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Osaka Math. J.
14 (1962), 253-263.

ON THE ORTHOGONAL INVERSE EXPANSION WITH AN APPLICATION TO THE MOMENTS OF ORDER STATISTICS

By

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1. Introduction

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from distribution function $F(x)$. We rearrange the sample in ascending order of magnitude to get

$$X_{1/n} \leqq X_{2/n} \leqq \cdots \leqq X_{n/n}\,.$$

If we put $U=F(X)$, the random variable U is distributed uniformly in the interval [0, 1]. We may regard X as a function of U and expand formally in Taylor series of U. According to Kendall [5] p. 328, K. Pearson had already considered the following inverse expansion in order to calculate the moments of order statistics:

$$(1.1)\qquad X_{r/n} = F^{-1}(U_{r/n}) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}\frac{d^\nu}{du^\nu} F^{-1}(u)\Big|_{u=u_p} (U_{r/n}-u_p)^\nu\,,$$

where
$$u_p = \frac{r}{n+1} = \mathrm{E}(U_{r/n})\,.$$

We shall call (1. 1) *K. Pearson's inverse expansion.* Clark and Williams [1], David and Johnson [2] and Siddiqui [9] applied this expansion to obtain approximate values of the moments of order statistics.

On the other hand, Plackett [7] proved that there is a universal upper bound for $\mathrm{E}(X_{n/n}-X_{1/n})/\sigma$, where σ is the standard deviation with respect to $F(x)$. Though essentially the same as above, Moriguti [6] presented an upper bound for $\mathrm{E}(X_{n/n})$ when $F(x)$ is restricted to be symmetric with fixed mean and variance. Gumbel [3] and Hartley and David [4] generalized the Moriguti's results to any not necessarily symmetric, population. Their discussion is based on the method of variation so that the special consideration is required to see whether the stationary solution gives in fact maximum or not.

In this paper we introduce the orthogonal inverse expansion instead of K. Pearson's one and derive universal upper bounds as its first term in the expansion. At the same time, if we know the distribution function $F(x)$, the above orthogonal expansion gives some methods of obtaining approximate values of $\mathrm{E}(X_{r/n})$ and $\mathrm{E}(X^2_{r/n})$ etc. together with its error from the exact value.

In section 2 we state some preliminary results concerning orthogonal functions for later use. The theory of $L^2(0, 1)$ space is fundamental. In section 3 we apply the results in section 2 to evaluating the moments of order statistics and in section 4 the normal case is presented as a numerical example.

2. Some preliminary results concerning orthogonal functions

Results mentioned in this section without proof may be referred, for example, to Sansone [8].

Proposition 1*. *Let H_p be a pre-Hilbert space and let $\{\varphi_\nu\}_{\nu=1,2,\cdots}$ be an orthonormal system in H_p. Put $a_\nu=(f, \varphi_\nu)$, $b_\nu=(g, \varphi_\nu)$ for any element f, g in H_p, then*

$$|(f, g)-\sum_{\nu=1}^{k} a_\nu b_\nu| \leqq \{||f||^2-\sum_{\nu=1}^{k} a_\nu^2\}^{1/2}\{||g||^2-\sum_{\nu=1}^{k} b_\nu^2\}^{1/2}, \tag{2.1}$$

where equality holds if and only if f, g, $\varphi_1, \varphi_2, \cdots, \varphi_k$ are linearly dependent.

Proof. From Schwartz' inequality follows

$$|(f-\sum_{\nu=1}^{k} a_\nu\varphi_\nu,\ g-\sum_{\nu=1}^{k} b_\nu\varphi_\nu)| \leqq ||f-\sum_{\nu=1}^{k} a_\nu\varphi_\nu||\cdot||g-\sum_{\nu=1}^{k} b_\nu\varphi_\nu||\,.$$

This is equivalent to (2.1). Equality holds if and only if there exist λ and μ, not all zero, such that

$$\lambda(f-\sum_{\nu=1}^{k} a_\nu\varphi_\nu)+\mu(g-\sum_{\nu=1}^{k} b_\nu\varphi_\nu) = 0\,.$$

Hence

$$\lambda f+\mu g = \sum_{\nu=1}^{k} (\lambda f+\mu g, \varphi_\nu)\varphi_\nu\,. \tag{2.2}$$

This means that f, g, $\varphi_1, \varphi_2, \cdots, \varphi_k$ are linearly dependent.

