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ON CERTAIN QUADRATIC FORMS RELATED TO SYMMETRIC RIEMANNIAN SPACES

BY

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In his recent works [8] [9], Matsushima investigates the Betti numbers of a certain type of compact locally symmetric riemannian manifolds and he shows that lower dimensional Betti numbers of such a manifold can be determined if we know that certain quadratic forms are positive definite. In order to verify the last condition, it is sufficient to compute the minimal eigenvalue of a linear transformation Q which is defined in terms of the curvature tensor of the manifold. Let M be the universal covering manifold of the manifold in question. In the case that M is an irreducible symmetric bounded domain in $\boldsymbol{C}^n$, the eigenvalues of Q were computed by Calabi-Vesentini [4] and Borel [2], and Betti numbers are given by Matsushima himself. In the paper [7], the authors have treated the case that the compact form of M is a group manifold. The purpose of the present paper is to study the eigenvalues of Q for the case which remains to be treated, i.e. the case that M is a non-compact irreducible symmetric riemannian manifold not isomorphic to a bounded domain in $\boldsymbol{C}^n$. Applying the results of Matsushima, we see that the p-th Betti number b_p of the compact locally symmetric space treated here is equal to that of the compact form of M if the ratio $p/\dim M$ is sufficiently small; in particular, b_1 vanishes in most cases. Our results give also informations about the Betti numbers of G/Γ where G is a semi-simple Lie group without compact simple factors and Γ is a discrete subgroup of G with compact quotient space G/Γ.

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§1. Preliminaries.

The notations settled in this section will be used throughout this paper. We denote by $\mathfrak{g}$ a real semi-simple Lie algebra and by $\varphi_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$, Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. We say that the pair

$(\mathfrak{g}, \mathfrak{k})$ is a *symmetric pair*, if the vector space $\mathfrak{g}$ admits a direct sum decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ such that $[\mathfrak{k}, \mathfrak{m}]\subset\mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}]\subset\mathfrak{k}$, By the *symmetric square* $\mathfrak{m}\vee\mathfrak{m}$ we mean the vector space of the symmetric tensors belonging to $\mathfrak{m}\otimes\mathfrak{m}$. The restriction of $\varphi_{\mathfrak{g}}$ to $\mathfrak{m}$ is a non-degenerate inner product on $\mathfrak{m}$, and this gives rise to non-degenerate inner products on $\mathfrak{m}\otimes\mathfrak{m}$ and $\mathfrak{m}\vee\mathfrak{m}$, which are denoted by $\langle\ ,\ \rangle_{\mathfrak{m}\otimes\mathfrak{m}}$ and $\langle\ ,\ \rangle_{(\mathfrak{g},\mathfrak{k})}$ respectively. For brevity we write $\langle\ ,\ \rangle$ instead of $\langle\ ,\ \rangle_{(\mathfrak{g},\mathfrak{k})}$. We define a linear endomorphism Q of the tensor space $\mathfrak{m}\otimes\mathfrak{m}$ by the formula

$$\langle Q(X\otimes Y),\ Z\otimes U\rangle_{\mathfrak{m}\otimes\mathfrak{m}} = \varphi_{\mathfrak{g}}([Y, U],\ [Z, X]) \tag{1.1}$$

for any $X, Y, Z, U\in\mathfrak{m}$, The subspace $\mathfrak{m}\vee\mathfrak{m}$ is stable under Q, and we shall denote also by Q the restriction of Q to $\mathfrak{m}\vee\mathfrak{m}$. It is obvious that Q is a self-adjoint operator on $\mathfrak{m}\vee\mathfrak{m}$ with respect to the inner product $\langle\ ,\ \rangle$. Now, following Matsushima [9], we define a quadratic forms H^r $(r=1, 2, \cdots)$ on $\mathfrak{m}\vee\mathfrak{m}$ by putting

$$H^r(\xi) = \frac{b_{(\mathfrak{g},\mathfrak{k})}}{r}\langle\xi, \xi\rangle+\langle Q\xi, \xi\rangle \tag{1.2}$$

for any $\xi\in\mathfrak{m}\vee\mathfrak{m}$, where

$$2b_{(\mathfrak{g},\mathfrak{k})} = \operatorname*{Min}_{X\in\mathfrak{k}}\ \{1+\varphi_{\mathfrak{k}}(X, X)\ ;\ \varphi_{\mathfrak{g}}(X, X) = -1\} \tag{1.3}$$

and $\varphi_{\mathfrak{k}}$ is the Killing form of $\mathfrak{k}$. Matsushima [8] defines also a quadratic form H on $\mathfrak{m}\vee\mathfrak{m}$ which coincides with H^1 in the case that $\mathfrak{g}$ is non-compact simple and $\mathfrak{k}$ is a compact semi-simple Lie algebra. In the following, we shall write Q, H^r, H also as $Q_{(\mathfrak{g},\mathfrak{k})}$, $H^r{}_{(\mathfrak{g},\mathfrak{k})}$, $H_{(\mathfrak{g},\mathfrak{k})}$ respectively; if $\mathfrak{g}$ is a non-compact simple Lie algebra, $\mathfrak{k}$ means always the subalgebra corresponding to a maximal compact subgroup of the adjoint group of $\mathfrak{g}$. The theorems of Matsushima are now stated as follows:

