}\times(-\infty, \infty)$.

Let M_p, $p=0, \cdots, q$, be positive constants such that for $p=0, \cdots, q$

$$\sup_{\substack{x\in\Omega \\ -\infty<t<\infty}} |a_{l-k,\alpha}^{(p)}(x, t)|\leqq M_p, \qquad |\alpha|\leqq 2m-kd, \quad k=0, \cdots, l-1,$$

$$\sup_{\substack{x\in\Omega \\ -\infty<t<\infty}} |D_x^\kappa b_{j,l_j-k,\beta}^{(p)}(x, t)|\leqq M_p,$$

$$\sup_{x\in\Omega}\int_{-\infty}^{\infty} |(\eta b_{j,l_j-k,\beta}^{(p)})^\wedge(x, \lambda)|\,d\lambda\leqq M_p,$$

$$\sup_{x\in\Omega}\int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/d} |(\eta b_{j,l_j-k,\beta}^{(p)})^\wedge(x, \lambda)|\,d\lambda\leqq M_p \quad \text{if} \quad m_j>0,$$

$$|\beta|\leqq m_j-kd, \quad k=0, \cdots, l_j, \quad |\kappa|\leqq 2m-m_j, \quad j=1, \cdots, m,$$

where η is an arbitrary smooth real valued function with a compact support of some fixed length such that

$\eta(t)=1$ in some interval of length $2\delta_0+1$, $|D_t^k\eta(t)|\leqq K_1$ for $k=0, \cdots, l+1$, $-\infty<t<\infty$, with some fixed constant K_1.

Lemma 4.1. *Suppose for some q the assumption* (VI$_q$) *and* (VII$_q$) *are satisfied in addition to the assumptions in section* 1. *Suppose $f(x, t)$ is q times continuously differentiable in t when it is considered as a function with values in $L^2(\Omega)$. If u is a solution of*

$$A(x, t, D_x, D_t)u(x, t) = f(x, t), \quad x\in\Omega, \ -\infty<t<\infty, \tag{4.1}$$

$$B_j(x, t, D_x, D_t)u(x, t) = 0, \quad x\in\partial\Omega, \ -\infty<t<\infty, \quad j=1, \cdots, m, \tag{4.2}$$

such that $\|D_t^{i+p}u(t)\|_{(l-i)q}$, $i=0, \cdots, l$, $p=0, \cdots, q$, are locally square integrable, then for any a, b such that $0<b-a\leqq 2\delta_0$ and δ, δ' such that $0<\delta'<\delta<\delta_0$, we have

$$\sum_{k=0}^{l}\left(\int_{a+\delta}^{b-\delta}\|D_t^{l-k}u^{(q)}(t)\|_{kd}^2dt\right)^{1/2}\leqq C_{28}\left\{\left(\int_{a+\delta'}^{b-\delta'}\|f^{(q)}(t)\|_0^2dt\right)^{1/2}\right.$$
$$+K\sum_{r=0}^{q-1}\binom{q}{r}M_{q-r}\sum_{k=0}^{l}\left(\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k}u^{(r)}(t)\|_{kd}^2dt\right)^{1/2}$$
$$+K\sum_{r=0}^{q}\binom{q}{r}M_{q-r}\sum_{k=0}^{l-1}\sum_{i=1}^{l-k}\frac{1}{(\delta-\delta')^i}\left(\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k-i}u^{(r)}(t)\|_{kd}^2dt\right)^{1/2} \tag{4.3}$$
$$\left.+\left(\int_{a+\delta'}^{b-\delta'}\|u^{(q)}(t)\|_0^2dt\right)^{1/2}\right\}.$$

Proof. $u^{(q)}(x, t)$ is a solution of

$$A(x, t, D_x, D_t)u^{(q)}(x, t) = F(x, t), \qquad x\in\Omega,\ -\infty<t<\infty,$$
$$B_j(x, t, D_x, D_t)u^{(q)}(x, t) = G_j(x, t), \qquad x\in\partial\Omega,\ -\infty<t<\infty,$$

where letting $A_{l-k}^{(q-r)}$ and $B_{j,l_j-k}^{(q-r)}$ stand for the operators obtained by differentiating in t $(q-r)$ times the corresponding coefficients of A_{l-k} and B_{j,l_j-k} respectively

$$F = f^{(q)}-\sum_{r=0}^{q-1}\sum_{k=0}^{l}\binom{q}{r}A_{l-k}^{(q-r)}(x, t, D_x)D_t^ku^{(r)},$$

$$G_j = -\sum_{r=0}^{q-1}\sum_{k=0}^{l_j}\binom{q}{r}B_{j,l_j-k}^{(q-r)}(x, t, D_x)D_t^ku^{(r)}, \qquad j=1, \cdots, m.$$

By the assumptions $F(t)\in L^2(\Omega)$ and $G_j(t)\in H_{2m-m_j}(\Omega)$, $j=1, \cdots, m$, for each t, hence we can apply Lemma 3.3 to $u^{(q)}$. The estimation of $\int_{a+\delta'}^{b-\delta'}\|F(t)\|_0^2dt$ and $\int_{a+\delta'}^{b-\delta'}\|G_j(t)\|_{2m-m_j}^2dt$, $j=1, \cdots, m$, is straightforward. As in the proof of Lemma 3.3 we get

$$\left(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/d}\|\widehat{\varphi G_j}(\lambda)\|_0)^2d\lambda\right)^{1/2}$$
$$\leqq C_{29}K\sum_{r=0}^{q-1}\binom{q}{r}M_{q-r}\left\{\sum_{k=0}^{l}\left(\int_{a+\delta'}^{b-\delta'}\|D_t^{l-k}u^{(r)}(t)\|_{kd}^2dt\right)^{1/2}\right.$$
$$\left.+\sum_{k=0}^{l-1}\sum_{i=1}^{l-k}\frac{1}{(\delta-\delta')^i}\left(\int_{a+\delta'}^{b-\delta'}\|D_t^{l-1-i}u^{(r)}(t)\|_{kd}^2dt\right)^{1/2}\right\}.$$

Thus the proof of the present lemma is completed.

With the aid of Lemma 4.1 we can proceed by difference quotient argument to prove

Theorem 4.1. *Suppose the assumptions* (I)～(V), (VI_q) *and* (VII_q) *are satisfied for some $q>0$ and f is a q times continuously differentiable function of t with values in $L^2(\Omega)$. Let u be a solution of* (4.1)–(4.2). *If $\sum_{k=0}^{l}\|D_t^{l-k}u(t)\|_{kd}$ is locally square integrable in t, then so is $\sum_{k=0}^{l}\|D_t^{l-k+q}u(t)\|_{kd}$;*

especially u *is a* $l+q-1$ *times continuously differentiable function of* t *with values in* $L^2(\Omega)$.