REMARK 1. Proposition 1 is useful to the effect that when approximating inner product (f, g) in H_p by $\sum_{\nu=1}^{k} a_\nu b_\nu$, its error is given only by

* The proposition together with its proof was improved by a comment of Mr. S. Ikeda, Nihon University.

coefficients $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ just used in approximation.

REMARK 2. Proposition 1 holds whenever $\{\varphi_\nu\}_{\nu=1,2,\cdots}$ is an orthonormal system in H_p. But if further it is a complete orthonormal system, the right side of (2.1) converges to zero as k tends to infinity. Hence we can approximate as exactly as we please by choosing k sufficiently large.

EXAMPLE 1. Examples of Hilbert space. Let

$$\text{(2.3)}\qquad \begin{cases} L^2(0,1) = \left\{ f(u) \,\middle|\, \int_0^1 f^2(u)du < \infty \right\}, \\ L_e^2(0,1) = \left\{ f(u) \,\middle|\, \int_0^1 f^2(u)du < \infty \text{ and } f(u) = f(1-u) \right\}, \\ L_0^2(0,1) = \left\{ f(u) \,\middle|\, \int_0^1 f^2(u)du < \infty \text{ and } f(u) = -f(1-u) \right\}, \end{cases}$$

then $L^2(0,1)$ space is known to be a Hilbert space with respect to the inner product $(f, g) = \int_0^1 f(u)g(u)du$. It is easily seen that either $L_e^2(0,1)$ or $L_0^2(0,1)$ also constitutes a Hilbert space with respect to the same inner product as above.

Proposition 2. *Let X be a random variable having distribution function $F(x)$. Suppose that $F(x)$ is absolutely continuous with respect to the Lebesgue measure with finite second order moment. Put $u = F(x)$, then*

$$\text{(2.4)}\qquad x(u) = F^{-1}(u) \in L^2(0,1).$$

If further $F(x)$ is symmetric, then

$$\text{(2.5)}\qquad x(u) \in L_0^2(0,1).$$

Proof. Since $F(x)$ is absolutely continuous, there exists the derivative du/dx. Hence

$$\int_0^1 x^2(u)du = \int_{-\infty}^{\infty} x^2 \frac{du}{dx} dx < \infty.$$

If further $F(x)$ is symmetric or $F(-x) = 1 - F(x)$, then

$$-x(u) = x(1-u),$$

hence by (2.3) $x(u) \in L_0^2(0,1)$.

REMARK 3. By Proposition 2 $x(u)$ belongs to $L^2(0,1)$ space, and so we can expand this by the orthonormal system in $L^2(0,1)$. If, in addition, the system is taken complete, this expansion converges to $x(u)$ strongly in $L^2(0, 1)$, though it does not mean pointwise convergence.

REMARK 4. In Proposition 2 the inverse function $x(u)$ is considered in spite of the non-uniqueness in determining $F^{-1}(u)$. But, since the family of non-degenerate intervals on which $F(x)$ is constant is countable, $x(u)$ is defined almost everywhere in u with respect to the Lebesgue measure in $[0, 1]$.

EXAMPLE 2. Examples of complete orthonormal system in $L^2(0, 1)$.

(i) Legendre polynomials.

$$\varphi_\nu(u) = \frac{\sqrt{2\nu+1}}{\nu!}\frac{d^\nu}{du^\nu}u^\nu(u-1)^\nu \quad (\nu = 0, 1, 2, \cdots). \tag{2.6}$$

(ii) Trigonometric functions.

$$1, \ \frac{\cos 2\pi u}{\sqrt{2}}, \ \frac{\cos 4\pi u}{\sqrt{2}}, \cdots$$
$$\frac{\sin 2\pi u}{\sqrt{2}}, \ \frac{\sin 4\pi u}{\sqrt{2}}, \cdots$$

Each of these series of functions is well known to constitute a complete orthonormal system in $L^2(0, 1)$.

