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AMITSUR'S COMPLEX FOR INSEPARABLE FIELDS[1)]

By

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To K. Shoda with deepest respect

0. Introduction and notations

In [3], Amitsur introduced a complex defined for any extension field F of a field C and proved that the second cohomology group, $H^2(F/C)$, of this complex is isomorphic to the Brauer group of central simple C-algebras split by F in case F is algebraic over C. Thus the nth cohomology group of Amitsur's complex provides a kind of higher dimensional analogue of the Brauer group. If F is normal, separable over C with Galois group G, Amitsur showed that this analogue is exactly $H^n(G, F^*)$, the nth cohomology of G with coefficients in the multiplicative group of F. These results were generalized to commutative rings in [12]. At the other extreme, if F is purely inseparable of exponent one over C, the Brauer group and $H^2(F/C)$ can be described as a group of Lie algebra extensions of the algebra of C-derivations of F. Thus in this case one would hope that the nth cohomology group of Amitsur's complex should provide some further information on this Lie algebra. The present paper, using a result of A. J. Berkson, shows that in fact this is not the case, that purely inseparable extensions play a small role in determining Amitsur's cohomology groups. Specifically, if $C \subset K \subset F$ is a tower of commutative rings of characteristic p and every element of K has p^eth power in C for some fixed positive integer e, then there is an exact sequence of Amitsur cohomology

$$\cdots \to H^n(K/C) \to H^n(F/C) \to H^n(F/K) \to H^{n+1}(K/C) \to \cdots$$

Berkson's result asserts that if C and K are fields and $e=1$, then $H^n(K/C)=0$ for $n \neq 2$, which then immediately extends to the case of arbitrary e, and moreover proves $H^n(F/C)=H^n(F/K)$ for $n \neq 2$. Furthermore, for $n=2$, we have an exact sequence

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$$0 \to H^2(K/C) \to H^2(F/C) \to H^2(F/K) \to 0$$

which gives new proofs of several known results on Brauer groups. This is the principal content of sections 4 and 6. Sections 2 and 3 contain the necessary lemmas, many of which are of intrinsic interest. In section 5 we show how the same spectral sequence techniques used in section 4 simplify the proofs in [4], and demonstrate that the short exact sequences derived there are, in the case of purely inseparable fields, the early terms of our long exact sequence. Section 7 provides some partial connection between $H^n(F/C)$ and $H^n(F_s/C)$ where F_s is the maximal separable subfield of F. Section 1 is a parenthetical section computing some homology groups, as contrasted with the cohomology groups of the rest of the paper.

Throughout this paper C will denote a commutative ring with unit. We shall be concerned primarily with commutative C-algebras (also with unit) which we shall denote by F, K, L. The only exceptions to this commutativity will be in sections 4, 6 and 7, where we shall use A for a central separable C-algebra.

Tensor products will always be tensor products over C unless otherwise indicated. Repeated tensor products will be denoted by exponents: $F^n = F \otimes_C F \otimes_C \cdots \otimes_C F$ to n factors (F^0 means C).

For any commutative ring F we denote by F^* the group of units of F.

With these notations, we recall several definitions connected with Amitsur's complex [3], [8, p. 15], [12]:

Let F be a C-algebra (commutative, as per our conventions) and for each $n=0, 1, \cdots$ define C-algebra homomorphisms $\varepsilon_i : F^n \to F^{n+1}$ ($i=1, 2, \cdots, n+1$) by $\varepsilon_i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n$. If $n=0$, the lone ε is the unit map $C \to F$ defined by $c \to c \cdot 1$. These ε's then also send the multiplicative group of units F^{n*} into $(F^{n+1})^*$. We define

$$\Delta_n^+ : F^n \to F^{n+1} \quad \text{by} \quad \Delta_n^+ = \Sigma(-1)^{i+1}\varepsilon_i$$

which makes the sequence of groups $\cdots \to \{0\} \to C \to F \to F^2 \to \cdots$ and mappings Δ^+ ($\Delta_n^+ = 0$ for $n < 0$) into a complex[2)] [3, Th. 5.1] which we denote $\mathfrak{C}^+(F/C)$. Amitsur's complex proper is the multiplicative analogue:

$$\Delta_n : F^{n*} \to (F^{n+1})^* \quad \text{is defined by} \quad \Delta_n(x) = \Pi\varepsilon_i(x)^{(-1)^{i+1}}$$

which makes the sequence of groups $\cdots \to \{1\} \to C^* \to F^* \to F^{2*} \to \cdots$ and mappings Δ into a complex [3, Th. 5.1] which we denote $\mathfrak{C}(F/C)$.

We denote by $H^n(F/C)$ (resp. $H^n(F/C)^+$) $\operatorname{Ker}\Delta_{n+1}/\operatorname{Im}\Delta_n$ (resp.

2) The indices of groups in a complex run from $-\infty$ to ∞.

$\operatorname{Ker}\Delta_{n+1}^{+}/\operatorname{Im}\Delta_n^{+}$). We write these cohomology groups additively. They obviously vanish when $n \leq -2$.

If F is a finitely generated free C-module, Amitsur [4] has also defined homology groups $H_n(F/C)$ as follows: In this context, F^{n+1} is a free, finitely generated module over $\mathcal{E}_i F^n$ so that the usual norm function is available (norm_i of an element x of F^{n+1} is the determinant of the $(\mathcal{E}_i F^n)$-linear endomorphism $y \to xy$ of F^{n+1}); furthermore, in this context, $\mathcal{E}_i$ is a monomorphism, and we define $\nu_i(x)$ to be $\mathcal{E}_i^{-1}$ ($\text{norm}_i\ x$), and $\delta_n(x) = \Pi \nu_i(x)^{(-1)^{i+1}}$. This gives a complex [4, § 1]

$$\to F^{n*} \to \cdots \to F^{2*} \to F^{*} \to C^{*} \to \{1\} \to \cdots$$

with mappings δ, and we denote by $H_n(F/C)$ the homology group $\operatorname{Ker}\delta_n/\operatorname{Im}\delta_{n+1}$.

Our primary interest is in the cohomology groups H^n, but we begin with a short section calculating some homology groups H_n. Amitsur showed in [4] that if F is a normal separable extension field of C with Galois group G, then $H_n(F/C) = H_n(G, F^*)$; we treat the other extreme, where F is a purely inseparable field extension of C of finite degree.

1. Homology for purely inseparable fields

We begin with a calculation of norms for purely inseparable rings.

Lemma 1.1. *Let C be a field of characteristic $p \neq 0$ and F a purely inseparable extension field with $[F:C] = q$. If C' is any commutative C-algebra, if $F' = F \otimes C'$ and if N' denotes the norm from F' to C', then $N'(x) = x^q$ for all x in F'.*

Proof. If F' were a field, the lemma would result from [14, p. 91, (15)]. We reduce to this case by the familiar device of computing N' for a "general element". Specifically, let $X_1, \cdots, X_q$ be independent independent indeterminates over C, let $C_1 = C(X_1, \cdots, X_q)$ and $F_1 = F \otimes C_1$. Then by [14, Corollary p. 186] F_1 is an integral domain. Since $q = [F_1 : C_1] = [F : C] < \infty$, F_1 is a field, clearly purely inseparable over C_1. Let N_1 denote the norm from F_1 to C_1, and let $X = \Sigma e_j \otimes X_j$ where $e_1, \cdots, e_q$ form a basis of F over C. Since the basis $\{e_j \otimes 1\}$ of F_1 over C_1 is equally a basis of $F_2 = F \otimes C_2$ over $C_2 = C[X_1, \cdots, X_q]$, the matrix of the endomorphism produced by multiplication by X in F_1 is the same as the matrix of the endomorphism produced in F_2, so that $N_1(X)$ is also the norm of X in F_2 over C_2. In fact, if we replace $X_1, \cdots, X_q$ by elements $x_1, \cdots, x_q$ of any commutative C-algebra C', this same matrix specializes to the matrix of multiplication by $x = \Sigma e_j \otimes x_j$ in $F' = F \otimes C'$

over C', and the corresponding norm $N'(x)$ is just the specialization of $N_1(X)$. Now $N_1(X)=X^q$ by the result for fields just quoted. This equality of polynomials still holds in C_2, and in C' after specializing the X_j. Since X^q specializes to x^q, this shows $N'(x)=x^q$, as desired.

Theorem 1.2. *Let F be a purely inseparable extension field of a field C with $[F:C]=q$. Let R_n be the radical of F^n. Then the Amitsur homology groups are the following (all groups written multiplicatively)*

$$H_n(F/C) \cong 1+R_{n+1} \qquad \text{if } n \text{ is even}$$
$$H_n(F/C) \cong F^*/F^{*q}\times(1+R_{n+1}) \qquad \text{if } n \text{ is odd.}$$

Proof. We apply Lemma 1.1 to $C'=\varepsilon_i F^n$, so that F' is isomorphic to F^{n+1}. We conclude that $\text{norm}_i\, x=x^q$, which is in C and independent of i. Thus $\nu_i(x)=x^q$; and $\delta_n(x)=1$ if n is odd, $\delta_n(x)=x^q$ if n is even.