5. Analytcity in t

In this section we prove the abstract analyticity in t of solutions of (4.1)-(4.2) assuming that the coefficients of A, $\{B_j\}$ as well as f are all analytic functions of t.

Theorem 5.1. *Suppose in addition to the assumptions* (I)$\sim$(V)

$$\begin{gathered} a_{l-k,\alpha}(x, t), \quad |\alpha| \leqq 2m-kd, \ k=0, \cdots, l-1, \\ D_x^\kappa b_{j,l_j-k,\beta}(x, t), \quad |\beta| \leqq m_j-kd, \ k=0, \cdots, l_j, \ |\kappa| \leqq 2m-m_j, \\ j=1, \cdots, m, \end{gathered}$$

are analytic functions of t *in* $\bar{\Omega}\times(-\infty, \infty)$ *with* t*-derivatives of all orders continuous in* $\bar{\Omega}\times(-\infty, \infty)$. *If* f *is analytic in* t *when considered as a function with values in* $L^2(\Omega)$, *then so is the solution* u *of* (4.1)-(4.2).

Proof. By Theorem 4.2 $\sum_{k=0}^{l} \|D_t^{l-k} u^{(q)}(t)\|_{kd}$ is locally square integrable. Write

$$N_\delta(v) = \sum_{k=0}^{l} \left(\int_{a+\delta}^{b-\delta} \|D_t^{l-k} v(t)\|_{kd}^2 dt \right)^{1/2}$$

whenever the right side is finite. Let $0<b-a\leqq \min(2\delta_0, 1)$. By the assumptions it is easy to see that there exist constants M and L_0 such that

$$M_p \leqq M_0 M^p p!, \tag{5.1}$$

$$\left(\int_a^b \|f^{(p)}(t)\|_0^2 dt \right)^{1/2} \leqq L_0 M^p p! \tag{5.2}$$

for any non-negative integer p. Now we apply Lemma 4.1 with $\delta=(q+1)\varepsilon$ and $\delta'=q\varepsilon$, where ε is some small positive number. Noting

$$\begin{aligned} & N_{q\varepsilon}(u^{(r)}) \leqq N_{(r+1)\varepsilon}(u^{(r)}) \qquad \text{if} \quad 0\leqq r\leqq q-1, \\ & N_{q\varepsilon}(u^{(r-i)}) \leqq N_{(r-i+1)\varepsilon}(u^{(r-i)}) \qquad \text{if} \quad l\leqq r\leqq q, \ 1\leqq i\leqq l, \\ & \left(\int_{a+q\varepsilon}^{b-q\varepsilon} \|u^{(q)}(t)\|_0^2 dt \right)^{1/2} \leqq N_{(q-l+1)\varepsilon}(u^{(q-l)}) \qquad \text{if} \quad q>l, \end{aligned} \tag{5.3}$$

we get when $q>l$

$$\begin{aligned} & \varepsilon^{q+l} N_{(q+1)\varepsilon}(u^{(q)}) \leqq C_{28} \{ \varepsilon^{q+l} L_0 M^q q! \\ & + \sum_{r=0}^{q-1} \frac{q!}{r!} M_0 M^{q-r} \varepsilon^{q+l} N_{(r+1)\varepsilon}(u^{(r)}) + \sum_{r=l}^{q} \frac{q!}{r!} M_0 M^{q-r} \sum_{i=1}^{l} \varepsilon^{q+l-i} N_{(r-i+1)\varepsilon}(u^{(r-i)}) \end{aligned}$$

$$+\sum_{r=0}^{l-1}\frac{q!}{r!}M_0M^{q-r}\sum_{i=1}^{l}\varepsilon^{q+l-i}\sum_{k=0}^{l-i}\left(\int_{a+q\varepsilon}^{b-q\varepsilon}\|D_t^{l-k-i}u^{(r)}(t)\|_{k\alpha}^2dt\right)^{1/2} \tag{5.3}$$
$$+\varepsilon^{q+l}N_{(q-l+1)\varepsilon}(u^{(q-l)})\}\ .$$

We prove that there exists a positive constant L such that for any positive integer p

$$\varepsilon^{p+l}N_{(p+1)\varepsilon}(u^{(p)})\leqq L^{l+p+1}\ . \tag{5.4}$$

It is clear that (5.4) is true for $p=0,\cdots,l$ if L is sufficiently large. Suppose (5.4) is known to hold for $r=0,\cdots,q-1$ with some $q>l$. If $q\varepsilon>1$ the left member of (5.3) vanishes. If $q\varepsilon\leqq 1$

$$\begin{aligned}&\sum_{r=0}^{q-1}q!(r!)^{-1}M_0M^{q-r}\varepsilon^{q+l}N_{(r+1)\varepsilon}(u^{(r)})\\&\quad\leqq M_0M^q\sum_{r=0}^{q-1}q!(r!)^{-1}\varepsilon^{q-r}M^{-r}\varepsilon^{r+l}N_{(r+1)\varepsilon}(u^{(r)})\\&\quad\leqq M_0M^qL^{l+1}\sum_{r=0}^{q-1}(LM^{-1})^r\leqq 2M_0ML^{l+q}\end{aligned}$$

if $L\geqq 2M$. Estimating similarly the other terms we get if $L\geqq\max(2M,1)$

$$\begin{aligned}\varepsilon^{q+l}N_{(q+1)\varepsilon}(u^{(q)})&\leqq C_{28}\{L_0M^qq!+2M_0ML^{l+q}+4M_0L^{q+l}\\&+M_0M^q\sum_{r=0}^{l-1}M^{-r}\sum_{i=1}^{l}\sum_{k=0}^{l-i}\left(\int_a^b\|D_t^{l-k-i}u^{(r)}(t)\|_{k\alpha}^2dt\right)^{1/2}+L^{q+1}\}\ .\end{aligned}$$

Therefore if L is sufficiently large it follows that (5.4) holds also for $p=q$. If $0<\delta<\delta_0$ and $(q+1)\varepsilon=\delta$, then from (5.4) it follows

$$\sum_{k=0}^{l}\left(\int_{a+\delta}^{b-\delta}\|D_t^{l-k}u^{(q)}(t)\|_{k\alpha}^2dt\right)^{1/2}\leqq L^{l+q+1}(q+1)^{q+l}\delta^{-q-l}$$

which implies the desired analyticity of $u(t)$.