Theorem A ([8] Theorem 1). *Let G be a semi-simple Lie group, all of whose simple factors are non-compact. Suppose that, for each simple factor G_i of G, the quadratic form $H_{(\mathfrak{g}_i,\mathfrak{k}_i)}$ associated to the Lie algebra $\mathfrak{g}_i$ of G_i is positive definite. Let Γ be a discrete subgroup of G with compact quotient space G/Γ. Then the first Betti number of G/Γ is equal to* 0.

Theorem B ([9] Theorem 1 and §9). *Let M be a simply connected, irreducible symmetric riemannian manifold which is non-compact and non-euclidean. Let G be the identity component of the group of all isometries of M. Let Γ be a discrete subgroup of G with compact quotient G/Γ and without element of finite order different from the identity, so that Γ is a* *discontinuous group of isometries of M with compact quotient M/Γ. Let M_u be the compact form of M. If the quadratic form $H^r{}_{(\mathfrak{g},\mathfrak{k})}$ is positive definite, then the r-th Betti number b_r of M/Γ is equal to the r-th Betti number $b_r(M_u)$ of M_u.*

We shall see when the quadratic forms $H_{(\mathfrak{g},\mathfrak{k})}$ and $H^r{}_{(\mathfrak{g},\mathfrak{k})}$ are positive definite for a real non-compact simple Lie algebra $\mathfrak{g}$. In the case that the center of $\mathfrak{k}$ is not trivial, this question is already discussed by Matsushima [8] [9]. We shall therefore confine ourselves to the case that the center of $\mathfrak{k}$ reduces to (0) so that $\mathfrak{k}$ is a compact semi-simple algebra. We know that $H=H^1$ in this case. Thus our results on H^r, given in §4, are sufficient to apply Theorems A and B for this case.

In order that the quadratic form H^r is positive definite, it is necessary and sufficient that the absolute value of the minimal eigenvalue of Q is strictly smaller than $\frac{b_{(\mathfrak{g},\mathfrak{k})}}{r}$. Now, under the assumption that $\mathfrak{g}$ is non-compact simple and $\mathfrak{k}$ is semi-simple, the value $b_{(\mathfrak{g},\mathfrak{k})}$ is computed as follows:

(a) If $\mathfrak{k}$ is simple, then we have by [8]

$$b_{(\mathfrak{g},\mathfrak{k})}=\frac{\dim \mathfrak{m}}{4 \dim \mathfrak{k}} \tag{1.4}$$

(b) When $\mathfrak{k}$ is semi-simple and has the same rank as $\mathfrak{g}$,

$$2b_{(\mathfrak{g},\mathfrak{k})}=\operatorname{Min}\{1-\varphi_{\mathfrak{g}}(\alpha_g,\ \alpha_g)/\varphi_{\mathfrak{k}}(\alpha_k,\ \alpha_k)\,;\alpha \text{ is a root of } \mathfrak{k}^c\} \tag{1.5}$$

where α_g or α_k is the contravariant representative of α with respect to the Killing form of $\mathfrak{g}$ or $\mathfrak{k}$ respectively.

(c) $(A_3,\ D_2)$ and $(D_l,\ B_q\times B_{l-q-1})$ are the only symmetric pairs for which $\mathfrak{k}$ is semi-simple and is not of the same rank as $\mathfrak{g}$. One finds

$$\begin{aligned} &2b=\frac{3}{4}\quad \text{for}\quad (\mathfrak{g},\mathfrak{k})=(A_3,\ D_2)\\ &2b=\frac{p}{2(l-1)}\quad \text{for}\quad (\mathfrak{g},\mathfrak{k})=(D_l,\ B_q\times B_{l-q-1}),\quad \text{and}\quad p<l, \end{aligned} \tag{1.6}$$

where $p=2q+1$

§ 2. Relation with the group manifold.