We let $\beta_n: F^n \rightarrow F^{n+1}$ be the ring homomorphism $x\rightarrow x^q$ (the image is actually in C, thought of as a subring of F^{n+1}), and let β_n^* be the group-homorphism of F^{n*} which is the restriction of β_n. Then according as n is even or odd, $H_n(F/C)$ is the kernel or the cokernel of β_{n+1}^*. These groups can best be calculated by introducing another ring homomorphism $\theta_n: F^n\rightarrow F$ defined by $\theta_n(f_1\otimes\cdots\otimes f_n)=f_1f_2\cdots f_n$; again θ_n^* denotes the restriction of θ_n to F^{n*}. Since $\operatorname{Ker}\beta_n\subset\operatorname{Ker}\theta_n=$ the ideal generated by all $f\otimes 1\otimes\cdots\otimes 1-1\otimes\cdots\otimes 1\otimes f\otimes 1\otimes\cdots\otimes 1$ (cf., for example, [7; X, Prop. 3.1]), which is in turn contained in $\operatorname{Ker}\beta_n$, we have $\operatorname{Ker}\theta_n=\operatorname{Ker}\beta_n$ which is a nil ideal with residue class ring F. Thus $\operatorname{Ker}\theta_n=\operatorname{Ker}\beta_n=R_n$ and $\operatorname{Ker}\theta_n^*=\operatorname{Ker}\beta_n^*=1+R_n$, proving the theorem for even n.

Furthermore, $F^{n*}=X_n\times\operatorname{Ker}\theta_n^*$ where $X_n=F^*\otimes 1\otimes\cdots\otimes 1$, since θ_n^* induces an isomorphism $X_n\rightarrow F^*$. Since $\beta_n^*(X_n)\subset X_{n+1}$ and $\beta_n^*(\operatorname{Ker}\theta_n^*)=1$, we have $\operatorname{Coker}\beta_n^*=(X_{n+1}/\beta_n^*X_n)\times\operatorname{Ker}\theta_n^*\cong(F^*/F^{*q})\times\operatorname{Ker}\theta_n^*$, completing the proof.

2. Faithful flatness and the acyclicity of $\mathfrak{C}^+$

Following Bourbaki [6, Ch. I, § 3] we say a C-module F is *faithfully flat* provided a sequence $X\rightarrow Y\rightarrow Z$ of C-modules and mappings is exact if and only if the induced sequence $X\otimes F\rightarrow Y\otimes F\rightarrow Z\otimes F$ is exact. If F is a C-algebra, then F is a faithfully flat C-module if and only if

(2.1) *The unit map $\nu: C\rightarrow F$ (defined by $\nu(c)=c\cdot 1$) is a monomorphism and its cokernel $F/C\cdot 1$ is a flat C-module.* [6, Ch. I, § 3, Prop. 9] (This condition makes (C, F) a flat couple in the sense of Serre [13].)

If F is any C-algebra which is a flat C-module and whose unit map is a split monomorphism (i.e., there exists a C-module map $\mu: F\rightarrow C$

such that $\mu\circ\nu$ is the identity) then F is faithfully flat, because then Coker ν is a direct summand in the flat module F.

In [12, Lemma 4.1] the hypothesis that F has a split unit map was used to prove the acyclicity of $\mathfrak{C}^+(F/C)$, which, together with the assumption that F is flat, was then used to investigate $\mathfrak{C}(F/C)$. The preceding remark suggests that this acyclicity might be proved assuming only that F is faithfully flat. In the present paper we need this and slightly more, as in the next two lemmas. The main idea of the proof (Lemma 2.2), which consists of regarding $F\otimes F$ as an $(F\otimes 1)$-algebra with split unit map, is due to Grothendieck [8, Lemme 1.1, p. 18].

Let $F\to F'$ be a homomorphism of C-algebras with kernel U. This induces a homomorphism $F^n\to F'^n$ commuting with the ε's, hence a mapping of complexes $\mathfrak{C}^+(F/C)\to\mathfrak{C}^+(F'/C)$. The kernel of this mapping is a subcomplex $\mathfrak{C}^+(U)$ of $\mathfrak{C}^+(F/C)$ whose nth term U_n is the canonical image in F^{n+1} of $(U\otimes F\cdots\otimes F)+(F\otimes U\otimes\cdots\otimes F)+\cdots+(F\otimes\cdots\otimes F\otimes U)$ $(U_n=\{0\}$ for $n<0)$. This formula for U_n also shows directly that $\mathfrak{C}^+(U)$ is a subcomplex, because each ε carries U_n into U_{n+1}, hence so does Δ^+. Since each ε is an algebra homomorphism, it will also carry any power U_n^i of U_n into U_{n+1}^i. Thus the sequence of groups $\cdots, U_0^i, U_1^i, \cdots$ also forms a subcomplex, which we call $\mathfrak{C}^+(U)^i$. We let $\mathfrak{C}^+(U)^0$ mean $\mathfrak{C}^+(F/C)$. These subcomplexes form a chain

$$\mathfrak{C}^+(F/C)=\mathfrak{C}^+(U)^0\supset\mathfrak{C}^+(U)\supset\mathfrak{C}^+(U)^2\supset\cdots \tag{2.2}$$

If U is nilpotent, then each U_n is nilpotent (though of higher index) and so this chain of subcomplexes is finite in each dimension. Note that the nth term of $\mathfrak{C}^+(U)^i$ consists of all sums of terms $f_1\otimes\cdots\otimes f_n$ in F^{n+1} such that $f_k\in U^{j(k)}$ with $\Sigma_k j(k)\geq i$.

Lemma 2.1. *Let F be a C-algebra with a split unit map, i.e., with a C-module homomorphism $\varphi: F\to C$ such that $\varphi\circ\nu$ is the identity (i.e., $\varphi(c\cdot 1)=c$). Let U be an ideal in F such that $\nu\circ\varphi(U^j)\subset U^j$ for all j. Then $\mathfrak{C}^+(U)^i$ is acyclic for all $i\geq 0$ (*[12, Lemma 4.1] *is the case $U=F$, $i=0$).*

Proof. As in [12, Lemma 4.1] or [8, A4], $\mathfrak{C}^+(F/C)$ has a contracting homotopy $s: F^{n+1}\to F^n$ defined by $s(f_1\otimes\cdots\otimes f_n)=\varphi(f_1)f_2\otimes f_3\otimes\cdots\otimes f_n$. The factor $\varphi(f_1)f_2$ may equally well be written $(\nu\circ\varphi)(f_1)f_2$. Thus if $f_k\in U^{j(k)}$ with $\Sigma_k j(k)\geq i$, then $\varphi(f_1)f_2\in U^{j(1)+j(2)}$ and $s(f_1\otimes\cdots\otimes f_n)\in U_{n-1}^i$. This shows $s(U_n^i)\subset U_{n-1}^i$ so that s is also a homotopy in $\mathfrak{C}^+(U)^i$, proving the latter is acyclic.

Lemma 2.2. *Let $F\to F'$ be an epimorphism of C-algebras with kernel*

U. If either F is faithfully flat over C and $U=FU_0$ with $U_0\subset C$ or if F' is faithfully flat over C (which implies $U\cap C\cdot 1=0$), then $\mathfrak{C}^+(U)^i$ is acyclic for all $i\geq 0$.

Proof. Take $K=F$ or F' in the two cases envisioned in the lemma, so that K is an F-module in a natural way and is faithfully flat over C. It will suffice to show $K\otimes\mathfrak{C}^+(U)^i$ is acyclic. Since K is flat, $K\otimes\mathfrak{C}^+(U)$ is the kernel of $K\otimes\mathfrak{C}^+(F/C)\rightarrow K\otimes\mathfrak{C}^+(F'/C)$ and the latter two complexes may be identified[3] with $\mathfrak{C}^+(K\otimes F/K)$ and $\mathfrak{C}^+(K\otimes F'/K)$. This identifies $K\otimes\mathfrak{C}^+(U)$ with $\mathfrak{C}^+(V)$ where V is the kernel of $K\otimes F\rightarrow K\otimes F'$, i.e., $V=K\otimes U$. Similarly, $K\otimes\mathfrak{C}^+(U)^i$ is identified with $\mathfrak{C}^+(V)^i$. To apply Lemma 2.1 to the K-algebra $K\otimes F$, we notice that the unit map $k\rightarrow k\otimes 1$ is split, the reverse map being $\varphi(\Sigma k_i\otimes f_i)=\Sigma k_i f_i$. We need to check that $(\nu\circ\varphi)(V^i)\subset V^i$. In the first case where $K=F$, we have $V^i=(K\otimes U_0F)^i=U_0^i(K\otimes F)$, $\varphi(V^i)=U_0^i\varphi(K\otimes F)=U_0^iK$, $(\nu\circ\varphi)(V^i)=U_0^iK\otimes 1=K\otimes U_0^i\subset V^i$. In the second case where $K=F'$, we have $V^i=(F'\otimes U)^i=F'\otimes U^i$, $\varphi(V^i)=F'U^i\subset F'U=0$. This completes the proof.