6. Analyticity in all variables

We conclude this paper by showing that the solutions of (0.1)–(0.2) are analytic in all variables in $\bar{\Omega}\times(-\infty,\infty)$ if the coefficients, the known function and the boundary of Ω are all analytic. In order to avoid an inessential complication we confine ourselves to the case $l=1$, hence the problem (0.1)–(0.2) is reduced to

$$D_tu(x,t)+A(x,t,D_x)u(x,t)=f(x,t)\,,\qquad x\in\Omega\,, \tag{6.1}$$
$$B_j(x,t,D_x)u(x,t)=0\,,\qquad x\in\partial\Omega\,,\quad j=1,\cdots,m\,. \tag{6.2}$$

Here $A(x,t,D_x)=\sum_{|\alpha|\leqq 2m}a_\alpha(x,t)D_x^\alpha$ and $B_j(x,t,D_x)=\sum_{|\beta|\leqq m_j}b_{j,\beta}(x,t)D_x^\beta$ $(j=1,\cdots,m)$ are differential operators in x of order $2m$ and m_j respectively all of which do not contain D_t.

Our assumptions are restated as follows:

ASSUMPTIONS. (I′) for each t $\pm D_y^{2m}+A(x, t, D_x)$ is an elliptic operator in $(x, y)\in\Gamma$ of order $2m$, and the Complementing Condition is satisfied by $(\pm D_y^{2m}+A(x, t, D_x), \{B_j(x, t, D_x)\}_{j=1}^m, \Gamma)$.

(II′) The boundary $\partial\Omega$ of Ω is an analytic manifold.

(III′) All the functions $a_\alpha(x, t)$, $|\alpha|\leqq 2m$, $b_{j,\beta}(x, t)$, $|\beta|\leqq m_j$, $j=1, \cdots, m$, $f(x, t)$ are analytic in $(x, t)\in\bar\Omega\times(-\infty, \infty)$.

Before proving the main result we note that if the coefficients of A, $\{B_j\}$, the function f and the boundary of Ω are all infinitely differentiable, then the solution is infinitely differentiable up to the boundary. This statement can be proved by differentiating (6. 1)-(6. 2) in t successively or by starting from

$$\|v\|_{2m+k,\Gamma}\leqq C^{(k)}\{\|(\pm D_y^{2m}+A(x, D_x))v\|_{k,\Gamma} + \sum_{j=1}^m\langle B_j(x, D_x)v\rangle_{2m-m_j+k,\partial\Gamma}+\|v\|_{0,\Gamma}\} \tag{6. 3}$$

instead of the one with $k=0$ in the proof of Lemma 2. 1. Thus it suffices to verify that the Cauchy data of u on the boundary are analytic, since once this has been proved we can apply Holmgren's theorem to show the analyticity of u near the boundary and the interior analyticity of u is easier to be proved. Furthermore we want to notice here that the analyticity of u as a function of t with values in $H_{2m+k}(\Omega)$ follows for any $k>0$ under the present assumptions for the same reason that implied the infinite differentiability of u above.

By means of an analytic transformation we may suppose that the origin is located on a part of $\partial\Omega$ which is contained in the hyperplane $x_n=0$, and we shall prove that the Cauchy data of u are analytic near the origin $x=0$, $t=0$. In what follows we denote by $C_{29}, \cdots, C_{42}$ constants dependent only on the assumption (I′) as well as certain smoothness properties of the coefficients of A, $\{B_j\}$ and the boundary $\partial\Omega$.

We shall employ the following semi-norms and norms:

$$|v|_i^2 = |v|_{i,\Omega}^2 = \sum_{|\kappa|=i}\int_\Omega |D_x^\kappa v(x)|^2 dx\,,$$

$$|v|_{i,r}^2 = \sum_{|\kappa|=i}\int_{\substack{|x|<r\\ x_n>0}} |D_x^\kappa v(x)|^2 dx\,,$$

$$\|v\|_{k,r}^2 = \sum_{i=0}^k |v|_{i,r}^2\,.$$

We may take positive numbers c_0, c_1 in such a manner that we have

$$|v|_i \leqq c_0 |v|_j^{i/j} |v|_0^{(j-i)/j} + c_1 |v|_0 , \tag{6.4}$$
$$|v|_{i,r} \leqq c_0 |v|_{j,r}^{i/j} |v|_{0,r}^{(j-i)/j} + c_1 r^{-i} |v|_{0,r} , \tag{6.5}$$

if $0<i<j\leqq 2m$ and $0<r$. Furthermore following M. Schechter [11] we use the boundary semi-norms for a function defined on $\partial\Omega$:

$$[\varphi]_{k,\partial\Omega} = \inf |v|_{k,\Omega} ,$$

where the infinum is taken over all functions v coinciding ϕ on $\partial\Omega$.

First we consider the case in which all the operators A, B_j, $j=1, \cdots, m$, have only highest order derivatives with constant coefficients:

$$A(x, t, D_x) = A(D_x) = \sum_{|\alpha|=2m} a_\alpha D_x^\alpha ,$$
$$B_j(x, t, D_x) = B_j(D_x) = \sum_{|\beta|=m_j} b_{j,\beta} D_x^\beta .$$

We begin with some estimation of a function u which satisfies

$$D_t u(x, t) + A(D_x) u(x, t) = f(x, t) , \quad x \in \Omega , \tag{6.6}$$
$$B_j(D_x) u(x, t) = g_j(x, t) , \quad x \in \partial\Omega, \ j=1, \cdots, m . \tag{6.7}$$

By Theorem 3.1 of [11] we have

$$|v|_{2m,\Gamma} \leqq C_{29}(|(\pm D_y^{2m} + A(D_x))v|_{0,\Gamma} + \sum_{j=1}^{m} [B_j(D_x)u]_{2m-m_j,\partial\Gamma} + |v|_{0,\Gamma})$$

for any function v which is defined and smooth in $\bar{\Gamma}$ and vanishes for $|y|>1$. Hence the same argument as in section 2 yields

Lemma 6.1. *If u satisfies* (6.6)–(6.7) *and has a compact support, then we have*

$$\sum_{k=0}^{2m} \int_{-\infty}^{\infty} (|\lambda|^{(2m-k)/2m} |\hat{v}(\lambda)|_k)^2 d\lambda \leqq C_{30} \Big\{ \int_{-\infty}^{\infty} |f(t)|_0^2 dt$$
$$+ \sum_{j=1}^{m} \int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\hat{g}_j(\lambda)|_0)^2 d\lambda + \sum_{j=1}^{m} \int_{-\infty}^{\infty} |g_j(t)|_{2m-m_j}^2 dt$$
$$+ \sum_{j=1}^{m} \int_{-\infty}^{\infty} |g_j(t)|_0^2 dt + \int_{-\infty}^{\infty} |u(t)|_0^2 dt \Big\} .$$