Given a symmetric pair $(\mathfrak{g},\mathfrak{k})$ with non-compact simple $\mathfrak{g}$ and compact $\mathfrak{k}$, we wish to compare the minimal eigenvalue of $Q_{(\mathfrak{g},\mathfrak{k})}$ with that of Q for symmetric pair $(\mathfrak{g}_u\times\mathfrak{g}_u,\ \mathfrak{g}_u)$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the Cartan decomposition. Denoting by $\mathfrak{g}^c$ the complexification of $\mathfrak{g}$, the space $\mathfrak{g}_u=\mathfrak{k}+\sqrt{-1}\,\mathfrak{m}$ is also a compact simple subalgebra of $\mathfrak{g}^c$. The pair $(\mathfrak{g}_u, \mathfrak{k})$ is also a symmetric pair, the *compact form* of $(\mathfrak{g}, \mathfrak{k})$. By an obvious identification $\sqrt{-1}\,\mathfrak{m}\vee\sqrt{-1}\,\mathfrak{m}=\mathfrak{m}\vee\mathfrak{m}$, one obtains the equality $Q_{(\mathfrak{g},\mathfrak{k})}=-Q_{(\mathfrak{g}_u,\mathfrak{k})}$ directly from the definition (1.1). In order to verify that $H^r{}_{(\mathfrak{g},\mathfrak{k})}$ is positive definite it is thus sufficient to show that $b_{(\mathfrak{g},\mathfrak{k})}/r$ is strictly greater than the maximal eigenvalue of $Q_{(\mathfrak{g}_u,\mathfrak{k})}$. Let σ be the injective homomorphism of $\mathfrak{g}_u$ into $\mathfrak{g}_u\times\mathfrak{g}_u$ defined by $\sigma(X)=(X, X)$ for any $X\in\mathfrak{g}_u$. We define another injection $\tau:\mathfrak{g}_u\rightarrow\mathfrak{g}_u\times\mathfrak{g}_u$ by $\tau(X)=(X, -X)/\sqrt{2}$; τ is isometric with respect to the Killing forms. The pair $(\mathfrak{g}_u\times\mathfrak{g}_u, \sigma(\mathfrak{g}_u))$, briefly denoted by $\mathfrak{u}$, is a symmetric pair with the Cartan decomposition $\mathfrak{g}_u\times\mathfrak{g}_u=\sigma(\mathfrak{g}_u)+\tau(\mathfrak{g}_u)$

Lemma 2.1. *The notations being as above, if* κ *(resp.* κ_u*) is the maximal eigenvalue of* $Q_{(\mathfrak{g}_u,\mathfrak{k})}$ *(resp.* $Q_{\mathfrak{u}}$*), then we have the inequality*: $\kappa\leqslant 2\kappa_u$

Proof. If $\tau\vee\tau$; $\mathfrak{g}_u\vee\mathfrak{g}_u\rightarrow\tau(\mathfrak{g}_u)\vee\tau(\mathfrak{g}_u)$ denotes the mapping naturally induced by τ and if π denotes the orthogonal projection of $(\tau\vee\tau)(\mathfrak{g}_u\vee\mathfrak{g}_u)$ onto $(\tau\vee\tau)(\mathfrak{m}\vee\mathfrak{m})$ then one obtains

$$\pi\cdot Q_{\mathfrak{u}}\cdot(\tau\vee\tau)=\frac{1}{2}(\tau\vee\tau)Q_{(\mathfrak{g}_u,\mathfrak{k})}$$

In fact, one has

$$\begin{aligned}&2\langle Q_{\mathfrak{u}}(\tau(X)\vee\tau(X)),\ \tau(Y)\vee\tau(Y)\rangle_{\mathfrak{u}}\\&=2\varphi_{\mathfrak{g}_u\times\mathfrak{g}_u}([\tau(X),\tau(Y)],\ [\tau(Y),\ \tau(X)])\\&=\varphi_{\mathfrak{g}_u\times\mathfrak{g}_u}(\tau([X,\ Y]),\ \tau([Y,\ X]))\\&=\varphi_{\mathfrak{g}_u}([X,\ Y],\ [Y,X])\\&=\langle Q_{(\mathfrak{g}_u,\mathfrak{k})}(X\vee X),\ Y\vee Y\rangle_{(\mathfrak{g}_u,\mathfrak{k})}\\&=\langle(\tau\vee\tau)\cdot Q_{(\mathfrak{g}_u,\mathfrak{k})}(X\vee X),\ \tau(Y)\vee\tau(Y)\rangle_{\mathfrak{u}}\end{aligned}$$

for any $X, Y\in\mathfrak{m}$. For any eigenvector ξ of $Q_{(\mathfrak{g}_u,\mathfrak{k})}$ corresponding to κ, one therefore finds

$$\begin{aligned}&2\kappa_u\langle\xi,\ \xi\rangle_{(\mathfrak{g}_u,\mathfrak{k})}=2\kappa_u\langle(\tau\vee\tau)(\xi),\ (\tau\vee\tau)(\xi)\rangle_{\mathfrak{u}}\\&\geqslant 2\langle Q_{\mathfrak{u}}((\tau\vee\tau)(\xi)),\ (\tau\vee\tau)(\xi)\rangle_{\mathfrak{u}}\\&=\langle Q_{(\mathfrak{g}_u,\mathfrak{k})}(\xi),\ \xi\rangle_{(\mathfrak{g}_u,\mathfrak{k})}=\langle\kappa\xi,\ \xi\rangle_{(\mathfrak{g}_u,\mathfrak{k})}=\kappa\langle\xi,\ \xi\rangle_{(\mathfrak{g}_u,\mathfrak{k})}\end{aligned}$$

Lemma 2.2. *The number* $2\kappa_u$ *in lemma* 2.1 *equals* $\varphi_{\mathfrak{g}_u}(\vartheta, \vartheta)$ *where* ϑ *is the highest root of* $\mathfrak{g}_u$

For the proof, see [7].