3. Reduction of $\mathfrak{C}(F/C)$

We now pass to the multiplicative complex $\mathfrak{C}(F/C)$. An obvious corollary of Lemma 2.2 in low dimensions is

Lemma 3.1. *If F is a faithfully flat C-algebra then $H^{-1}(F/C)=H^0(F/C)=0$.*

Proof. Lemma 2.2 with $U=0$ and $i=0$ asserts that $H^n(F/C)^+=0$ for all n. The cases $n=-1$ and $n=0$ may be expressed thus: The unit map $\nu: C\rightarrow F$ is a monomorphism; and if $x\in F$ then $\varepsilon_1x=\varepsilon_2x$ if and only if $x\in\nu(C)$. If we restrict to units, we see that $\nu: C^*\rightarrow F^*$ is a monomorphism; and if $x\in F^*$ then $(\varepsilon_1x)(\varepsilon_2x)^{-1}=1$ if and only if $x\in\nu(C^*)$, which proves Lemma 3.1.

The results on $\mathfrak{C}^+$ in section 2 can also be made to yield results on $H^n(F/C)$ for larger n by the following device: Just as we had a chain of additive complexes (2.2), we can define a chain of multiplicative complexes

$$\mathfrak{C}(F/C)=\mathfrak{C}(U)^0\supset\mathfrak{C}(U)\supset\mathfrak{C}(U)^2\supset\cdots \tag{3.1}$$

for any ideal U in F. If U is the kernel of the homomorphism $F\rightarrow F'$, $\mathfrak{C}(U)^i$ is to be the complex whose nth term is the multiplicative group

3) For example, $K\otimes F^n=(K\otimes F)\otimes_K(K\otimes F)\otimes_K\cdots\otimes_K(K\otimes F)$ to n factors. In fact, we shall have much occasion to identify $K\otimes F^n$ with the n-fold tensor product of $K\otimes F$ over K.

of units contained in $1+U_n^i$ where U_n, as before, is the kernel of $F^{n+1}\to F'^{n+1}$. Again $\mathfrak{C}(U)$ is the kernel of $\mathfrak{C}(F/C)\to\mathfrak{C}(F'/C)$ since a unit x in F^{n+1} maps to 1 if and only if $x-1$ maps to 0 (i.e., $x-1\in U_n$). If U is a nilpotent ideal, then so is U_n^i, and the nth term of $\mathfrak{C}(U)^i$ is just $1+U_n^i$; furthermore, in each dimension, the chain (3.1) will terminate with $\{1\}$ in a finite number of steps.

The device we use is the fact that the factor complexes in the chain (3.1) are isomorphic to those of the chain (2.2):

Lemma 3.2. *If U is an ideal in F, there is an isomorphism of complexes*

$$\mathfrak{C}(U)^i/\mathfrak{C}(U)^{i+1}\cong\mathfrak{C}^+(U)^i/\mathfrak{C}^+(U)^{i+1}$$

for each $i\geq 1$. The isomorphism is induced by the mapping $1+u\to u$ for $u\in U_n^i$.

Proof. Clearly if u and v are in U_n^i then $(1+u)(1+v)\in 1+(u+v)+U_n^{i+1}$. Thus the given mapping induces an isomorphism of $(1+U_n^i)/(1+U_n^{i+1})$ to U_n^i/U_n^{i+1}. That this commutes with the boundary operators is direct and immediate.

Proposition 3.3. *Let $F\to F'$ be an epimorphism of C-algebras with nilpotent kernel U. Assume either that F is faithfully flat over C and $U=FU_0$ with $U_0\subset C$ or that F' is faithfully flat. Then the mapping of complexes $\mathfrak{C}(F/C)\to\mathfrak{C}(F'/C)$ induces an isomorphism*

$$H^n(F/C)\cong H^n(F'/C)\quad \textit{for all } n.$$

Proof. By Lemma 2.2 $\mathfrak{C}^+(U)^i$ is acyclic for every i. Using the homology sequence corresponding to the sequence $0\to\mathfrak{C}^+(U)^{i+1}\to\mathfrak{C}^+(U)^i\to\mathfrak{C}^+(U)^i/\mathfrak{C}^+(U)^{+1}\to 0$, we deduce the acyclicity of $\mathfrak{C}^+(U)^i/\mathfrak{C}^+(U)^{i+1}$, which is isomorphic to $\mathfrak{C}(U)^i/\mathfrak{C}(U)^{i+1}$ by Lemma 3.2. Returning via the analogous homology sequence for the complexes $\mathfrak{C}$, we have $H^n(\mathfrak{C}(U)^i)\cong H^n(\mathfrak{C}(U)^{i+1})$ for all n and all $i\geq 1$. But for each n there is a j such that $1+U_n^j=\{1\}$, so that $H^n(\mathfrak{C}(U)^j)=0$. These facts together imply $H^n)\mathfrak{C}(U)^{j-1})=\cdots=H^n(\mathfrak{C}(U))=0$. The homology sequence corresponding to the exact sequence $1\to\mathfrak{C}(U)\to\mathfrak{C}(F/C)\to\mathfrak{C}(F'/C)\to 1$ then shows that $\mathfrak{C}(F/C)$ and $\mathfrak{C}(F'/C)$ have isomorphic homology.

Theorem 3.4. *Let F be a commutative C-algebra and F' a commutative C'-algebra, each faithfully flat (so that C and C' may be considered to be subrings of F and F'; cf. (2.1)). Suppose that $F\to F'$ is a ring epimorphism with nilpotent kernel U which carries C onto C'. Then the* *natural mapping of complexes* $\mathfrak{C}(F/C)\to\mathfrak{C}(F'/C')$ *induces an isomorphism of cohomology*

$$H^n(F/C)\cong H^n(F'/C') \quad \textit{for all } n.$$

Proof. We factor the given epimorphism as follows

$$F\to F\otimes_C C'\to F'.$$

The first of these mappings is a mapping of C-algebras with kernel $F(U\cap C)$ so that the first set of hypotheses in Proposition 3.3 is satisfied. The second mapping is a mapping of C'-algebras satisfying the second set of hypotheses in Proposition 3.3. Two applications of Proposition 3.3 thus complete the proof.

We shall need an easy generalization of Theorem 3.4.

Corollary 3.5. *Let* $\{\alpha\}$ *be a directed system,* $\{C_\alpha\}$ *a directed set of rings,* $\{F_\alpha\}$ *and* $\{F'_\alpha\}$ *directed sets of* C_α*-algebras with mappings* $F_\alpha\to F'_\alpha$ *for each* α *that commute with the mappings of the directed sets. Assume that* F_α *is faithfully flat over* C_α, F'_α *is faithfully flat over* C'_α (*=image of* C_α *under the mapping* $F_\alpha\to F'_\alpha$) *and* $\operatorname{Ker}(F_\alpha\to F'_\alpha)$ *is nilpotent. Let* F, F', C' *denote the direct limits of* F_α, C_α, F'_α, C'_α. *Then there is an induced isomorphism of cohomology*

$$H^n(F/C)\cong H^n(F'/C') \quad \textit{for all } n.$$

Proof. Since tensor product commutes with direct limits, we have $F^n=\varinjlim F^n_\alpha$ whence $F^{n*}=\varinjlim F^{n*}_\alpha$ [if $x\in F^{n*}$ there are x_α, y_α in F^n_α mapping on x and x^{-1} respectively; then $x_\alpha y_\alpha-1$ maps to 0 so for some $\beta>\alpha$, $x_\alpha y_\alpha-1$ maps to 0 in F^n_β; the image of x_α in F^n_β is a unit which maps to x in F^n] and $\mathfrak{C}(F/C)=\lim\mathfrak{C}(F_\alpha/C_\alpha)$. Since homology commutes with direct limits [7, Prop. 9.3*], we have $H^n(F/C)\cong\lim H^n(F_\alpha/C_\alpha)\cong\varinjlim H^n(F'_\alpha/C'_\alpha)\cong H^n(F'/C')$.

REMARK. It is clear that we could also generalize Proposition 3.3 to direct limits in a similar fashion.

Corollary 3.6. *Let* F *be a faithfully flat* C*-algebra and let* $K\to K'$ *be an epimorphism of* C*-algebras with nilpotent kernel. Then the mapping of complexes* $\mathfrak{C}(K\otimes F/K)\to\mathfrak{C}(K'\otimes F/K')$ *induces an isomorphism of cohomology*

$$H^n(K\otimes F/K)\cong H^n(K'\otimes F/K') \quad \textit{for all } n.$$

Proof. $K\otimes F$ and $K'\otimes F$ are faithfully flat K- and K'-algebras respectively, since they verify (2.1). Since the kernel of $K\otimes F\to K'\otimes F$

is still nilpotent, Theorem 3.4 is applicable and yields the result.