EXAMPLE 3. Examples of complete orthonormal system in $L_e^2(0, 1)$ and $L_0^2(0, 1)$. Let $\varphi_\nu(u)$ be Legendre polynomial in Example 2.

(2.7)
- (i) $\{\varphi_{2\nu}(u)\}_{\nu=0,1,2,\cdots}$ constitutes a complete orthonormal system in $L_e^2(0, 1)$.
- (ii) $\{\varphi_{2\nu+1}(u)\}_{\nu=0,1,2,\cdots}$ constitutes a complete orthonormal system in $L_0^2(0, 1)$.

Proof. From the definition (2.6) of $\varphi_\nu(u)$, we have

$$\varphi_\nu(1-u) = (-1)^\nu\varphi_\nu(u).$$

Hence

$$\varphi_{2\nu}(u) \in L_e^2(0, 1) \qquad \text{for } \nu = 0, 1, 2, \cdots$$

and

$$\varphi_{2\nu+1}(u) \in L_0^2(0, 1) \quad \text{for } \nu = 0, 1, 2, \cdots.$$

To see the completeness of $\{\varphi_{2\nu}\}_{\nu=0,1,2,\cdots}$ in $L_e^2(0, 1)$, we select an arbitrary f in $L_e^2(0, 1)$, then

$$||f||^2 = \sum_{\nu=0}^{\infty}(f, \varphi_{2\nu})^2 + \sum_{\nu=0}^{\infty}(f, \varphi_{2\nu+1})^2,$$

because of the completeness of $\{\varphi_\nu\}_{\nu=0,1,2,\cdots}$ in $L^2(0,1)$. Moreover

$$(f,\varphi_{2\nu+1})=\int_0^1 f(u)\varphi_{2\nu+1}(u)du$$
$$=\int_0^1 f(1-u)\varphi_{2\nu+1}(1-u)du = -(f,\varphi_{2\nu+1}),$$

and hence
$$(f,\varphi_{2\nu+1})=0 \qquad (\nu=0,1,2,\cdots).$$

Thus
$$\|f\|^2=\sum_{\nu=0}^{\infty}(f,\varphi_{2\nu})^2 .$$

This proves our assertion (i). An analogous argument shows that (ii) is also true. We can also constitute a complete orthonormal system in either $L_0^2(0,1)$ or $L_e^2(0,1)$ from the trigonometric functions in Example 2.

3. Universal upper bound and approximation for the moments of order statistics.

We now apply the results of section 2 to evaluating the moments of order statistics.

Theorem 1. *Let the distribution function $F(x)$ be absolutely continuous with respect to the Lebesgue measure with mean μ and variance σ^2. Let $\varphi_0=1,\varphi_1,\cdots,\varphi_k$ be any orthonormal system in $L^2(0,1)$ and put*

$$a_\nu=\int_0^1 x(u)\varphi_\nu(u)du ,$$
$$b_\nu=\frac{1}{B(r,n-r+1)}\int_0^1 u^{r-1}(1-u)^{n-r}\varphi_\nu(u)du ,$$

then

$$\text{(3.1)}\qquad |\mathrm{E}(X_{r/n})-\mu-\sum_{\nu=1}^{k}a_\nu b_\nu|$$
$$\leqq\{\sigma^2-\sum_{\nu=1}^{k}a_\nu^2\}^{1/2}\left\{\frac{B(2r-1,2n-2r+1)}{[B(r,n-r+1)]^2}-1-\sum_{\nu=1}^{k}b_\nu^2\right\}^{1/2}.$$

Proof. The expected value of the order statistic $X_{r/n}$ is given by the formula:

$$\text{(3.2)}\qquad \mathrm{E}(X_{r/n})=\frac{1}{B(r,n-r+1)}\int_{-\infty}^{\infty}x[F(x)]^{r-1}[1-F(x)]^{n-r}p(x)dx ,$$

where $p(x)$ is the derivative of $F(x)$. Put $u=F(x)$ and transform the variable x to u, then

$$\text{(3.3)}\qquad \mathrm{E}(X_{r/n})=\frac{1}{B(r,n-r+1)}\int_0^1 x(u)u^{r-1}(1-u)^{n-r}du .$$