Let $r>0$, $\delta>0$ be such that $r+\delta<\rho$ and let $\zeta(x)$, $\psi(t)$ be smooth functions satisfying $\zeta(x)=1$ for $|x|<r$, $\zeta(x)=0$ for $|x|>r+\delta$, $\psi(t)=1$ for $|t|<r$, $\psi(t)=0$ for $|t|>r+\delta$ and for all x or t

$$|D_x^\kappa \zeta(x)| \leqq K\delta^{-|\kappa|} , \quad 1 \leqq |\kappa| \leqq 2m , \tag{6.8}$$
$$|D_t \psi(t)| \leqq K\delta^{-1} . \tag{6.9}$$

If u is a solution of (6.6)–(6.7), then $v(x, t) = \psi(t)\zeta(x)u(x, t)$ is a function with a compact support satisfying

$$D_t v + A(D_x)v = F(x, t)\,, \qquad x \in \Omega\,,$$
$$B_j(D_x)v = G_j(x, t)\,, \qquad x \in \partial\Omega,\ j = 1, \cdots, m\,,$$

where

$$F = \psi\zeta f + \psi'\zeta u + \psi \sum_{|\alpha|=2m} a_\alpha \sum_{\alpha'<\alpha} D_x^{\alpha-\alpha'}\zeta \cdot D_x^{\alpha'} u\,, \tag{6.10}$$

$$G_j = \psi\zeta g_j + \psi \sum_{|\beta|=m_j} b_{j,\beta} \sum_{\beta'<\beta} D_x^{\beta-\beta'}\zeta \cdot D_x^{\beta'} u\,. \tag{6.11}$$

By the same argument as that of section 2 we can prove

$$\begin{aligned}\Big(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\hat{G}_j(\lambda)|_0)^2 d\lambda\Big)^{1/2} &\leqq \Big(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\widehat{\psi g_j}(\lambda)|_{0,r+\delta})^2 d\lambda\Big)^{1/2} \\ &+ C_{31} M_{0,0} \sum_{k=0}^{m_j-1} \frac{K}{\delta^{m_j-k}} \Big(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\widehat{u\psi}(\lambda)|_{k,r+\delta})^2 d\lambda\Big)^{1/2},\end{aligned} \tag{6.12}$$

where $M_{0,0}$ is a constant such that $|a_\alpha| \leqq M_{0,0}$, $|b_{j,\beta}| \leqq M_{0,0}$ for all α, β and j. Nothing $|\lambda|^{(2m-m_j)/2m} \leqq \varepsilon|\lambda| + \varepsilon^{-(2m-m_j)/m_j}$ and using (6.5) we can easily show that for $0 \leqq k < m_j$

$$\begin{aligned}|\lambda|^{(2m-m_j)/2m} |w|_{k,r+\delta} &\leqq c_0\varepsilon |\lambda|^{(2m-m_j)/2m} |w|_{m_j,r+\delta} \\ &+ (c_0\varepsilon^{-k/(m_j-k)}\varepsilon_1 + c_1(r+\delta)^{-k}\varepsilon_2)|\lambda|\,|w|_{0,r+\delta} \\ &+ (c_0\varepsilon^{-k/(m_j-k)}\varepsilon_1^{(2m-m_j)/m_j} + c_1(r+\delta)^{-k}\varepsilon_2^{-(2m-m_j)/m_j})|w|_{0,r+\delta}\,,\end{aligned} \tag{6.13}$$

$$\begin{aligned}|\lambda|^{(2m-m_j)/2m} |w|_{m_j,r+\delta} &\leqq c_0 |w|_{2m,r+\delta} \\ &+ (c_0 + c_1\varepsilon_3(r+\delta)^{-m_j})|\lambda|\,|w|_{0,r+\delta} + c_1(r+\delta)^{-m_j}\varepsilon^{-(2m-m_j)/m_j}|w|_{0,r+\delta}\,.\end{aligned} \tag{6.14}$$

If we combine (6.13) and (6.14) after a suitable choice of ε_1, ε_2, ε_3, we get

$$\begin{aligned}|\lambda|^{(2m-m_j)/2m} |w|_{k,r+\delta} &\leqq c_0^2\varepsilon |w|_{2m,r+\delta} + c_2\varepsilon|\lambda|\,|w|_{0,r+\delta} \\ &+ \{(c_0+c_1)\varepsilon^{-(2m-m_j+k)/(m_j-k)} + (c_0+c_1)c_1\varepsilon(r+\delta)^{-2m}\}|w|_{0,r+\delta}\end{aligned} \tag{6.15}$$

which $c_2 = c_0^2 + c_0c_1 + c_0 + c_1$. Using (6.15) with $\varepsilon = \delta^{m_j-k}/L$, $L \geqq 1$, we obtain

$$\begin{aligned}&\frac{1}{\delta^{m_j-k}} \Big(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\widehat{\psi u}(\lambda)|_{k,r+\delta})^2 d\lambda\Big)^{1/2} \\ &\leqq \frac{c_0^2}{L} \Big(\int_{-\infty}^{\infty} |\widehat{\psi u}(\lambda)|_{2m,r+\delta}^2 d\lambda\Big)^{1/2} + \frac{c_2}{L} \Big(\int_{-\infty}^{\infty} (|\lambda|\,|\widehat{\psi u}(\lambda)|_{0,r+\delta})^2 d\lambda\Big)^{1/2} \\ &+ \Big((c_0+c_1)L^{(2m-m_j+k)/(m_j-k)}\delta^{-2m} + \frac{(c_0+1)c_1}{L(r+\delta)^{2m}}\Big)\Big(\int_{-\infty}^{\infty} |\widehat{\psi u}(\lambda)|_{0,r+\delta}^2 d\lambda\Big)^{1/2}.\end{aligned} \tag{6.16}$$