4. Spectral sequences and the basic exact sequence

Let K and F be C-algebras and denote by $E_0^{m,n}$ the group of units in $K^m \otimes F^n$, $m, n \geq 0$. If m or n is negative, $E_0^{m,n}$ is to be the trivial group. If we fix m and vary n, we get exactly the terms in $\mathfrak{C}(K^m \otimes F/K^m)$ (cf. footnote[3]); the corresponding coboundary operator we denote by $\Delta_F^{m,n}$. If we fix n and vary m, we get $\mathfrak{C}(K \otimes F^n/F^n)$; its coboundary operator we denote by $\Delta_K^{m,n}$. These two operators make $E_0^{m,n}$ a double complex [4, Th. 3.1] [7, p. 60]. As usual, $E^k = \Sigma_{m+n=k+1} E_0^{m,n}$ is an ordinary complex whose kth cohomology group we denote by $H^k(K, F/C)$. Corresponding to this double complex are two spectral sequences, both converging to $H^k(K, F/C)$ [7, XV §6]. For the first of these, the E_1 term is just the homology of E_0 with coboundary Δ_F:

$$'E_1^{m,n} = H^{n-1}(K^m \otimes F/K^m) .$$

On this $'E_1$, Δ_K induces a coboundary operator $\bar{\Delta}_K^{m,n} : {}'E_1^{m,n} \to {}'E_1^{m+1,n}$; the corresponding homology is $'E_2$. The second spectral sequence reverses the roles of Δ_K and Δ_F:

$$''E_1^{m,n} = H^{m-1}(K \otimes F^n/F^n)$$

and $''E_2$ is the homology $''E_1$, using the operator $\bar{\Delta}_F$ induced by Δ_F.

The restriction map ρ_n: $H^n(F/C) \to H^n(K \otimes F/K)$, introduced by Amitsur [4, §2] as a generalization of the ordinary restriction map of Galois cohomology is induced by the mapping $f \to f \otimes 1$ from F^n to $F^n \otimes K$, and is exactly the mapping $\bar{\Delta}_K^{0,n+1} : {}'E_1^{0,n+1} \to {}'E_1^{1,n+1}$. We shall use the shorter notation ρ_n instead of $\bar{\Delta}_K^{0,n+1}$ when convenient. We shall use the same notation, ρ_n, occasionally for the corresponding mapping with K replaced by K^m. Note that $'E_2^{0,n}$ is just $\operatorname{Ker} \rho_{n-1} : H^{n-1}(F/C) \to H^{n-1}(K \otimes F/C)$, since $'E_1^{m,n} = 0$ when m or n is negative, so that $'E_2^{0,n} = \operatorname{Ker} ({}'E_1^{0,n} \to {}'E_1^{1,n}) / \operatorname{Im} ({}'E_1^{-1,n} \to {}'E_1^{0,n}) = \operatorname{Ker} \rho_{n-1}$.

Furthermore, $\operatorname{Ker} \rho_2$ has a simple interpretation: Under suitable hypotheses, $H^2(F/C)$ may be identified with the Brauer group of central separable C-algebras with F as splitting ring [12, Th. 3]; with this identification, ρ_2 becomes the mapping which associates with such an algebra A the K-algebra $K \otimes A$ (which, of course, is split by $K \otimes F$). Thus $\operatorname{Ker} \rho_2$ is the subgroup of the Brauer group determined by the C-algebras split by both F and K [4, Proof of Th. 3.2].

One final introductory remark: We use only the most elementary properties of the later terms $'E_r$ and $''E_r$ in the spectral sequence [7, XV], viz., (4.1) $'E_{r+1}^{m,n} = \operatorname{Ker}('E_r^{m,n} \to 'E_r^{m+r,n-r+1})/\operatorname{Im}('E_r^{m-r,n+r-1} \to 'E_r^{m,n})$ (*exactly what the mappings are plays no role*);

(4.2) $$'E_\infty^{m,n} = 'E_r^{m,n} \quad \textit{forall large } r;$$

(4.3) *$H^k(K, F/C)$ has a chain of subgroups with corresponding factor groups $\{'E_\infty^{m,n} \mid m+n = k+1\}$.*

Immediate corollaries of these are

(4.4) $$'E_r^{m,n} = 0 \quad \textit{implies} \quad 'E_s^{m,n} = 0 \quad \textit{for} \quad s = r, r+1, \cdots, \infty;$$

(4.5) *if, among the $'E_\infty^{m,n}$ with $m+n=k+1$ there is only one nonzero one, then this one is isomorphic to $H^k(K, F/C)$;*

(4.6) *$'E_\infty^{m,n} = 'E_r^{m,n}$ if all $'E_r^{i,j}$ vanish for $i+j=m+n+1$, $i \geq m+r$, and also vanish for $i+j=m+n-1$, $i \leq m-r$.*

The same properties hold for $''E$ except that the roles of m and n must be interchanged. For example, we shall need (4.6) for $''E$ only in case $r=1$ $n=0$; it reads

(4.6′) *$''E_\infty^{m,0} = ''E_1^{m,0}$ if all $''E_1^{i,j} = 0$ for $i+j = m+1$, $j \geq 1$ (and all $''E_1^{i,j} = 0$ for $i+j = m-1$, $j \leq -1$, but this is automatic).*

Proposition 4.1. *Let K and F be C-algebras. Suppose that F is faithfully flat over C and that the homomorphism $\theta: K^m \to K$ given by $\theta(k_1 \otimes k_2 \otimes \cdots \otimes k_m) = k_1 k_2 \cdots k_m$ has a nilpotent kernel for each $m > 1$. Then there is an exact sequence*

$$\cdots \xrightarrow[\rho_{n-1}]{} H^{n-1}(K \otimes F/K) \to H^n(K, F/C) \longrightarrow H^n(F/C) \xrightarrow[\rho_n]{} H^n(K \otimes F/K) \to \cdots$$

Proof. Since F^n is also faithfully flat, Corollary 3.6 asserts that the map $\theta \otimes 1: K^m \otimes F^n \to K \otimes F^n$ induces isomorphisms

$$\theta^{m,n}: \quad 'E_1^{m,n} \longrightarrow 'E_1^{1,n}$$

for $m \geq 1$. To compute $'E_2$, we compute the operator on $'E_1^{1,n}$ into which $\overline{\Delta}_K$ is carried by these θ's. Since the following diagram commutes

$$\begin{tikzcd} K^m \otimes F^n \arrow[r, "\theta \otimes 1"] \arrow[d, "\varepsilon_i \otimes 1"'] & K \otimes F^n \arrow[d, "1"] \\ K^{m+1} \otimes F^n \arrow[r, "\theta \otimes 1"'] & K \otimes F^n \end{tikzcd}$$

and $\Delta_K^{m,n}$ is an alternating product of $m+1$ $\mathcal{E}$'s, we have a commutative diagram

$$\begin{array}{ccc} {}'E_1^{m,n} & \xrightarrow{\theta^{m,n}} & {}'E_1^{1,n} \\ {\scriptstyle \overline{\Delta}_K^{m,n}}\downarrow & & \downarrow{\scriptstyle d_m} \\ {}'E_1^{m+1,n} & \xrightarrow[\theta^{m+1,n}]{} & {}'E_1^{1,n} \end{array}$$

where d_m is the identity for even m and the zero map for odd m. Thus the complex $\{'E_1^{m,n}, \overline{\Delta}_K^{m,n}, m=0, 1, \cdots\}$ is isomorphic to the complex $\{'E_1^{1,n}, d_m, m=0, 1, \cdots\}$, which is acyclic for $m>1$. It follows that $'E_2^{m,n}=0$ for $m>1$, and the map $\overline{\Delta}_K^{1,n}: {}'E_1^{1,n}\to{}'E_1^{2,n}$ is zero. Thus $'E_2$ has only two nonzero columns, viz., $m=0$ and $m=1$. By (4.4) and (4.6), $'E_\infty^{m,n}={}'E_2^{m,n}$ and by (4.3) there is an exact sequence

$$(4.7) \qquad 0\to {}'E_2^{1,n}\to H^n(K, F/C)\to {}'E_2^{0,n+1}\to 0 .$$

Furthermore, since $\overline{\Delta}_K^{1,n}=0$ as above, $'E_2^{1,n}={}'E_1^{1,n}/\mathrm{Im}\,\overline{\Delta}_K^{0,n}=\mathrm{Coker}\,\rho_{n-1}$ and $'E_2^{0,n}=\mathrm{Ker}\,\overline{\Delta}_K^{0,n}=\mathrm{Ker}\,\rho_{n-1}$. Thus the short exact sequences (4.7) can be combined into the long exact sequence of the Proposition.

Lemma 4.2. *Let F be a K-algebra and K a C-algebra. Then*

$$H^k(K, F/C)\cong H^k(K/C) \quad \textit{for all } k .$$

Proof. This is a slight generalization of [4, Theorem 3.3] and is proved in fundamentally the same way: The mapping $u: K^m\otimes F^n\to K^{m-1}\otimes F^n$ defined by $u(k_1\otimes\cdots\otimes k_m\otimes f_1\otimes\cdots\otimes f_n)=k_1\otimes\cdots\otimes k_{n-1}\otimes k_n f_1\otimes\cdots\otimes f_n$ is a contracting homotopy in $\mathfrak{C}(K\otimes F^n/F^n)$. Thus $''E_1^{m,n}=0$ when $n\geq 1$. (The double complex has exact rows except for the row $n=0$; the spectral sequence $''E$ collapses.) Therefore, by (4.4), (4.5) and (4.6), $H^k(K, F/C)$ $={}''E_\infty^{k+1,0}={}''E_1^{k+1,0}=H^k(K\otimes C/C)$.