Now put

$$f = x(u)\,, \qquad g = \frac{1}{B(r,\, n-r+1)} u^{r-1}(1-u)^{n-r}\,, \tag{3.4}$$

and apply Proposition 1 to (3.3). We have

$$\begin{cases} \|f\|^2 = \int_0^1 [x(u)]^2 du = \int_{-\infty}^{\infty} x^2 p(x)dx = \mu^2+\sigma^2\,, \\ \|g\|^2 = \dfrac{1}{[B(r,\, n-r+1)]^2}\int_0^1 u^{2r-2}(1-u)^{2n-2r}du \\ \qquad = \dfrac{B(2r-1,\, 2n-2r+1)}{[B(r,\, n-r+1)]^2}\,, \\ a_0 = \int_0^1 x(u)\varphi_0(u)du = \mu\,, \\ b_0 = \int_0^1 g(u)\varphi_0(u)du = 1\,. \end{cases} \tag{3.5}$$

So f and g belong to $L^2(0, 1)$ and we obtain (3.1) by substituting the above relations into (2.1).

Corollary. *For any distribution function absolutely continuous with respect to the Lebesgue measure with mean zero and variance one,*

$$|\mathrm{E}(X_{r/n})| \leq \left\{\frac{B(2r-1,\, 2n-2r+1)}{[B(r,\, n-r+1)]^2}-1\right\}^{1/2}, \tag{3.6}$$

where equality holds if and only if

$$x(u) = \pm\left\{\frac{B(2r-1,\, 2n-2r+1)}{[B(r,\, n-r+1)]^2}-1\right\}^{-1/2}\left\{1-\frac{1}{B(r,\, n-r+1)}u^{r-1}(1-u)^{n-r}\right\}. \tag{3.7}$$

Proof. (3.6) follows from (3.1) by considering in Theorem 1 the case where $\{\varphi_\nu\}_{\nu=0,1,2,\dots}$ consists of only φ_0. (3.7) follows from (2.2) and (3.4), where constants λ and μ in (2.2) are determined by the condition var. $X=1$.

These results (3.6) and (3.7) were already obtained by Hartley and David [4] by the method of variation. As remarked there, there exists no distribution satisfying the condition (3.7) in case $r \neq 1$, n, since the monotonicity of $x(u)$ in the interval [0, 1] is violated. If $r=1$ or n, (3.7) yields the distribution attaining the universal upper bound, which is the results due to Gumbel [3]. In case $r=n$,

$$
F(x)=\begin{cases} 1, & x \geqq \sqrt{2n-1} \\ \left[\dfrac{1}{n}\left(1+\dfrac{n-1}{\sqrt{2n-1}}x\right)\right]^{1/(n-1)}, & -\dfrac{\sqrt{2n-1}}{n-1} \leqq x \leqq \sqrt{2n-1} \\ 0, & x \leqq -\dfrac{\sqrt{2n-1}}{n-1}. \end{cases} \tag{3.8}
$$

We shall turn according to Moriguti [6] to the case where the distribution is known to be symmetric. The universal upper bound is then expected to be smaller than for the general distribution. In order to calculate the moments of order statistics in normal sample, this symmetric case is useful as indicated in section 4.