Substituting (6.16) in (6.12) and then applying Plancherel's theorem we get

$$\left(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/2m}|\hat{G}_j(\lambda)|_0)^2 d\lambda\right)^{1/2}$$
$$\leqq\left(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/2m}|\psi\hat{g}_k(\lambda)|_{0,r+\delta})^2 d\lambda\right)^{1/2}$$
$$+C_{32}M_{0,0}KL^{-1}\left\{\left(\int_{-r-\delta}^{r+\delta}|u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}+\left(\int_{-r-\delta}^{r+\delta}|D_t u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}$$
$$+C_{32}M_{0,0}K(K(M\delta)^{-1}+L^{2m-1}\delta^{-2m})\left(\int_{-r-\delta}^{r+\delta}|u(t)|_{0,r+\delta}^2 dt\right)^{1/2}.$$

Similarly noting that if $L\geqq 1$

$$\sum_{k=0}^{2m-m_j}\delta^{-(2m-m_j-k)}|w|_{k,r+\delta}\leqq C_{33}(|w|_{2m-m_j,r+\delta}+\delta^{-(2m-m_j)}|w|_{0,r+\delta}), \tag{6.17}$$

$$\sum_{k=0}^{2m-1}\delta^{-(2m-k)}|w|_{k,r+\delta}\leqq C_{34}(L^{-1}|w|_{2m,r+\delta}+L^{2m-1}\delta^{-2m}|w|_{0,r+\delta}), \tag{6.18}$$

we obtain

$$\left(\int_{-\infty}^{\infty}|G_j(t)|_{2m-m_j}^2 dt\right)^{1/2}\leqq C_{35}K\left\{\left(\int_{-r-\delta}^{r+\delta}|g_j(t)|_{2m-m_j,r+\delta}^2 dt\right)^{1/2}\right.$$
$$+\delta^{-2m+m_j}\left(\int_{-r-\delta}^{r+\delta}|g_j(t)|_{0,r+\delta}^2 dt\right)^{1/2}+M_{0,0}L^{-1}\left(\int_{-r-\delta}^{r+\delta}|u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}$$
$$\left.+M_{0,0}L^{2m-1}\delta^{-2m}\left(\int_{-r-\delta}^{r+\delta}|u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}.$$

$\int_{-\infty}^{\infty}|F(t)|_0^2 dt$ and $\int_{-\infty}^{\infty}|G_j(t)|_0^2 dt$ can be estimated in a similar manner. Thus with the aid of Lemma 6.1 we obtain

Lemma 6.2. *If u is a solution of* (6.6)–(6.7), *then for each $r>0$, $\delta>0$ such that $r+\delta<\rho$ we have*

$$\left(\int_{-r}^{r}|D_t u(t)|_{0,r}^2 dt\right)^{1/2}+\left(\int_{-r}^{r}|u(t)|_{2m,r}^2 dt\right)^{1/2}$$
$$\leqq C_{36}\left[\left(\int_{-r-\delta}^{r+\delta}|f(t)|_{0,r+\delta}^2 dt\right)^{1/2}+\sum_{j=1}^{m}\left(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/2m}|\psi\hat{g}_j(\lambda)|_{0,r+\delta})^2 d\lambda\right)^{1/2}\right.$$
$$+K\sum_{j=1}^{m}\left(\int_{-r-\delta}^{r+\delta}|g_j(t)|_{2m-m_j,r+\delta}^2 dt\right)^{1/2}+\sum_{j=1}^{m}K\delta^{-2m+m_j}\left(\int_{-r-\delta}^{r+\delta}|g_j(t)|_{0,r+\delta}^2 dt\right)^{1/2}$$
$$+M_{0,0}KL^{-1}\left\{\left(\int_{-r-\delta}^{r+\delta}|D_t u(t)|_{0,r+\delta}^2 dt\right)^{1/2}+\left(\int_{-r-\delta}^{r+\delta}|u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}\right\}$$
$$\left.+M_{0,0}K(K(L\delta)^{-1}+L^{2m-1}\delta^{-2m})\left(\int_{-r-\delta}^{r+\delta}|u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right],$$

where ψ is the function depending on r, δ as was defined after Lemma 6.1.

From now on we shall distinguish the normal variable x_n from the tangential space variables $x'=(x_1,\cdots,x_{n-1})$ and by ∇^p we denote any derivative of order p with respect to x', For the sake of convenience we express Leibnitz formula as follows:

$$\nabla^p(fg) = \sum_{p'=0}^{p}\binom{p}{p'}\nabla^{p-p'}f\cdot\nabla^{p'}g\,,$$

although some comment would be necessary in doing so. We denote by $A^\#$ and $B_j^\#$ the principal parts of A and B_j respectively:

$$A^\#(x, t, D_x) = \sum_{|\alpha|=2m} a_\alpha(x, t)D_x^\alpha\,,$$
$$B_j^\#(x, t, D_r) = \sum_{|\beta|=m_j} b_{j,\beta}(x, t)D_x^\beta\,, \quad j=1, \cdots, m\,.$$

If u is a solution of (6.1)–(6.2), then for $D_t^q\nabla^p u$ we have

$$D_tD_t^q\nabla^p u + A^\#(0, 0, D_x)D_t^q\nabla^p u = F_{p,q}(x, t)\,, \quad x\in\Omega\,, \tag{6.19}$$

$$B_j^\#(0, 0, D_x)D_t^q\nabla^p u = G_{j,p,q}(x, t)\,, \quad x\in\partial\Omega,\ j=1, \cdots, m\,. \tag{6.20}$$

Here

$$\begin{aligned}F_{p,q} = {}& D_t^q\nabla^p f + \sum_{|\alpha|=2m}(a_\alpha(0, 0)-a_\alpha(x, t))D_x^\alpha D_t^q\nabla^p u\\ &-\sum_{|\alpha|<2m} a_\alpha D_x^\alpha D_t^q\nabla^p u - \sum{}'\binom{q}{q'}\binom{p}{p'}\sum_{|\alpha|\leqq 2m} D_t^{q-q'}\nabla^{p-p'}a_\alpha D_x^\alpha D_t^{q'}\nabla^{p'}u\,,\end{aligned} \tag{6.21}$$

$$\begin{aligned}G_{j,p,q} = {}& \sum_{|\beta|=m_j}(b_{j,\beta}(0, 0)-b_{j,\beta}(x, t))D_x^\beta D_t^q\nabla^p u\\ &-\sum_{|\beta|<m_j} b_{j,\beta}D_x^\beta D_t^q\nabla^p u - \sum{}'\binom{q}{q'}\binom{p}{p'}\sum_{|\beta|\leqq m_j} D_t^{q-q'}\nabla^{p-p'}b_{j,\beta}D_p^\beta D_t^{q'}\nabla^{p'}u\,,\end{aligned} \tag{6.22}$$