Theorem 4.3. *Let F be a K-algebra and K be a C-algebra and assume F is faithfully flat over both K and C. Assume further that the mapping $\theta: K^m\to K$ has a nilpotent kernel as in Proposition 4.1. Then there is an exact sequence*

$$\cdots\to H^{n-1}(F/K)\to H^n(K/C)\to H^n(F/C)\to H^n(F/K)\to\cdots$$

Proof. By Proposition 4.1 and Lemma 4.2 we see that we need only show $H^n(F/K)\cong H^n(K\otimes F/K)$. Let N be the kernel of $\theta: K\otimes K\to K$, so that $0\to N\to K\otimes K\to K\to 0$ is exact. Consider this as a sequence of K-modules, with K acting on N and on $K\otimes K$ as $1\otimes K$; tensor over

K with F to get an exact sequence $0 \to N \otimes_K F \to K \otimes F \to F \to 0$. Thus the kernel of the $(1 \otimes K)$-algebra epimorphism $K \otimes F \to F$ is nilpotent, and Proposition 3.3 gives the required isomorphism of homology groups.

REMARK. It is not difficult to trace the mappings through the various isomorphisms in our proofs to see that the mapping $H^n(K/C) \to H^n(F/C)$ is the analogue of the lift mapping in Galois cohomology and is induced by the natural mapping $K^{n+1} \to F^{n+1}$ (in turn induced by $K \to F$); and the mapping $H^n(F/C) \to H^n(F/K)$ is the analogue of the restriction and is induced by the natural mapping $F \otimes \cdots \otimes F \to F \otimes_K \cdots \otimes_K F$.

Corollary 4.4. *Suppose $\{K_\alpha\}$ is a directed set of C-algebras with direct limit K, and suppose that the kernel of $\theta: K_\alpha^m \to K_\alpha$ is nilpotent for every m and every α. If F is any faithfully flat C-algebra then the sequence in Proposition* 4.1 *is exact.*

If, besides, F is a K-algebra (hence a K_α-algebra for each α) and is faithfully flat over each K_α, then the exact sequence in Theorem 4.3 *is exact.*

Proof. As in Corollary 3.5, the complexes $\mathfrak{C}(K_\alpha/C)$ form a directed system with limit $\mathfrak{C}(K/C)$. Similarly $\mathfrak{C}(K_\alpha \otimes F/K_\alpha)$ has direct limit $\mathfrak{C}(K \otimes F/K)$, the double complex $\{E_0^{m,n}\}$ defined for K_α and F has direct limit equal to the double complex defined for K and F, and $\mathfrak{C}(F/K_\alpha)$ has direct limit $\mathfrak{C}(F/K)$ [7; VI, Ex. 17]. Since direct limit commutes with homology, $H^n(K/C)$, $H^n(K, F/C)$, and $H^n(F/K)$ are the direct limits of $H^n(K_\alpha/C)$, $H^n(K_\alpha, F/C)$ and $H^n(F/K_\alpha)$ respectively. Since direct limit is an exact functor, the exact sequences for F and K_α given by Proposition 4.1 and Theorem 4.3 are carried by direct limits into the corresponding exact sequence for F and K.

5. Remarks on [4]

The techniques in the previous section can also be used to prove many theorems in [4], resulting in shorter proofs and in slightly weaker hypotheses. We reproduce some of these shorter proofs here, since the spectral sequence technique demonstrates that Amitsur's exact sequences are exactly the same as the early terms in the exact sequences of the preceding section. All these exact sequences are inspired by those of Hochschild-Serre [11, Th. 2 and §5] read down to Galois cohomology.

Proposition 5.1. [4, Theorem 3.2]. *Let F be a faithfully flat C-algebra and K any C-algebra. Then $H^0(K, F/C) = 0$ and $H^1(K, F/C) \cong \operatorname{Ker} \rho_1$ where $\rho_1: H^1(F/C) \to H(K \otimes F/K)$ is as defined at the beginning of* §4. *If*

$H^1(K\otimes F/K) = H^1(K^2\otimes F/K^2) = 0$ (or, more generally, $H^1(F/C)\to H^1(K\otimes F/K)\to H^1(K^2\otimes F/K^2)\to H^1(K^3\otimes F/K^3)$ is exact, all the mappings being of the form ρ_1), then also $H^2(K, F/C)\cong\operatorname{Ker}\rho_2$ where ρ_2: $H^2(F/C)\to H^2(K\otimes F/K)$.

Proof. By Lemma 3.1, $'E_1^{m,0}={}'E_1^{m,1}=0$. Hence $'E_1^{m,n}=0$ when $m+n=1$; by (4.4) and (4.5), $H^0(K, F/C)=0$. When $m+n=2$, $'E_1^{m,n}=0$ except for $'E_1^{0,2}$. Thus by (4.4) and (4.5), $H^1(K, F/C)={}'E_\infty^{0,2}$ which, by (4.6), equals $'E_2^{0,2}=\operatorname{Ker}\rho_1$. The last hypotheses of the Proposition assert $'E_1^{1,2}={}'E_1^{2,2}=0$ (or, more generally, $'E_2^{1,2}={}'E_2^{2,2}=0$) so that when $m+n=3$, $'E_2^{m,n}=0$ except $'E_2^{0,3}=\operatorname{Ker}\rho_2$. Similarly, $H^2(K, F/C)={}'E_\infty^{0,3}={}'E_2^{0,3}=\operatorname{Ker}\rho_2$.

REMARKS. 1. By [4, Th. 3.8], $H^1(K^m\otimes F/K^m)=0$ whenever K^m is a finite direct sum of (not necessarily Noetherian) local rings. Thus the hypotheses in Proposition 5.1 are surely satisfied if C is a field and F and K are finite dimensional C-algebras, so that in this case $H^1(K, F/C)=0$ and $H^2(K, F/C)=\operatorname{Ker}\rho_2$.

2. For the significance of $\operatorname{Ker}\rho_2$, see the comments above Proposition 4.1.

Proposition 5.2. [4, Corollary 3.4] *If F and K are C-algebras and if $H^m(K\otimes F^n/F^n)=0$ whenever $n\neq 0$ and $m+n$ is either k or $k+1$, then $H^k(K, F/C)\cong H^k(K/C)$.*

Proof. The assumptions translate to; $''E_1^{m,n}=0$ when $m+n=k+1$ or $k+2$ and $n\geq 1$. By (4.4), (4.5) and (4.6'), $H^k(K, F/C)={}''E_\infty^{k+1,0}={}''E_1^{k+1,0}=H^k(K/C)$.

For the next proposition we use the notation $H^2(K\otimes F/K)^0$ for the kernel of the mapping $\bar{\Delta}_K^{1,3}$: $'E_1^{1,3}\to{}'E_1^{2,3}$ namely, the set of elements x in $H^2(K\otimes F/K)$ $(={}'E_1^{1,3})$ such that $\varepsilon_1^*x=\varepsilon_2^*x$ where the ε_i^* are the mappings induced by the ε_i: $K\to K^2$ used in defining $\mathfrak{C}(K/C)$.

Proposition 5.3 [4, Theorems 4.1 and 4.2]. *Let F and K be faithfully flat C-algebras. If*

$$\begin{aligned}H^1(K\otimes F/F) &= H^1(K\otimes F^2/F^2) = H^2(K\otimes F/F) = H^1(K\otimes F/K)\\ &= H^1(K^2\otimes F/K^2) = 0\end{aligned}$$

then there is an exact sequence

$$0\to H^2(K/C)\to H^2(F/C)\xrightarrow{\rho_2} H^2(K\otimes F/K)\,.$$

If, besides, $H^1(K^3\otimes F/K^3)=H^1(K\otimes F^3/F^3)=H^2(K\otimes F^2/F^2)=H^3(K\otimes F/F)=0$, this can be extended to an exact sequence

$$0 \to H^2(K/C) \to H^2(F/C) \xrightarrow{\rho_2} H^2(K \otimes F/K)^0 \to H^3(K/C) \to H^3(F/C).$$

Proof. Proposition 5.1 with F and K interchanged shows that

$$H^2(K, F/C) = \operatorname{Ker}(\rho_2' : H^2(K/C) \to H^2(K \otimes F/F)) = H^2(K/C).$$

Also, Proposition 5.1 in its original form shows

$$0 \to H^2(K, F/C) \to H^2(F/C) \xrightarrow{\rho_2} H^2(K \otimes F/K)$$

is exact. These two facts prove the first conclusion of Proposition 5.3.