Theorem 2. *Let the distribution function $F(x)$ be symmetric and absolutely continuous with respect to the Lebesgue measure with finite variance σ^2. Let $\varphi_0, \varphi_1, \cdots$ be Legendre polynomials cited in Example 2. Put*

$$
a_\nu = \int_0^1 x(u)\varphi_\nu(u)du
$$

and

$$
b_\nu = \frac{1}{B(r, n-r+1)}\int_0^1 u^{r-1}(1-u)^{n-r}\varphi_\nu(u)du\,,
$$

then

$$
\begin{aligned}
|\mathrm{E}(X_{r/n})-\sum_{\nu=0}^{k} a_{2\nu+1}b_{2\nu+1}| &\leqq \{\sigma^2-\sum_{\nu=0}^{k} a_{2\nu+1}^2\}^{1/2} \\
&\times \left\{\frac{B(2r-1, 2n-2r+1)-B(n, n)}{2[B(r, n-r+1)]^2}-\sum_{\nu=0}^{k} b_{2\nu+1}^2\right\}^{1/2}.
\end{aligned} \tag{3.9}
$$

If further $F(x)$ has the finite fourth order moment, then

$$
\begin{aligned}
|\mathrm{E}(X_{r/n}^2)-\sigma^2-\sum_{\nu=1}^{k} a'_{2\nu}b_{2\nu}| &\leqq \{\mathrm{E}(X^4)-\sigma^4-\sum_{\nu=1}^{k} a'^2_{2\nu}\}^{1/2} \\
&\times \left\{\frac{B(2r-1, 2n-2r+1)+B(n, n)}{2[B(r, n-r+1)]^2}-1-\sum_{\nu=1}^{k} b_{2\nu}^2\right\}^{1/2},
\end{aligned} \tag{3.10}
$$

where

$$
a'_\nu = \int_0^1 [x(u)]^2\varphi_\nu(u)du\,.
$$

Proof. Since $F(x)$ is symmetric, the inverse function $x(u)$ belongs to $L_0^2(0, 1)$ by Proposition 2. Then by Example 3

$$
a_{2\nu} = (x(u), \varphi_{2\nu}(u)) = 0 \qquad \nu = 0, 1, 2, \cdots.
$$

If we define f and g by (3.4) and apply (2.1) to these functions where $\nu=1, 3, \cdots, 2k+1, 2, 4, 6, \cdots$, then

$$(3.11)\qquad |(f, g)-\sum_{\nu=0}^{k} a_{2\nu+1}b_{2\nu+1}| \leqq \{\|f\|^2-\sum_{\nu=0}^{k} a_{2\nu+1}^2\}^{1/2} \times \{\|g\|^2-\sum_{\nu=0}^{k} b_{2\nu+1}^2-\sum_{\nu=0}^{\infty} b_{2\nu}^2\}^{1/2}.$$

So far as we concern symmetric population, we can thus improve the upper bound by the amount $\sum_{\nu=0}^{\infty} b_{2\nu}^2$. Put

$$(3.12)\qquad g^*(u) = \frac{1}{B(r, n-r+1)} u^{n-r}(1-u)^{r-1},$$

then

$$b_{2\nu} = (g, \varphi_{2\nu}) = \frac{1}{2}(g+g^*, \varphi_{2\nu}).$$

Since

$$g(u)+g^*(u) \in L_e^2(0, 1)$$

and $\{\varphi_{2\nu}\}_{\nu=0,1,2,\dots}$ constitute a complete orthonormal system in $L_e^2(0, 1)$ by Example 3,

$$(3.13)\qquad \sum_{\nu=0}^{\infty} b_{2\nu}^2 = \frac{1}{4}\|g+g^*\|^2 = \frac{1}{2[B(r, n-r+1)]^2}\{B(2r-1, 2n-2r+1)+B(n, n)\}.$$

Substituting (3.13) and (3.5) (in this case $\mathrm{E}(X)=0$) into (3.11), we obtain (3.9). Since $\sum_{\nu=0}^{\infty} b_\nu^2=\|g\|^2$, it follows

$$\sum_{\nu=0}^{\infty} b_{2\nu+1}^2 = \frac{1}{2[B(r, n-r+1)]^2}\{B(2r-1, 2n-2r+1)-B(n, n)\},$$

and analogous argument leads to (3.10).