where $\sum'$ means that the summation extends over all (p', q') satisfying $0\leqq p'\leqq p$, $0\leqq q'\leqq q$ expect for $(p, q)=(p', q')$. If $\eta(t)$ is a function such that $\eta(t)=1$ for $|t|\leqq r+\delta$, $\eta(t)=0$ for $|t|>2(r+\delta)$, and $|D_t^k\eta(t)|\leqq K(r+\delta)^{-k}$ for $k=1, 2$, then

$$\begin{aligned}\psi G_{j,p,q} = {}& \sum_{|\beta|=m_j}\gamma_{j,\beta}D_x^\beta\nabla^p(\psi D_t^q u) - \sum_{|\beta|<m_j}\eta b_{j,\beta}D_x^\beta\nabla^p(\psi D_t^q u)\\ &-\sum{}'\binom{q}{q'}\binom{p}{p'}\sum_{|\beta|\leqq m_j}\eta D_t^{q-q'}\nabla^{p-p'}b_{j,\beta}\cdot D_x^\beta\nabla^{p'}(\psi D_t^{q'}u)\,,\end{aligned}$$

where $\gamma_{j,\beta}(x, t)=\eta(t)(b_{j,\beta}(0, 0)-b_{j,\beta}(x, t))$. As in section 3 we can easily show that there exists a constant M_1 such that if $|x|\leqq r+\delta$ and $m_j>0$

$$\int_{-\infty}^{\infty}|\hat{\gamma}_{j,\beta}(x, \lambda)|\,d\lambda\leqq M_1(r+\delta) \tag{6.23}$$

$$\int_{-\infty}^{\infty}|\lambda|^{(2m-m_j)/2m}|\hat{\gamma}_{j,\beta}(x, \lambda)|\,d\lambda\leqq M_1(r+\delta)^{m_j/2m}\,. \tag{6.24}$$

Let $M_{p,q}$ be positive numbers such that for all α, β, κ, j with $|\alpha|\leqq 2m$, $|\beta|\leqq m_j$, $|\kappa|\leqq 2m-m_j$, $j=1, \cdots, m$

$$\left.\begin{aligned}
&\sup_{x,t} |D_t^q \nabla^p a_\alpha(x, t)| \leqq M_{p,q} , \\
&\sup_{x,t} |D_x^\kappa D_t^q \nabla^p b_{j,\beta}(x, t)| \leqq M_{p,q} , \\
&\sup_{x} \int_{-\infty}^{\infty} |(\eta D_t^q \nabla^p b_{j,\beta})^\wedge(x, \lambda)| d\lambda \leqq M_{p,q} , \\
&\sup_{x} \int_{-\infty}^{\infty} |\lambda|^{(2m-m_j)/2m} |(\eta D_t^q \nabla^p b_{j,\beta})^\wedge(x, \lambda)| d\lambda \leqq M_{p,q} ,
\end{aligned}\right\} \tag{6.25}$$

where sup is taken over $\Omega\times(-\infty, \infty)$ or Ω. Then it follows from (6.23), (6.24), (6.25) and Lemma 3.2 that

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |\widehat{\psi G}_{j,p,q}(\lambda)|_{0,r+\delta})^2 d\lambda\right)^{1/2} \\
&\leqq C_{37}\Big[M_1(r+\delta)\left(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} |(\nabla^p \psi D_t^q u)^\wedge(\lambda)|_{m_j,r+\delta})^2 d\lambda\right)^{1/2} \\
&+M_1(r+\delta)^{m_j/2m}\left(\int_{-\infty}^{\infty} |(\nabla^p \psi D_t^q u)^\wedge(\lambda)|_{m_j,r+\delta}^2 d\lambda\right)^{1/2} \\
&+M_{0,0}\left(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} \|(\nabla^p \psi D_t^q u)^\wedge(\lambda)\|_{m_j-1,r+\delta})^2 d\lambda\right)^{1/2} \\
&+M_{0,0}\left(\int_{-\infty}^{\infty} \|(\nabla^p \psi D_t^q u^\wedge(\lambda)\|_{m_j-1,r+\delta}^2 d\lambda\right)^{1/2} \\
&+\sum{}'\binom{q}{q'}\binom{p}{p'} M_{p-p',q-q'}\Big\{\left(\int_{-\infty}^{\infty} (|\lambda|^{(2m-m_j)/2m} \|(\nabla^{p'} \psi D_t^{q'} u)^\wedge(\lambda)\|_{m_j,r+\delta})^2 d\lambda\right)^{1/2} \\
&+\left(\int_{-\infty}^{\infty} \|(\nabla^{p'} \psi D_t^{q'} u)^\wedge(\lambda)\|_{m_j,r+\delta}^2 d\lambda\right)^{1/2}\Big\}\Big].
\end{aligned} \tag{6.26}$$

The following inequalities follow from (6.5), (6.15), (6.14):

$$\begin{aligned}
(r+\delta)^{m_j/2m} |w|_{m_j,r+\delta} &\leqq c_0(r+\delta) |w|_{2m,r+\delta} \\
&+ (c_0+c_1(r+\delta)^{-(2m-1)m_j/2m}) |w|_{0,r+\delta}
\end{aligned} \tag{6.27}$$

$$\begin{aligned}
&|\lambda|^{(2m-m_j)/2m} \|w\|_{m_j-1,r+\delta} + \|w\|_{m_j-1,r+\delta} \\
&\leqq C_{38}(r+\delta)(|w|_{2m,r+\delta} + |\lambda|\,|w|_{0,r+\delta} + (r+\delta)^{-2m} |w|_{0,r+\delta}) ,
\end{aligned} \tag{6.28}$$

$$\begin{aligned}
&|\lambda|^{(2m-m_j)/2m} \|w\|_{m_j,r+\delta} + \|w\|_{m_j,r+\delta} \\
&\leqq C_{38}(|w|_{2m,r+\delta} + |\lambda|\,|w|_{0,r+\delta} + (r+\delta)^{-2m} |w|_{0,r+\delta}) .
\end{aligned} \tag{6.29}$$

We shall use the following notations:

$$d_{p,q}(u, r) = \max\left\{\left(\int_{-r}^{r} |D_t^{q+1} \nabla^p u(t)|_{0,r}^2 dt\right)^{1/2} + \left(\int_{-r}^{r} |D_t^q \nabla^p u(t)|_{2m,r} dt\right)^{1/2}\right\}$$

$$e_{p,q}(f, r) = \max\left(\int_{-r}^{r} |D_t^q \nabla^p f(t)|_{0,r}^2 dt\right)^{1/2}$$

for $p, q=0, 1, \cdots, 0<r<\rho$, where the maximum is taken over all derivatives ∇^p of order p.