Finally, we examine $'E_\infty^{m,n}$ with $m+n=4$. Lemma 3.1 asserts that $'E_1^{m,0} = 'E_1^{m,1} = 0$ for all m. Then by (4.4), $'E_\infty^{4,0} = 'E_\infty^{3,1} = 0$. By hypothesis and (4.4), $0 = H^1(K^2 \otimes F/K^2) = 'E_1^{2,2} = 'E_\infty^{2,2}$. The last hypothesis on $'E$ is $0 = H^1(K^3 \otimes F/K^3) = 'E_1^{3,2}$ which allows us to compute $'E_\infty^{1,3}$ by (4.6) with $r=2$ since $'E_2^{i,j} = 0$ automatically for $i \leq 1-2$. Thus $'E_\infty^{1,3} = 'E_2^{1,3} = \operatorname{Ker} \overline{\Delta}_K^{1,3} / \operatorname{Im} \rho_2 = H^2(K \otimes F/K)^0 / \operatorname{Im} \rho_2$. Lastly, by (4.1), $'E_{r+1}^{0,4} = \operatorname{Ker}('E_r^{0,4} \to 'E_r^{r,5-r}) / \{0\} \subset 'E_r^{0,4}$, so that $'E_\infty^{0,4} \subset 'E_1^{0,4}$. By (4.3) we get an exact sequence $0 \to 'E_\infty^{1,3} \to H^3(K, F/C) \to 'E_\infty^{0,4} \to 0$, which, in view of the facts just computed for the $'E_\infty$, gives an exact sequence

$$H^2(F/C) \xrightarrow{\rho_2} H^2(K \otimes F/K)^0 \to H^3(K, F/C) \to H^3(F/C).$$

The last set of hypotheses in the Proposition allows us to use Proposition 5.2 to show $H^3(K, F/C) \cong H^3(K/C)$, which completes the proof.

REMARKS. 1. As we remarked before, if C is a field and K and F are finite-dimensional over C, all the H^1's in the hypotheses of Proposition 5.3 will vanish. If, besides, F is a K-algebra, the remaining hypotheses will also be satisfied ([4, Theorem 2.9] or rather the fact, proved in the proof of Lemma 4.2, that $''E_1^{m,n} = H^m(K \otimes F^n/F^n) = 0$ when $n \neq 0$).

2. In case F is a normal, separable extension field of C and K an intermediate field, the exact sequences in the proposition reduce to the Hochschild-Serre sequences for Galois cohomology [11, §5].

3. From the constructions involved, it is clear that these exact sequences coincide with the early portions of the long exact sequence of Theorem 4.3, when the latter is applicable.

6. Inseparable fields

In the rest of this paper we assume C is a field of characteristic $p \neq 0$. If F is an extension field, we shall use the notation $F^{\cdot p^e}$ to denote $\{x^{p^e} \mid x \in F\}$ to distinguish this from the p^e-fold tensor product F^{p^e}. We recall that F is said to be purely inseparable over C of exponent e if $F^{\cdot p^e} \subset C$ but $F^{\cdot p^{e-1}} \not\subset C$. A key result in this direction has been proved by A. J. Berkson [5, Th. 4]: *Let F be a purely inseparable extension field of C of exponent one with $[F:C]$ finite. Then $H^n(F/C)=0$ for all* $n \neq 2$. It follows from [3] or [12] that $H^2(F/C)$ may be identified with $B(F/C)$, the Brauer group of central simple C-algebras split by F, so that $H^2(F/C)$ is usually not zero. Berkson's result extends immediately, using Theorem 4.3:

Theorem 6.1. *Let F be a purely inseparable extension field of C with finite exponent. Then $H^n(F/C)=0$ for all $n \neq 2$.*

Proof. F is a union of subfields F_α with $[F_\alpha : C]$ finite (and, of course, F_α is purely inseparable with finite exponent). As in Corollary 3.5, we have $H^n(F/C)=\varinjlim H^n(F_\alpha/C)$, so it suffices to prove the theorem when F is finite dimensional over C. We proceed by induction on the exponent of F. The case of exponent one is Berkson's theorem. Now let $K=F^{\cdot p}C$ so that K is a field of smaller exponent over C, and F is purely inseparable of exponent one over K. The pure inseparability of K implies that K^m has only one simple homomorphic image; namely K, so that the kernel of $\theta: K^m \to K$ is the radical of K^m, hence nilpotent. Thus Theorem 4.3 applies to yield exact sequences $H^n(K/C) \to H^n(F/C) \to H^n(F/K)$ whose extreme terms vanish for $n \neq 2$ by the induction hypothesis. Thus $H^n(F/C)=0$ for $n \neq 2$.

Corollary 6.2. *Let K be a purely inseparable extension field of C with finite exponent and let F be a commutative K-algebra. Then $H^n(F/C) \cong H^n(F/K)$ for all $n \neq 2$. For $n=2$, we have an exact sequence $0 \to H^2(K/C) \to H^2(F/C) \to H^2(F/K) \to 0$.*

Proof. If $[K:C]$ is infinite, the kernel of $K^m \to K$ wil be nil but not nilpotent, so Theorem 4.3 does not apply. But Corollary 4.4 does and, together with the fact that $H^n(K/C)=0$ for $n \neq 2$, yields the result.

Corollary 6.3. *Let F be a normal, but not necessarily separable extension field of C with $[F:C]$ finite. Let G be the group of C-algebra antomorphisms of F. Then $H^n(F/C) \cong H^n(G, F^*)$ for all $n \neq 2$.*

Proof. Let K be the maximal purely inseparable extension of C which is contained in F. Then F is normal and separable over K [14, Cor. 3, p. 74] and its Galois group is G. By Corollary 6.2, $H^n(F/C) \cong H^n(F/K)$ and the latter is $H^n(G, F^*)$ by [3, Th. 6.1] or [12, Th. 1].

We now identify $H^2(F/C)$ with the Brauer group $\mathcal{B}(F/C)$ of all central simple C-algebras split by F [3, Th. 5.4] or [12, Th. 3]. It is straightforward to trace the mappings $\mathfrak{B}(K/C)\to\mathfrak{B}(F/C)$ and $\mathfrak{B}(F/C)\to\mathfrak{B}(F/K)$ in the exact sequence of Corollary 6.2. The first associates to a central simple C-algebra A the same algebra (if A is split by K, it is split by F since $K\subset F$); the second associates to A the K-algebra $K\otimes A$. From this it is already clear that $0\to\mathfrak{B}(K/C)\to\mathfrak{B}(F/C)\to\mathfrak{B}(F/K)$ is exact. In fact, we shall think of $\mathfrak{B}(K/C)$ as a subgroup of $\mathfrak{B}(F/C)$. The major content of Corollary 6.2 is the fact that the last of these mappings is an epimorphism. Since every central simple C-algebra is split by some F containing K, we have

Corollary 6.4. *Let K be a purely inseparable extension field of C with finite exponent. Then every central simple K-algebra is similar to an algebra of the form $K\otimes A$ for some central simple C-algebra A. If A is split by F over C for some F containing K, then, $K\otimes A$ is split by F over K. The correspondence $A\to K\otimes A$ induces an epimorphism $\mathfrak{B}(F/C)\to\mathfrak{B}(F/K)$; $\mathfrak{B}(F/K)\cong\mathfrak{B}(F/C)/\mathfrak{B}(K/C)$.*

REMARK. Corollary 6.4 is due to Hochschild [10, Th. 5]. Amitsur attempted a proof using his complex, but the proof is invalid because of a gap in [3, Lemma 8.1]. This lemma is also correct and provable by means of Amitsur's complex; in fact it appears as Corollary 7.6 (c) below.

Corollary 6.5 (cf. [10, p. 140]). *If F is a purely inseparable field extension of C with finite exponent, then $\mathfrak{B}(F/C)$ contains a finite chain of subgroups $\mathfrak{B}(F/C)\supset\mathfrak{B}(CF^{\cdot p}/C)\supset\mathfrak{B}(CF^{\cdot p^2}/C)\supset\cdots$ with factor groups isomorphic to Brauer groups of extensions of exponent one, viz., $\mathfrak{B}(F^{\cdot p^i}/C)/\mathfrak{B}(F^{\cdot p^{i+1}}/C)\cong\mathfrak{B}(CF^{\cdot p^i}/CF^{\cdot p^{i+1}})$. If $[F:C]<\infty$, the latter Brauer groups can be described in terms of certain Lie algebra extensions* [9, Th. 6] *or* [3, Th. 6].

Corollary 6.6 (cf. [1, Ch. VII, Ths. 26 and 28]). *If $K_1,\cdots,K_t$ are purely inseparable extension fields of C, each with finite exponent, and if $F=\Pi_i K_i$ is the (necessarily unique) composite of the K_i, then $\mathfrak{B}(F/C)=\Pi\mathfrak{B}(K_i/C)$.*

Proof. Clearly it is sufficient to deal with the composite of two fields $K_1=K$ and $K_2=L$. By writing K and L as unions of extension fields of finite degree, we see that it is also sufficient to treat the case where K and L are finite over C. We proceed by induction on the maximum of the exponents of K and L. If this maximum is one, the result is contained in [12, p. 354, Corollary]. The induction needs a lemma:

Lemma 6.7. *If* $\mathfrak{B}(KL/C)=\mathfrak{B}(K/C)\mathfrak{B}(L/C)$ *for all* K, L *and* C *with* K *and* L *of exponent* $<e$ *over* C, *and if* K' *and* L' *are purely inseparable extensions of* C' *with exponents* $<e$ *and if* C' *is purely inseparable of finite exponent over* C, *then* $\mathfrak{B}(K'L'/C)=\mathfrak{B}(K'/C)\mathfrak{B}(L'/C)\mathfrak{B}(C'/C)$.