Corollary. *For any distribution function which is symmetric and absolutely continuous with respect to the Lebesgue measure with variance* 1,

$$(3.14)\qquad |\mathrm{E}(X_{r/n})| \leqq \frac{1}{\sqrt{2}\,B(r, n-r+1)}\{B(2r-1, 2n-2r+1)-B(n, n)\}^{1/2}.$$

where equality holds if and only if

$$(3.15)\qquad x(u) = \pm\frac{1}{\sqrt{2}}\{B(2r-1, 2n-2r+1)-B(n, n)\}^{-1/2} \times \{u^{r-1}(1-u)^{n-r}-u^{n-r}(1-u)^{r-1}\}.$$

Proof. (3.14) follows from (3.9) by considering in Theorem 2 the case where $\{\varphi_\nu\}_{\nu=0,1,2,\dots}$ consists of only $\{\varphi_{2\nu}\}_{\nu=0,1,2,\dots}$. From (2.2) equality holds in (3.14) if and only if

$$x(u) = \text{const}\cdot\{g(u)-\sum_{\nu=0}^{\infty} b_{2\nu}\varphi_{2\nu}(u)\}\ ,$$

where $g(u)$ and $b_{2\nu}$ are defined in (3.4) and Theorem 2. If we denote $g^*(u)$ by (3.12), $g(u)+g^*(u)$ belongs to $L_e^2(0, 1)$. Hence

$$g(u)+g^*(u) = 2\sum_{\nu=0}^{\infty} b_{2\nu}\varphi_{2\nu}(u)\ ,$$

and

$$x(u) = \text{const}\cdot\{u^{r-1}(1-u)^{n-r}-u^{n-r}(1-u)^{r-1}\}\ .$$

(3.15) is derived by determining the constant from the condition $||x(u)||=1$.

(3.14) and (3.15) coincide with the results of Moriguti [6], though he deals only with the case $r=n$.

4. Expected values of order statistics in normal sample

As an application of the orthogonal inverse expansion we shall deal with a standard normal distribution and calculate $\mathrm{E}(X_{r/n})$ by Theorem 2, where the right side of the formula (3.9) gives the deviation from the exact value. For this purpose we make use of Legendre polynomials cited in Example 2, of which the first six are as follows:

$$\begin{aligned}
\varphi_0(u) &= 1\ ,\\
\varphi_1(u) &= \sqrt{3}(2u-1)\ ,\\
\varphi_2(u) &= \sqrt{5}(6u^2-6u+1)\ ,\\
\varphi_3(u) &= \sqrt{7}(20u^3-30u^2+12u-1)\ ,\\
\varphi_4(u) &= 3(70u^4-140u^3+90u^2-20u+1)\ ,\\
\varphi_5(u) &= \sqrt{11}(252u^5-630u^4+560u^3-210u^2+30u-1)\ .
\end{aligned} \tag{4.1}$$

Put in general

$$\varphi_\nu(u) = \sum_{i=0}^{\nu} a_{\nu,i}u^i\ ,$$

then the coefficients a_ν and b_ν are given by

$$\begin{aligned}
a_\nu &= \sum_{i=0}^{\nu} a_{\nu,i}\int_0^1 u^i x(u)du\\
&= \sum_{i=0}^{\nu} \frac{a_{\nu,i}}{i+1}\mathrm{E}(X_{i+1/i+1})\ ,
\end{aligned} \tag{4.2}$$

$$b_\nu = \sum_{i=0}^{\nu} a_{\nu,i}\frac{r(r+1)\cdots(r+i-1)}{(n+1)(n+2)\cdots(n+i)}\ . \tag{4.3}$$

Now we have

(4. 4)

$$\int_0^1 ux(u)\,du = \frac{1}{2\sqrt{\pi}}, \qquad \int_0^1 u^4x(u)\,du = \frac{1}{2\sqrt{\pi}}\left\{1-\frac{1}{\pi}\mathrm{Cos}^{-1}\left(-\frac{1}{3}\right)\right\},$$

$$\int_0^1 u^2x(u)du = \frac{1}{2\sqrt{\pi}}, \qquad \int_0^1 u^5x(u)\,du = \frac{1}{6}\mathrm{E}(X_{6/6}),$$

$$\int_0^1 u^3x(u)du = \frac{1}{4\pi\sqrt{\pi}}\mathrm{Cos}^{-1}\left(-\frac{1}{3}\right), \qquad = \frac{1}{6}\times 1.267206361,$$

where the value of $\mathrm{E}(X_{6/6})$ is found for example in Teichroew [10]. From (4. 2), (4. 3) and (4. 4) we can calculate $\mathrm{E}(X_{r/n})$ by (4. 9) to get Table 1 and Table 2 corresponding to the case $n=10$ and $n=20$, respectively. From these Tables, we can see that the approximation is good uniformly with r. This may be foreseen since the orthogonal expansion gives the best approximation in the mean.