It follows from the above arguments that $H_{(\mathfrak{g},\mathfrak{k})}$ is positive difinite if $b_{(\mathfrak{g},\mathfrak{k})}>\varphi_{\mathfrak{g}_u}(\vartheta, \vartheta)$. The table I gives the values $b_{(\mathfrak{g},\mathfrak{k})}$ and $\varphi_{\mathfrak{g}_u}(\vartheta, \vartheta)$ for all symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ with non-compact simple $\mathfrak{g}$ and compact semi-simple $\mathfrak{k}$. We get then.

Lemma 2. 3. *The quadratic form $H_{(\mathfrak{g},\mathfrak{k})}$ is positive definite for all $(\mathfrak{g}_u, \mathfrak{k})$ except for the cases AII $(l=2, 3)$, BII, DII, CII $(p=1, 2)$, GI, or FII.*

Table I.

	$(\mathfrak{g}_u, \mathfrak{k})$	$b_{(\mathfrak{g},\mathfrak{k})}$	$2\kappa_u=\varphi_{\mathfrak{g}_u}(\vartheta, \vartheta)$
A I	$(A_l, B_{l/2})$ $l\geqq 2$ $(A_l, D_{(l+1)/2})$ $l\geqq 3$	$(l+3)/4(l+1)$	$1/(l+1)$
A II	(A_{2l-1}, C_l)	$(l-1)/4l$	$1/2l$
B I	$(B_l, D_p\times B_{l-p})$ $l-1\geqq p\geqq 2$	$\mathrm{Min}\left(\frac{p}{2l-1}, \frac{2l-2p+1}{2(2l-1)}\right)$	$1/(2l-1)$
D I	$(D_l, D_p\times D_{l-p})$ $l\geqq 2p\geqq 4$	$p/2(l-1)$	$1/2(l-1)$
	$(D_l, B_p\times B_{l-p-1})$ $l-1\geqq 2p\geqq 2$	$(2p+1)/4(l-1)$	
B II	(B_l, D_l)	$1/2(2l-1)$	$1/(2l-1)$
D II	(D_l, B_{l-1})	$1/4(l-1)$	$1/2(l-1)$
C II	$(C_l, C_p\times C_{l-p})$ $2p\leqq l$	$p/2(l+1)$	$1/(l+1)$
E I	(E_6, C_4)	$7/24$	$1/12$
E II	$(E_6, A_1\times A_5)$	$1/4$	
E IV	(E_6, F_4)	$1/8$	
E V	(E_7, A_7)	$5/18$	$1/18$
E VI	$(E_7, A_1\times D_6)$	$2/9$	
E VIII	(E_8, D_8)	$4/15$	$1/30$
E IX	$(E_8, A_1\times E_7)$	$1/5$	
G I	$(G_2, A_1\times A_1)$	$1/4$	$1/4$
F I	$(F_4, A_1\times C_3)$	$5/18$	$1/9$
F II	(F_4, B_4)	$1/9$	

§ 3. The exceptional cases

In this section we shall determine the eigenvalues of $Q_{(\mathfrak{g},\mathfrak{k})}$ for the exceptional cases in lemma 2. 3 by the method employed in [7]. Given a symmetric pair $(\mathfrak{g}, \mathfrak{k})$, we write ρ for the linear isotropy representation

$\rho(X)=ad(X)|\mathfrak{m}$, $X\in\mathfrak{k}$. We may extend ρ and Q to a representation of $\mathfrak{k}^c$ on $\mathfrak{m}^c$ and to an endomorphism of $\mathfrak{m}^c\vee\mathfrak{m}^c$ respectively, which will be denoted by the same letters. Q commutes with $\rho\vee\rho$, the restriction of $\rho\otimes\rho$ to $\mathfrak{m}^c\vee\mathfrak{m}^c$. Let $\rho\vee\rho=\rho_1+\cdots+\rho_s$ be the decomposition of $\rho\vee\rho$ into irreducible representations. The highest weights $\Lambda_{(i)}$ of ρ_i are assumed to be in the order $\Lambda_{(i)}>\cdots>\Lambda_{(s)}$ with respect to some ordering: this assumption will be satisfied in the following cases. Each subspace W_i of $\mathfrak{m}^c\vee\mathfrak{m}^c$ spanned by the weight vectors of $\Lambda_{(i)}$ is stable under Q. We write $(TrQ)_i$ for the trace of Q restricted to W_i. We know in [7] that Q is represented by a scalar matrix in the irreducible invariant subspace of $\mathfrak{m}^c\vee\mathfrak{m}^c$ corresponding to ρ_i. If κ_i denote the unique eigenvalue of Q restricted to the subspace, then

$$(3.1)\qquad (TrQ)_i=\sum_{1\leqq j\leqq i} m_j(\Lambda_{(i)})\kappa_j\,;$$

where $m_j(\Lambda_{(i)})$ is the multiplicity of the weight $\Lambda_{(i)}$ in ρ_j.