Proof. By Corollary 6.4 $\mathfrak{B}(K'L'/C)$, $\mathfrak{B}(K'/C)$ and $\mathfrak{B}(L'/C)$ all contain $\mathfrak{B}(C'/C)$ and the respective factor groups are $\mathfrak{B}(K'L'/C')$, $\mathfrak{B}(K'/C')$ and $\mathfrak{B}(L'/C')$. By hypothesis, $\mathfrak{B}(K'L'/C')=\mathfrak{B}(K'/C')\mathfrak{B}(L'/C')$. Taking inverse images in $\mathfrak{B}(K'L'/C)$ gives the lemma.

We now complete the induction in the Corollary. Take K and L with maximum exponent e, $F=KL$, and apply the lemma with $C'=CF^{\cdot p}$, $K'=KF^{\cdot p}$ and $L'=LF^{\cdot p}$ to get $\mathfrak{B}(F/C)=\mathfrak{B}(KF^{\cdot p}/C)\mathfrak{B}(LF^{\cdot p}/C)\mathfrak{B}(CF^{\cdot p}/C)$. Apply the lemma again with $C'=CK^{\cdot p}$, $K'=K$, $L'=CF^{\cdot p}$ and yet again with $C'=CL^{\cdot p}$, $K'=CF^{\cdot p}$, $L'=L$ to get $\mathfrak{B}(F/C)=[\mathfrak{B}(K/C)\mathfrak{B}(CF^{\cdot p}/C)\mathfrak{B}(CK^{\cdot p}/C)][\mathfrak{B}(CF^{\cdot p}/C)\mathfrak{B}(L/C)\mathfrak{B}(CL^{\cdot p}/C)]\mathfrak{B}(CF^{\cdot p}/C)$. But $CK^{\cdot p}\subset K$ and $CL^{\cdot p}\subset L$, so $\mathfrak{B}(CK^{\cdot p}/C)\subset\mathfrak{B}(K/C)$ and $\mathfrak{B}(CL^{\cdot p}/C)\subset\mathfrak{B}(L/C)$; and by the induction hypothesis, $\mathfrak{B}(CF^{\cdot p}/C)=\mathfrak{B}(CK^{\cdot p}/C)\mathfrak{B}(CL^{\cdot p}/C)\subset\mathfrak{B}(K/C)\mathfrak{B}(L/C)$. Inserting these into the formula for $\mathfrak{B}(F/C)$, we get $\mathfrak{B}(F/C)=\mathfrak{B}(K/C)\mathfrak{B}(L/C)$.

7. Separable subfields

In this section F is an algebraic extension field of C and we adopt the following notations: F_s and F_i for the maximal separable and purely inseparable extensions of C contained in F; $q=p^e$ for the smallest power of the characteristic p such that $F^{\cdot q}\subset F_s$ (we assume that q is finite); and K for the field of all qth roots of elements of C. Note that $F_i\subset K$.

In section 4 we have given connections between $H^n(F/C)$ and $H^n(F/F_i)$. One would expect some connection also with $H^n(F_s/C)$. This section is devoted to giving a few such connections under special hypotheses, with applications to Brauer groups.

Raising to qth powers gives a ring homomorphism $F^n\to F_s^n$ (not an algebra homomorphism; it is not the identity on C) which commutes with the operators $\mathcal{E}_i$ and thus gives a mapping of complexes $\mathfrak{C}(F/C)\to\mathfrak{C}(F_s/C)$. This in turn induces the homomorphism $\kappa_n: H^n(F/C)\to H^n(F_s/C)$ which we intend to study. We factor κ_n thus:

$$(7.1)\qquad H^n(F/C)\to H^n(KF/K)\overset{\pi_n}{\to}H^n((KF)^{\cdot q}/K^{\cdot q})=H^n(F_s/C),$$

where the first mapping is induced by the mapping $F^n\to KF\otimes_K\cdots\otimes_K KF$ defined by $x_1\otimes\cdots\otimes x_n\to x_1\otimes_K\cdots\otimes_K x_n$, and the second mapping is induced by the mapping $KF\otimes_K\cdots\otimes_K KF\to(KF)^{\cdot q}\otimes_{K^{\cdot q}}\cdots\otimes_{K^{\cdot q}}(KF)^{\cdot q}$ given by $x_1\otimes_K\cdots\otimes_K x_n\to x_1^q\otimes_{K^{\cdot q}}\cdots\otimes_{K^{\cdot q}}x_n^q$. The second mapping is clearly an isomorphism of rings and thus π is an isomorphism. We prove the last fact $(KF)^{\bullet q}=F_s$, in a lemma.

Lemma 7.1. *With the notations above, $KF_s=KF$ and $(KF)^{\bullet q}=F_s$.*

Proof. Since $F_s=CF_s^{\bullet q}$ [14; II, Th. 8], we have $F_s=CF_s^{\bullet q}=(KF_s)^{\bullet q}\subset(KF)^{\bullet q}=CF^{\bullet q}\subset F_s$, so that all the inclusions are equalities and the lemma follows.

Corollary 7.2. *If $K\subset F$, then $F=KF_s=F_iF_s$.*

In this section we shall deal exclusively with the special case $F=F_iF_s$ (equivalently, F is separable over a purely inseparable extension of C; or $F\cong F_i\otimes F_s$ [1, Th. 2.31 and Lemma 7.7]; this includes the case where F is normal over C [14, p. 74, Cor. 3]). In this case we can factor the first mapping in (7.1) further:

$$H^n(F/C)\xrightarrow{\sigma_n}H^n(F/F_i)\xrightarrow{\rho_n}H^n(K\otimes_{F_i}F/K)=H^n(KF/K)\,, \tag{7.2}$$

where σ_n is the mapping in Theorem 4.3 with K there replaced by F_i (induced by $x_1\otimes\cdots\otimes x_n\to x_1\otimes_{F_i}\cdots\otimes_{F_i}x_n$), ρ_n is defined at the beginning of §4 (induced by $x_1\otimes_{F_i}\cdots\otimes_{F_i}x_n\to(1\otimes_{F_i}x_1)\otimes_K\cdots\otimes_K(1\otimes_{F_i}x_n)$), and the last equality follows from $K\otimes_{F_i}F=KF$ [1, Th. 2.31].

This factorization of κ_n with $n=2$ already explains a lemma of Amitsur [2, Lemma 4.1]:

Proposition 7.3. *Let A be a central simple C-algebra. The isomorphism $x\to x^q$ of K to C extends to an isomorphism of $A\otimes K$ to a central simple C-algebra A_1. Then A_1 is similar in the sense of Brauer to A^q.*

Proof. Let F be a separable splitting field of A so that $F_s=F$ and $F_i=C$. Consider the mapping κ_2 factored as in (7.1) and (7.2); since $\sigma_2=1$ here, we have $\kappa_2=\pi_2\circ\rho_2$. When we identify $H^2(F/C)$ with $\mathfrak{B}(F/C)$, etc., the mapping ρ_2 is identified with the correspondence $A\to A\otimes K$ of C-algebras to K-algebras and π_2 becomes the correspondence of K-algebras and C-algebras described in the proposition. Thus κ_2 is identified with the composite $A\to A\otimes K\to A_1$. But $\kappa_2(x)=x^q$ so that κ_2 is also identified with the correspondence $A\to A^q$. This proves A_1 and A^q are in the same Brauer class.

We can prove the most explicit result under the hypothesis $K\subset F$, which, according to Corollary 7.2, is still stronger than $F=F_iF_s$. In this case, $F_i=K$, so that ρ_n in (7.2) is the identity and $\kappa_n=\pi_n\circ\sigma_n$.

Theorem 7.4. *If $K\subset F$ then κ_n: $H^n(F/C)\to H^n(F_s/C)$ is an iso-*

morphism for $n \neq 2$ *and* κ_2 *is an epimorphism with kernel isomorphic to* $H^2(K/C)$.

Proof. Since π_n is an isomorphism, $\kappa_n = \pi_n \circ \sigma_n$ is an epimorphism if and only σ_n is, and $\operatorname{Ker} \kappa_n = \operatorname{Ker} \sigma_n$. But the required properties of σ_n are asserted by Corollary 6.2.

This theorem proves the following results on Brauer groups. Recall that $\mathfrak{B}(C)$ denotes the group of similarity classes of central simple C-algebras, and $\mathfrak{B}(F/C)$ is the subgroup defined by the algebras split by F.