Applying (6.27), (6,28), (6.29) to the right of (6.26) and then making use of Plancherel theorem we get

$$\left(\int_{-\infty}^{\infty}(|\lambda|^{(2m-m_j)/2m}|\widehat{\psi G_{j,p,q}}(\lambda)|_{0,r+\delta})^2 d\lambda\right)^{1/2} \leqq C_{39}(M_1+M_{0,0})(r+\delta)\times$$
$$\times\left\{d_{p,q}(u,r+\delta)+\left(\frac{K}{\delta}+\frac{1}{(r+\delta)^{2m}}\right)\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}$$
$$+C_{39}\sum'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}\left\{d_{p',q'}(u,r+\delta)\right.$$
$$\left.+\left(\frac{K}{\delta}+\frac{1}{(r+\delta)^{2m}}\right)\left(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}.$$

As is easily seen

$$\left(\int_{-r-\delta}^{r+\delta}|G_{j,p,q}(t)|_{2m-m_j,r+\delta}^2 dt\right)^{1/2}$$
$$\leqq C_{40}(M_1+M_{0,0})(r+\delta)\left\{\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}\right.$$
$$\left.+(r+\delta)^{-2m}\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p(t)|_{0\ r+\delta}^2 dt\right)^{1/2}\right\} \qquad (6.30)$$
$$+C_{40}\sum'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}\left\{\left(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}\right.$$
$$\left.+(r+\delta)^{1-2m}\left(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}.$$

It is not difficult to show that $\left(\int_{-r-\delta}^{r+\delta}|F_{p,q}(t)|_{0,r+\delta}^2 dt\right)^{1/2}$ is dominated by the sum of $e_{p,q}(f,r+\delta)$ and the right-hand side of (6.30) with C_{40} possibly replaced by another constant. Similarly

$$\sum_{j=1}^{m}\delta^{-2m+m_j}\left(\int_{-r-\delta}^{r+\delta}|G_{j,p,q}(t)|_{0,r+\delta}^2 dt\right)^{1/2}$$
$$\leqq C_{41}(M_1+M_{0,0})(r+\delta)\left\{\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}\right.$$
$$\left.+\frac{1}{\delta^{2m}}\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}$$
$$+C_{41}\sum'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}\left\{\left(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{2m,r+\delta}^2 dt\right)^{1/2}\right.$$
$$\left.+\frac{1}{\delta^{2m}}\left(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\right\}.$$

Thus with the aid of Lemma 6.2 we obtain

$$\begin{aligned}d_{p,q}(u, r) &\leqq C_{42}e_{p,q}(f, r+\delta)+C_{43}(r+\delta+L^{-1})d_{p,q}(u, r+\delta)\\&+\frac{C_{43}L^{2m-1}}{\delta^{2m}}\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\\&+C_{42}\sum{}'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}d_{p',q'}(u, r+\delta)\\&+\frac{C_{43}}{\delta^{2m}}\sum{}'\binom{q}{q'}\binom{p}{p'}M_{p-p',q-q'}\left(\int_{-r-\delta}^{r+\delta}|D_t^{q'}\nabla^{p'}u(t)|_{0,r+\delta}^2 dt\right)^{1/2},\end{aligned}$$

where C_{43} is a constant dependent only on the assumptions (I′), certain smoothness properties of the coefficients of A, $\{B_j\}$ and the boundary of Ω, and the constants $M_1, M_{0,0}, K$. We introduce the following notations:

$$N_{\rho,p,q}(u) = ((p+q)!)^{-1}\sup_{\rho/2\leqq r<\rho} d_{p,q}(u, r)(\rho-r)^{2m+p+q},$$
$$M_{\rho,p,q}(f) = ((p+q)!)^{-1}\sup_{\rho/2\leqq r<\rho} e_{p,q}(f, r)(\rho-r)^{2m+p+q}$$

for $p, q=0, 1, 2, \cdots$. Under the present assumptions there exist positive constants M_0, M such that

$$M_{p,q}\leqq M_0M^{p+q}p!\,q!\,,\qquad p, q=0, 1, 2, \cdots. \tag{6.31}$$

From the definition it follows that if $p\geqq 2m$

$$\left(\int_{-r-\delta}^{r+\delta}|D_t^q\nabla^p u(t)|_{0,r+\delta}^2 dt\right)^{1/2}\leqq(p+q-2m)!\,N_{\rho,p-2m,q}(u)(\rho-r-\delta)^{-p-q}. \tag{6.34}$$

Due to the previous remark $u(t)$ is analytic in t as a function with values in $H_{4m}(\Omega)$, hence there exist positive constants N_0, N such that

$$\left.\begin{array}{l}|D_t^q\nabla^p u(t)|_{0,\rho}\\|D_t^q\nabla^p u(t)|_{2m,\rho}\end{array}\right\}\leqq\|D_t^q u(t)\|_{4m,\Omega}\leqq N_0N^q q! \tag{6.34}$$

for $0\leqq p\leqq 2m$, $q=0, 1, 2, \cdots$. We may assume $N\geqq 2M$. Multiplying by $(\rho-r)^{2m}/(p+q)!$ both sides of (6.31) with $\rho/2\leqq r<\rho$ and $\delta=(\rho-r)/(p+q+1)$ and using (6.32), (6.33), (6.34) we obtain for $p\geqq 2m$

$$\begin{aligned}N_{\rho,p,q}(u)&\leqq C_{42}e^2M_{\rho,p,q}(f)+C_{43}e^2(\rho+L^{-1})N_{\rho,p,q}(u)\\&+C_{43}e^2L^{2m-1}\frac{(p+q)^{2m}(p+q+2m)!}{(p+q)!}N_{\rho,p-2m,q}(u)\\&+C_{42}e^2\sum{}'\frac{q!}{q'!}\frac{p!}{p'!}\frac{(p'+q')!}{(p+q)!}M_0(\rho M)^{p-p'+q-q'}N_{\rho,p',q'}(u)\\&+C_{43}e^2\sum_{p'\leqq 2m}{}'\frac{q!}{q'!}\frac{p!}{p'!}\frac{(p+q)^{2m}}{(p+q)!}(p'+q'-2m)!\,M_0(\rho M)^{p-p'+q-q'}N_{\rho,p'-2m,q'}(u)\\&+\sqrt{2}\,C_{43}e^2\sum_{p'<2m}\frac{p!\,q!}{p'!}\frac{(p+q)^{2m}}{(p+q)!}M_0M^{p-p'+q-q'}N^{q'}\rho^{p+q+1/2}.\end{aligned} \tag{6.35}$$