$$(3.1)'\qquad m_1(\Lambda_{(1)})=1 \quad\text{and so}\quad (TrQ)_1=\kappa_1.$$

Moreover, if $\mathfrak{g}$ is compact simple, which will be the case in the sequel, one has

$$(3.2)\qquad TrQ=(\dim\mathfrak{m})/4,\ \dim\mathfrak{m}=\dim\mathfrak{g}-\dim\mathfrak{k}\,,$$

(3.3) $-1/2$ is an eigenvalue with multiplicity one of Q. These formulas (3.1) to (3.3) are given in [7] and will be used to compute the eigenvalues of Q. Hereafter a representation and its highest weight will be denoted by the same letter.

(I) The case of $A\,II$: $(\mathfrak{g}_u,\ \mathfrak{k})=(A_{2l-1},\ C_l)$ $l\geqq 2$

First we try to find the highest weight of ρ and its weight vector: For an arbitrary square matrix X of degree $2l$, we divide X into four parts

$$X=\begin{pmatrix} A_X & B_X\\ C_X & D_X\end{pmatrix}$$

where A_X, B_X, C_X, D_X are square matrices with complex coefficients of degree l. $\mathfrak{g}^c$ is the Lie algebra $\mathfrak{sl}(2l,\boldsymbol{C})=\{X;\ Tr(A_X+D_X)=0\}$. If J denotes a $(2l,2l)$-matrix such that $A_J=D_J=0$ and $B_J=-C_J=E$ (= the unit matrix), $\mathfrak{k}^c$ is $\{X\in\mathfrak{sl}(2l,\boldsymbol{C})\,;\ {}^tXJ+JX=0\}$. Hence the orthogonal complement $\mathfrak{m}^c$ of $\mathfrak{k}^c$ consists of the matrices Y such that ${}^tYJ-JY=0$, or eqivalently, such that B_Y, C_Y are skew-symmetric matrices and one has ${}^tA_Y=D_Y$. An element X of $\mathfrak{k}^c$ operates on $\mathfrak{m}^c$ by $Y\in\mathfrak{m}^c\to\rho(X)Y=[X,\ Y]$ $=XY-YX$. On the other hand, the identity representation of $\mathfrak{k}^c$ operating on C^{2l} naturally induces a representation ρ' on $C^{2l}\wedge C^{2l}$ which we take as the space of skew-symmetric matrices of degree $2l$; $\rho'(X)Z=XZ+Z{}^tX$, Z=skew-symmetric. $\rho'(\mathfrak{k}^c)$ leaves J invariant, and ρ' is irreducible on the space $\mathfrak{n}$ of skew-symmetric matrices which are orthogonal to J. This irreducible representation, denoted also by ρ', has the highest weight $\Lambda_2=\lambda_1+\lambda_2$ if a Cartan subalgebra of $\mathfrak{k}^c$ consists of diagonal matrices, H, and a base (H_i) is so chosen that $H=\sum\lambda_iH_i$ in case that the (i, i)-element of A_H is λ_i, If $\alpha(Y)$ denotes the skew-symmetric matrix $-YJ$ for any Y in $\mathfrak{m}^c$, α; $\mathfrak{m}^c\to\mathfrak{n}$ is an isomorphism of $\mathfrak{k}^c$-modules. Thus the highest weight of ρ is also Λ_2, as was shown by E. Cartan. Let $e_i=(0,\cdots, 1,\cdots, 0)$ be the element of C^{2l} whose elements are all zero except that the i-th equals one, and $W\in\mathfrak{n}$ be the skew-symmetric matrix corresponding to $e_1\wedge e_2$, or equivalently such that $A_W=E_{12}-E_{21}$ and B_W, C_W. and D_W are zeros. Then W and therefore WJ is clearly the weight vector corresponding to Λ_2. For brevity we write X for WJ. The transposed ${}^tX$ is a weight vector of $-\Lambda_2$. By (3. 1)$'$, we obtain $\kappa_1=(TrQ)_1=\langle Q(X\otimes X), {}^tX\otimes{}^tX\rangle/\langle X\otimes X, {}^tX\otimes{}^tX\rangle$, which is computed with (1. 1). Thus we find

$$\kappa_1 = \varphi_{\mathfrak{g}^c}([X, {}^tX], [{}^tX, X])/\varphi_{\mathfrak{g}^c}(X, {}^tX)^2=1/4l$$