Corollary 7.5. *Associating to a central simple C-algebra A the algebra* $A^q = A \otimes \cdots \otimes A$ *induces an endomorphism* κ' *of* $\mathfrak{B}(C)$ (*raising to qth powers*) *which, for any separable extension field L of C sends* $\mathfrak{B}(KL/C)$ *onto* $\mathfrak{B}(L/C)$ *with kernel* $\mathfrak{B}(K/C)$.

Proof. Take $F = KL$, identify $H^2(F/C)$ with $\mathfrak{B}(F/C)$, etc., and apply Theorem 7.4.

Corollary 7.6. (a) *The endomorphism* κ' *of* $\mathfrak{B}(C)$ *in Corollary* 7.5 *is an epimorphism with kernel* $\mathfrak{B}(K/C)$.
(b) *The group* $\mathfrak{B}(C)$ *is divisible by q.*
(c) *Every central simple C-algebra A is similar to* A_1^q *for some central simple* A_1.
(d) A^q *is a matrix algebra if and only if A is split by K* (cf. [1, Th. 8.21]).

Proof. $\mathfrak{B}(C)$ is known to be the union of $\mathfrak{B}(L/C)$ as L ranges over the separable extensions of C; i.e., every central simple C-algebra is split by some separable extension field [1, p. 62 Cor.]. By Corollary 7.5, $\operatorname{Im} \kappa'$ contains every $\mathfrak{B}(L/C)$ so that κ' is an epimorphism. Moreover, if $x \in \mathfrak{B}(L/C)$ then also $x \in \mathfrak{B}(KL/C)$ so by Corollary 7.5, $x \in \operatorname{Ker} \kappa'$ only if $x \in \mathfrak{B}(K/C)$. This proves (a). The other conclusions are direct translations of (a).

REMARK. (c) is the promised corrected version of [3, Lemma 8.1].

If instead of $K \subset F$ we return to the weaker hypothesis $F = F_i F_s$, it is no longer true that κ_n is an isomorphism for $n \neq 2$. For example, if C is the field of formal power series in x and y over the field of two elements and F is the field of power series in $\sqrt{x}$ and y over the field of four elements, then F is normal over C with cyclic automorphism group G, so that $H^n(F/C)$ can be interpreted as the cohomology of G with coefficients in F^* and thence calculated more or less explicitly. The condition for κ_n to be an epimorphism when n is even turns out to be $C^* = F_i^{*2} N_{F_s/C}(F_s)$, which is false in this case.

Under one rather obvious hypothesis we can still prove κ_n is an isomorphism for $n \neq 2$. To do this we introduce the lift mapping λ_n: $H^n(F_s/C) \to H^n(F/C)$ induced by the injection of F_s^n into F^n. It is easily verified that $\lambda_n \circ \kappa_n$ (resp. $\kappa_n \circ \lambda_n$) is the operation of multiplication by q in $H^n(F/C)$ (resp. in $H^n(F_s/C)$)— or raising to qth powers if $H^n(F/C)$ is written multiplicatively.

Proposition 7.7. *If the degree of F_s over C is prime to the characteristic then, for all n, $\kappa_n \circ \lambda_n$ is an automorphism of $H^n(F_s/C)$ so that κ_n is an epimorphism $H^n(F/C) \to H^n(F_s/C)$ and λ_n is a monomorphism $H^n(F_s/C) \to H^n(F/C)$. If besides $F = F_iF_s$, then κ_n and λ_n are isomorphisms for $n \neq 2$.*

Proof. By [4, Th. 2.10], every element of $H^n(F_s/C)$ is annihilated by the degree $[F_s : C]$. Hence multiplication by q is an automorphism of $H^n(F_s/C)$, proving the first part of the proposition. If $F = F_iF_s = F_i \otimes F_s$, then $[F : F_i] = [F_s : C]$ and, for $n \neq 2$, $H^n(F/C) = H^n(F/F_i)$ by Corollary 6.2; thus the same argument applies to show that $\lambda_n \circ \kappa_n$ is an automorphism of $H^n(F/C)$, which proves λ_n and κ_n are isomorphisms.

REMARKS. 1. In the example mentioned above, not only is κ_n not an isomorphism, but neither is λ_n; however, $H^n(F/C)$ and $H^n(F_s/C)$ are in fact isomorphic for $n \neq 2$. We do not know whether such an isomorphism always exists for all F and C.

2. Using the same kind of argument, if $K \subset F$ and $[F_s : C]$ is a multiple of q, then $\lambda_n = 0$ for $n \neq 2$, since κ_n is an isomorphism by Theorem 7.4 and $\kappa_n \circ \lambda_n$ is the zero map on $H^n(F_s/C)$.

We conclude with an exact sequence involving the κ_n's. This exact sequence comes from the double complex dercribed in §4, but with F_i replacing C. From here on it will be convenient to change our conventions and consider all tensor products to be tensor products over F_i, including the iterated tensor products $F^n = F \otimes_{F_i} \cdots \otimes_{F_i} F$. Then besides the notations $E_0^{m,n}$, $''E_r^{m,n}$, etc. as in §4, we shall use the notation

$$H^k = {''E}_2^{3,k-2} = \operatorname{Ker} \overline{\Delta}_F^{3,k-2} / \operatorname{Im} \overline{\Delta}_F^{3,k-3} \quad \text{for } k \geq 3 .$$

This is the homology of the complex whose terms are ${''E}_1^{3,k-2} = H^2(K \otimes F^{k-2}/F^{k-2})$.

Proposition 7.8. *Let F be a field of finite degree over a purely inseparable extension F_i of finite exponent over C (equivalently, $F = F_iF_s$ with $[F_s : C]$ finite and F_i of finite exponent). Then there is an exact sequence*

$$H^3 \to H^3(F/C) \xrightarrow{\kappa_3} H^3(F_s/C) \to H^4 \to H^4(F/C) \xrightarrow{\kappa_4} \cdots$$

Proof. By Corollary 4.4 we have an exact sequence

$$H^n(K,\ F/F_i) \to H^n(F/F_i) \xrightarrow{\rho_n} H^n(K \otimes_{F_i} F/K) \cdots$$

But for $n \geq 3$ we have isomorphisms σ_n: $H^n(F/C) \to H^n(F/F_i)$ (Corollary 6.2) and π_n: $H^n(K \otimes_{F_i} F/K) \to H^n(F_s/C)$; furthermore according to (7.1) and (7.2), $\pi_n \circ \rho_n \circ \sigma_n = \kappa_n$. Thus we have an exact sequence

$$\cdots \to H^n(K,\ F/F_i) \to H^n(F/C) \xrightarrow{\kappa_n} H^n(F_s/C) \to \cdots$$

It remains to show $H^n(K, F/F_i) = H^n$. We compute this cohomology group of the double complex by computing $''E$. Since F is separable of finite degree over F_i, F^n is also separable and hence is a finite direct sum of separable extension fields of F_i, say $L_1 \oplus \cdots \oplus L_t$. Consequently $\mathfrak{C}(K \otimes F^n/F^n)$ is the direct product of complexes $\mathfrak{C}(K \otimes L_j/L_j)$ and $H^n(K \otimes_{F_i} F^n/F^n)$ is the direct sum of $H^n(K \otimes L_j/L_j)$. Since K is purely inseparable over F_i and L_j is separable over F_i, $K \otimes L_j$ is a field [1, Th. 2.31] and is purely inseparable over L_j. Then Corollary 6.2 implies that $''E_1^{m,n} = \Sigma_j H^{m-1}(K \otimes L_j/L_j) = 0$ for $m \neq 3$. Thus $''E_\infty^{m,n} = 0$ for $m \neq 3$ and $''E_\infty^{3,n} = ''E_2^{3,n}$ (by (4.6), say) which is H^{n+2}. By (4.5) $H^n(K, F/F_i) = H^n$.

Corollary 7.9. *If, besides the hypotheses of Proposition* 7.8, F *is normal over* F_i (*equivalently* F_s *is normal over* C) *with Galois group* G, *then* $H^n = H^{n-2}(G, H^2(K \otimes_{F_i} F/F))$ *so that we have an exact sequence*

$$\cdots \to H^{n-2}(G,\ H^2(K \otimes_{F_i} F/F)) \to H^n(F/C) \to H^n(F_s/C) \to \cdots \qquad (n \geq 3)\,.$$

Note also that $H^n(F/C) = H^n(G, F^*)$ *and* $H^n(F_s/C) = H^n(G, F_s^*)$.

Proof. Essentially the same proof as used in [12, Th. 1] to show $H^n(F_s/C) = H^n(G, F_s^*)$ will show the desired result here.

REMARK. If $F_i = K$ (equivalently, $K \subset F$), Proposition 7.8 recovers the fact that κ_n is an isomorphism for $n \geq 3$ (Theorem 7.4), since then $''E_1^{3,k-2} = H^2(K \otimes_{F_i} F^{k-2}/F^{k-2}) = H^2(F^{k-2}/F^{k-2}) = 0$. Thus also $H^k = ''E_2^{3,k-2} = 0$ proving κ_n is an isomorphism.

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