If we note

$$\frac{q!}{q'!}\frac{p!}{p'!}\frac{(p'+q')!}{(p+q)!}\leqq 1 \qquad \text{if} \quad 0\leqq p'\leqq p,\ 0\leqq q'\leqq q$$

$$\frac{(p+q)^{2m}(p+q-2m)!}{(p+q)!}\leqq(2m)^{2m} \qquad \text{if} \quad p\geqq 2m,\ q\geqq 0,$$

$$\frac{q!}{q'!}\frac{p!}{p'!}\frac{(p+q)^{2m}}{(p+q)!}(p'+q'-2m)!\leqq(2m)^{2m}(2m+1)^{p-p'} \qquad \text{if} \quad p\geqq p'\geqq 2m,$$

$$\frac{q!p!}{p'!}\frac{(p+q)^{2m}}{(p+q)!}\leqq(p+q)^{2m} \qquad \text{if} \quad p'\leqq 2m,$$

we obtain from (6.35)

$$\begin{aligned}N_{\rho,p,q}(u)&\leqq C_{42}e^2M_{\rho,p,q}(f)+C_{43}e^2(\rho+L^{-1})N_{\rho,p,q}(u)\\&+C_{42}e^2(2m)^{2m}L^{2m-1}N_{\rho,p-2m,q}(u)+C_{42}e^2M_0\sum{}'(\rho M)^{p-p'+q-q'}N_{\rho,p',q'}(u)\\&+C_{43}e^2(2m)^{2m}M_0\sum_{p'\geqq 2m}{}'(2m+1)^{p-p'}(\rho M)^{p-p'+q-q'}N_{\rho,p'-2m,q'}(u)\\&+\sqrt{2}\,C_{43}e^2M_0N_0(p+q)^{2m}\rho^{p+q+1/2}\sum_{p'<2m}M^{p-p'+q-q'}N^{q'}.\end{aligned}$$

Hence if ρ is so small and L is so large that $C_{43}e^2(\rho+L^{-1})\leqq 1/2$, we conclude

$$\begin{aligned}N_{\rho,p,q}(u)&\leqq C_{44}\{M_{\rho,p,q}(f)+L^{2m-1}N_{\rho,p-2m,q}(u)\\&+M_0\sum{}'(\rho M)^{p-p'+q-q'}N_{\rho,p',q'}(u)\\&+M_0\sum_{p'\leqq 2m}{}'(2m+1)^{p-p'}(\rho M)^{p-p'+q-q'}N_{\rho,p'-2m,q'}(u)\\&+M_0N_0(p+q)^{2m}\rho^{p+q+1/2}\sum_{p'<2m}N^{p-p'+q-q'}N_0^{q'}\}\end{aligned} \tag{6.36}$$

with some constant C_{44} of the same property as C_{43}. By assumption we have $M_{\rho,p,q}(f)\leqq R_0R^{p+q}$ for all p, q with some constants R_0, R. We want to show that there exist positive constants H_0, H such that

$$N_{\rho,p,q}(u)\leqq H_0H^{p+q} \tag{6.37}$$

for all $p, q=0, 1, 2, \cdots$. It follows from (6.34) that this is the case for $p\leqq 2m$, $q=0, 1, 2, \cdots$. By induction we can show that the same is the case also for $q=0$, $p=0, 1, 2, \cdots$. Hence we can proceed by induction with respect to $p+q$ to show that (6.37) is true for every p, q if H_0 and H are so large that

$$H\geqq 2(2m+1)\rho M,\quad 5C_{44}R_0\leqq H_0,\quad R\leqq H,$$
$$5C_{44}L^{2m-1}\leqq H^{2m},\quad 30C_{44}M_0\rho M\leqq H,$$
$$30(2m+1)C_{44}M_0\rho M\leqq H^{2m+1}$$
$$\log(20mC_{44}M_0N_0\sqrt{\rho}H_0^{-1})+2m\log s\leqq s\log H(\rho M)^{-1} \qquad \text{for} \quad s=1, 2, \cdots.$$

Due to the inequality

$$\int_{|x'|<r} |v(x', 0)|^2 dx' \leqq Z_2 r^{-1} \int_{\substack{|x|<r \\ x_n>0}} |v|^2 dx + Z_2 r \int_{\substack{|x|<r \\ x_n>0}} |\text{grad } v|^2 dx$$

which may be found in p. 282 of [10] where Z_2 depends only on n, we have

$$\begin{aligned}\Big(\int_{-r}^{r}\int_{|x'|<r} |v(x', 0, t)|^2 dx' dt\Big)^{1/2} &\leqq \Big(Z_2 r^{-1} \int_{-r}^{r}\int_{\substack{|x|<r \\ x_n>0}} |v|^2 dx dt\Big)^{1/2} \\ &+ \Big(Z_2 r \int_{-r}^{r}\int_{\substack{|x|<r \\ x_n>0}} |\text{grad}_x v|^2 dx dt\Big)^{1/2}.\end{aligned} \tag{6.38}$$

Applying (6.37) to $D_{x_n}^i D_t^q \nabla^p u$, $0 \leqq i < 2m$, and using (6.5), (6.38) we get

$$\Big(\int_{-r}^{r}\int_{|x'|<r} |D_{x_n}^i D_t^q \nabla^q u(x', 0, t)|^2 dx' dt\Big)^{1/2} \leqq \bar{H}_{i,r}\Big(\frac{H}{\rho - r}\Big)^{p+q} (p+q)! \tag{6.39}$$

for all p, q where

$$\begin{aligned}\bar{H}_{i,r} &= c_0((Z_2 r^{-1})^{1/2} + (Z_2 r)^{1/2}) H_0 (\rho - r)^{-2m} \\ &+ ((Z_2 r^{-1})^{1/2}(c_0 + c_1 r^{-i}) + (Z_2 r)^{1/2}(c_0 + c_1 r^{-i-1})) H_0 H^{-1} (\rho - r)^{-2m+1}.\end{aligned}$$

(6.39) shows that the Cauchy data of u on the boundary near the origin are analytic, hence with the aid of Holmgren's theorem we get

Theorem 6.1. *Under the assumptions* (I′), (II′), (III′) *any solution of* (6.1)–(6.2) *is analytic in* $(x, t) \in \bar{\Omega} \times (-\infty, \infty)$.

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