With the method in [7], $\Lambda_2\vee\Lambda_2$ turns out to be decomposed into two (resp. three) irreducible representations for $l=2$ (resp. 3). This fact, combined with (3. 1) to (3. 3), gives the eigenvalues of Q. Assume $l=3$, for instance. The three irreducible components of $\Lambda_2\vee\Lambda_2$ are $2\Lambda_2$, Λ_2 and Λ_0 (=the trivial representation of degree 1). The degree of $2\Lambda_2$ and Λ_2 are 90 and 14 respectively. Hence we have $7/2=\dim\mathfrak{m}/4=TrQ$ $=90\kappa_1+14\kappa_2+\kappa_3$, where $\kappa_1=1/12$ as above and $\kappa_3=-1/2$ by (3. 3). This shows $\kappa_2=-1/4$. So κ_1 is the largest eigenvalue of Q (as in the other cases). Since $b_{(\mathfrak{g},\mathfrak{k})}$ equals 1/6, for $(\mathfrak{g}, \mathfrak{k})$ whose compact form is (A_5, C_3), $b_{(\mathfrak{g},\mathfrak{k})}$ exceeds κ_1 and eventually $H_{(\mathfrak{g},\mathfrak{k})}$ is positive definite in this case.

(II) The case of CII: $(\mathfrak{g}_u, \mathfrak{k})=(C_l, C_p\times C_{l-p})$ $(p=1, 2)$. ρ is $\Lambda_1(C_p)$ $\otimes\Lambda_1(C_{l-p})$ and $\rho\vee\rho$ decomposed into the irreducible components shown in the first column of the following table.

$2\Lambda_1(C_p)\otimes 2\Lambda_1(C_{l-p})$	$\kappa_1=1/2(l+1)$
$\Lambda_2(C_p)\otimes\Lambda_2(C_{l-p})$	$\kappa_2=-1/2(l+1)$
$\Lambda_0(C_p)\otimes\Lambda_2(C_{l-p})$	$\kappa_3=-(p+1)/2(l+1)$
$\Lambda_2(C_p)\otimes\Lambda_0(C_{l-p})$	$\kappa_4=-(l-p+1)/2(l+1)$
$\Lambda_0(C_p)\otimes\Lambda_0(C_{l-p})$	$\kappa_5=-1/2$

For $p=1$, $-1/2(l+1)$ and $-(l+p+1)/2(l+1)$ must be omitted. Since $b_{(\mathfrak{g},\mathfrak{k})}=1/2(l+1)$ for $p=1$ and $b_{(\mathfrak{g},\mathfrak{k})}=1/(l+1)$ for $p=2$, we see that $H_{(\mathfrak{g},\mathfrak{k})}$ is non-negative but not positive definite for $p=1$ and positive definite for $p=2$.

(III) The case of $GI:(\mathfrak{g}_u, \mathfrak{k})=(G_2, A_1\times A_1)$

$$\rho = \Lambda_1(A_1)\otimes 3\Lambda_1(A_1)$$
$$\rho\vee\rho = 2\Lambda_1\otimes 6\Lambda_1+2\Lambda_1\otimes 2\Lambda_1+\Lambda_0\otimes 4\Lambda_1+\Lambda_0\otimes\Lambda_0$$

and the eigenvalues of $Q_{(\mathfrak{g}_u,\mathfrak{k})}$ are $1/4$, $-1/6$, $-1/4$, $-1/2$. Hence $H_{(\mathfrak{g},\mathfrak{k})}$ is non-negative but not positive definite.

(IV) The case of $FII:(\mathfrak{g}_u, \mathfrak{k})=(F_4, B_4)$

$$\rho = \Lambda_4(B_4)$$
$$\rho\vee\rho = 2\Lambda_4+\Lambda_1+\Lambda_0$$

The eigenvalues of Q are $1/18$, $-5/18$, $-1/2$. Therefore $H_{(\mathfrak{g},\mathfrak{k})}$ is positive definite.

(V) The case of $BDII:(\mathfrak{g}_u, \mathfrak{k})=(B_l, D_l)$ or (D_l, B_{l-1}). The symmetric Riemannian space $BDII$ is of the constant negative curvature. Since AII $(l=2)=A_3/C_2=D_3/B_2$, $AII(l=2)$ is contained in this case. If $\dim(BDII)=n$, we know $b_{(\mathfrak{g},\mathfrak{k})}=1/2(n-1)$ from Table I. On the other hand, the curvature tensor of the space of constant curvature is given by

$$R_{ijkl} = K(g_{il}g_{jk}-g_{ik}g_{jl})$$

From this relation and the fact that the Ricci tensor is $(-1/2)$-times metric tensor, we have $K=\frac{1}{2(n-1)}$. Therefore we see from the definition of $Q_{(\mathfrak{g},\mathfrak{k})}$ that

$$\begin{aligned} H_{(\mathfrak{g},\mathfrak{k})}(\xi) &= \frac{1}{2(n-1)}\sum_{i,l}\xi_{il}\xi^{il}+\frac{1}{2(n-1)}\Big[\Big(\sum_{i,l}g_{il}\xi^{il}\Big)^2-\sum_{i,l}\xi_{il}\xi^{il}\Big] \\ &= \frac{1}{2(n-1)}\Big(\sum_{i,l}g_{il}\xi^{il}\Big)^2 \end{aligned}$$

Hence $H_{(\mathfrak{g},\mathfrak{k})}$ is non-negative but not positive definite.