Table 1. Values of $\mathrm{E}(X_{r/10})$ in standard normal population.

r	first approximation a_1b_1	second approximation $a_1b_1+a_3b_3$	third approximation $a_1b_1+a_3b_3+a_5b_5$	exact value
10	$1.38\pm.17$	$1.527\pm.015$	$1.5384\pm.0005$	1.53875
9	$1.08\pm.09$	$1.030\pm.035$	$1.0032\pm.0026$	1.00136
8	$.77\pm.14$	$.651\pm.008$	$.6527\pm.0048$	.65606
7	$.46\pm.13$	$.357\pm.028$	$.3775\pm.0024$	.37576
6	$.15\pm.06$	$.113\pm.016$	$.1246\pm.0032$	.12267

Table 2. Values of $\mathrm{E}(X_{r/20})$ in standard normal population.

r	first approximation a_1b_1	second approximation $a_1b_1+a_3b_3$	third approximation $a_1b_1+a_3b_3+a_5b_5$	exact value
20	$1.53\pm.35$	$1.796\pm.082$	$1.856\pm.016$	1.867
19	$1.37\pm.17$	$1.468\pm.067$	$1.433\pm.032$	1.408
18	$1.21\pm.15$	$1.186\pm.075$	$1.131\pm.012$	1.131
17	$1.05\pm.16$	$.945\pm.057$	$.907\pm.018$	.921
16	$.89\pm.18$	$.739\pm.034$	$.734\pm.022$	.745
15	$.73\pm.20$	$.564\pm.037$	$.587\pm.013$	.590
14	$.56\pm.21$	$.413\pm.056$	$.454\pm.008$	.448
13	$.40\pm.19$	$.281\pm.065$	$.325\pm.019$	.315
12	$.24\pm.14$	$.163\pm.053$	$.197\pm.020$	.187
11	$.081\pm.051$	$.054\pm.021$	$.066\pm.008$	.062

Acknowledgement: the author is very grateful to Prof. J. Ogawa, Nihon University, and Dr. M. Okamoto, Osaka University for their kind advice and encouragement and also to Mr. S. Ikeda, Nihon University, for his helpful comments.

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(Received August 29, 1962)

References

[1] C. E. Clark and G. T. Williams: *The distribution of the members of an ordered sample*, Ann. Math. Statist. **29** (1958), 862–870.
[2] F. N. David and N. L. Johnson: *Statistical treatment of censored data*, Biometrika **41** (1954), 228–240.
[3] E. J. Gumbel: *The maxima of the mean largest value and of the range*, Ann. Math. Statist. **25** (1954), 76–84.
[4] H. O. Hartley and H. A. David: *Universal bounds for mean range and extreme observation*, Ann. Math. Statist. **25** (1954), 85–99.
[5] M. G. Kendall: The Advanced Theory of Statistics, vol. 1, Charles Griffin, London, 1958.
[6] S. Moriguti: *Extremal property of extreme value distributions*, Ann. Math. Statist. **22** (1951), 523–536.
[7] R. L. Plackett: *Limits of the ratio of mean range to standard deviation*, Biometrika **34** (1947), 120–122.
[8] G. Sansone: Orthogonal Functions, Interscience, New York, 1959.
[9] M. M. Siddiqui: *Approximation to the moments of the sample median*, Ann. Math. Statist. **33** (1962), 157–168.
[10] E. D. Teichroew: *Tables of expected values of order statistics and products of order statistics*, Ann. Math. Statist. **27** (1956), 410–426.