REMARK. For each symmetric pair $(\mathfrak{g}_u, \mathfrak{k})$ such that $\mathfrak{g}_u$ is simple and $\mathfrak{k}$ is of the same rank and semi-simple or simple, the system of the fundamental roots of $\mathfrak{k}^c$ is obtained from the diagram of the fundamental simplex of $\mathfrak{g}_u^c$ by omitting a vertex which we denote by the black vertex (cf. Borel-Siebenthal [3]). We can check case by case that the highest weight of the linear isotropy representation of $\mathfrak{k}^c$ coincides with the root $-\alpha$ where α is the root which corresponds to the black vertex.

§ 4. The Main Theorem.

In the following, we shall use the notation set down by Berger [1] for each simple Lie group.

Theorem 4.1. *The quadratic form $H^r_{(\mathfrak{g},\mathfrak{k})}$ is positive definite for the following values of r:*
(i) *if G is a complex simple Lie group with rank l,*

type of G	r	type of G	r
A_l	$r<\frac{l+1}{2}$	E_6	$r\leqslant 5$
B_l	$r\leqslant l-1$	E_7	$r\leqslant 8$
C_l	$r<\frac{l+1}{2}$	E_8	$r\leqslant 14$
D_l	$r\leqslant l-2$	F_4	$r\leqslant 4$
		G_2	$r\leqslant 1$

(ii) *if G is a real non-compact simple Lie group,*

type of G	r	type of G	r
$SL(l, \boldsymbol{R})$	$r<\frac{l+2}{4}$	E_7^1	$r\leqslant 4$
$SU^*(2l)$	$r<\frac{l-1}{2}$	E_7^2	$r\leqslant 3$
$SO^i(2l+1)$	$r<\mathrm{Min}\left(\frac{i}{2}, \frac{2l-i+1}{2}\right)$	E_8^1	$r\leqslant 7$
$SO^i(2l)$	$r<\frac{i}{2}\leqslant\frac{l}{2}$	E_8^2	$r\leqslant 5$
$Sp^i(l)$	$r<i\leqslant\frac{l}{2}$	G_2^*	$r<1$
E_6^1	$r\leqslant 3$	F_4^1	$r\leqslant 2$
E_6^2	$r\leqslant 2$	F_4^2	$r\leqslant 1$
E_6^4	$r\leqslant 1$		

Proof. If G is complex simple we have $b_{(\mathfrak{g},\mathfrak{k})}=1/4$ by [7]. And we know by lemma 2.2 the value of κ_u. (i) is a direct consequence of these facts. Next we consider the case of more general symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ with semi-simple $\mathfrak{k}$. If $\mathfrak{k}$ is of the same rank, we can easily calculate the maximal eigenvalue κ of $Q_{(\mathfrak{g}_u,\mathfrak{k})}$, using the method explained in § 3, and we obtain $\kappa=2\kappa_u$ for all cases other than *BII*, *DII*, *CII* and *FII*. As for *BII*, *DII*, *CII* and *FII*, we have $\kappa=\kappa_u$. If $\mathfrak{k}$ is not of the same rank, the computation of κ is very complicated but we know by Lemma 2.1 that $(b_{(\mathfrak{g},\mathfrak{k})}/r)-2\kappa_u>0$ implies $(b_{(\mathfrak{g},\mathfrak{k})}/r)-\kappa>0$. (ii) is an immediate consequence of these facts.

Using Theorem 4.1. and Theorem B, we have the following

Corollary. *Let G be a non-compact simple Lie group, K a maximal compact subgroup of G and Γ be a discrete subgroup of G without non-trivial element of finite order. Suppose that the quotient space G/Γ is compact. Then the r-th Betti number $b_r(K\backslash G/\Gamma)$ of a compact lacally symmetric space $K\backslash G/\Gamma$ equals the r-th Betti number $b_r(M_u)$ of the compact form M_u of $K\backslash G$ for the values of r which satisfy the condition in Theorem* 4.1.

§ 5. Betti numbers of G/Γ.

As a corollary of Theorem 4.1, we get

Theorem 5.1. *Let G be a connected semi-simple Lie group, each of whose simple factors is non-compact and not locally isomorphic to any of $SL(2, \boldsymbol{C})$, $SU^1(n)$, $SO^1(n)$, $Sp^1(n)$, G_2^*. Let Γ be a discrete subgroup with the compact quotient space G/Γ. Then the first Betti number of G/Γ vanishes.*

Proof. From the results of Matsushima [8] and of [7] and the consideration in §§ 2 and 3, the quadratic form H of each simple factor of G is positive definite. Therefore we see from Theorem A in § 1 that $b_1(G/\Gamma)=0$.

Theorem 5.2. *Let G be a complex simple Lie group with rank l and Γ be a discrete subgroup with the compact quotient space G/Γ. Let $b_i(G/\Gamma)$ be the i-th Betti number of G/Γ. Then*

(1) $b_2(G/\Gamma)=0$, *if* $l\geqslant 4$ *or if* $G=B_3$.

(2) $b_3(G/\Gamma)=2$ *and* $b_4(G/\Gamma)=0$, *if* $G=A_l$ $(l\geqslant 8)$, B_l $(l\geqslant 5)$, C_l $(l\geqslant 8)$, D_l $(l\geqslant 6)$, E_6, E_7, E_8 *or* F_4.

(3) $b_5(G/\Gamma)=1$ *or* 2 *if* $G=A_l$ $(l\geqslant 10)$; $b_5(G/\Gamma)=0$ *if* $G=B_l$ $(l\geqslant 6)$, C_l $(l\geqslant 10)$, D_l $(l\geqslant 7)$, E_6, E_7, *or* E_8.

Proof. Suppose that Γ has elements of finite order. Then we know in Selberg [10] that there exists a normal subgroup Γ_0 of Γ with finite index which contains no non-trivial element of finite order. Since G/Γ_0 is a finite covering of G/Γ, it is sufficient for $b_r(G/\Gamma)=0$ to show that $b_r(G/\Gamma_0)=0$. Hence we can suppose without loss of generality that Γ

contains no non-trivial element of finite order. Let K be a maximal compact subgroup of G. G/Γ is a principal fibre bundle over a compact locally symmetric space $K\backslash G/\Gamma$ with structure group K. From the hypothesis for the rank of G and from theorem 4.1, we see $b_i(K\backslash G/\Gamma) = b_i(K) = 0$ for $0 < i < 3$, since K is simple. Hence, by Serre [11], the following exact sequence of real cohomology groups associated to the above principal bundle is valid for the dimension $\leqslant 5$, since the structure group K is connected.

$$(5.1) \quad 0 \to H^1(K\backslash G/\Gamma) \to H^1(G/\Gamma) \to H^1(K) \to H^2(K\backslash G/\Gamma) \to H^2(G/\Gamma) \to H^2(K) \to \cdots \to H^5(K\backslash G/\Gamma) \to H^5(G/\Gamma) \to H^5(K)$$

(i) is a direct consequence of the above facts.

As for (2), we see from the hypothesis for the rank of G $b_4(K\backslash G/\Gamma) = b_4(K) = 0$, since K is simple (For the Betti numbers of compact simple Lie groups, see for instance [5], [6]). On the other hand, $H^3(K\backslash G/\Gamma) \cong H^3(K) \cong \boldsymbol{R}$. Therefore we get $H^3(G/\Gamma) \cong \boldsymbol{R} + \boldsymbol{R}$ from (5.1), which implies (2).

Using the above exact sequence, we know by the similar method that the later half of (2) and (3) are valid.

Theorem 5.3. *Let G be a non-compact real simple Lie group and Γ be a discrete subgroup of G with compact quotient space G/Γ. Then the second Betti number $b_2(G/\Gamma)$ of G/Γ equals zero if the type of G is E_6^i ($i=1,2$), E_7^i ($i=1, 2, 3$), E_8^i ($i=1,2$) or F_4^1, or if G is classical and satisfies the following conditions:*

type of G	l	txpe of G	l
$\mathrm{SL}(l+1, \boldsymbol{R})$	$l \geqslant 6$	$\mathrm{SO}^i(2l)$	$\frac{l}{2} \geqslant \frac{i}{2} > 2$
$\mathrm{SU}^*(2l)$	$l \geqslant 6$	$\mathrm{SO}^*(2l)$	$l \geqslant 7$
$\mathrm{SU}^i(l+1)$	$\frac{l+1}{2} \geqslant i \geqslant 5$	$\mathrm{Sp}(l, \boldsymbol{R})$	$l \geqslant 7$
$\mathrm{SO}^i(2l+1)$	$\mathrm{Min}\left(\frac{i}{2}, \frac{2l+1-i}{2}\right) > 2$	$\mathrm{Sp}^i(l)$	$\frac{l}{2} \geqslant i \geqslant 3$

Proof. We can suppose without loss of generality that Γ contains no non-trivial element of finite order. Let K be a maximal compact subgroup of G. First we consider the case where $K\backslash G$ has no complex structure. For this case, the first and the second Betti numbers of compact form M_u of $K\backslash G$ equal zero. From the hypothesis on G, we have $b_i(K\backslash G/\Gamma) = b_i(M_u) = 0$ for $0 < i < 3$. We have $b_i(K) = 0$ for $0 < i < 3$, since K is semi-simple. Therefore we get $b_2(G/\Gamma) = 0$ by the exact sequence (5.1). In the case where $K\backslash G$ has a complex structure, we know by Matsushima [9] that $H^1(G/\Gamma)=0$, $H^2(K\backslash G/\Gamma)\cong H^2(M_u)\cong \boldsymbol{R}$. On the other hand, we have $H^2(K)=0$. Therefore the exact sequence (5.1) are valid for the dimension <3, from which we obtain $H^2(G/\Gamma)=0$.

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