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On the Ideal Boundaries of Abstract Riemann Surfaces[1]

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Let R be a Riemann surface with positive boundary. Let $\{R_n\}$ $(n=0, 1, 2, \cdots)$ be its exhaustion with compact relative boundaries ∂R_n. We proved the following

Theorem.[2] *Let $R \notin O_g$ and $\in O_{HB}(O_{HD})$*[3]. *Then $R-R_0 \in O_{AB}(O_{AD})$.* We shall extend the above theorem.

Part I

Martin's topology.[4] Let $G(z, p_i)$ be the Green's function with pole at p_i. Put $K(z, p_i)=\dfrac{G_1(z, p_i)}{G_1(p_0, p_i)}$, where p_0 is a fixed point. Suppose $\{p_i\}$ is a divergent sequence of points. We call $\{p_i\}$ a fundamental sequence determining an ideal boundary point, if $\{K(z, p_i)\}$ converges uniformly in every compact domain of R. If $\{K(z, p_i)\}$ and $\{K(z, p_i')\}$ determine the same limit function, we say that $\{p_i\}$ and $\{p_i'\}$ define the same ideal boundary point. We denote by B the set of all the ideal boundary points. We define the distance between two points p and q of $R+B$ by

$$\sup_{z\in R_1}\left|\frac{K(z, p)}{1+K(z, p)}-\frac{K(z, q)}{1+K(z, q)}\right|=\delta(p, q)\ .$$

Let $K_{v_i}(z, p)$ be the lower envelope of superharmonic functions larger than $K(z, p)$ in v_i. Then R. S. Martin proved that $\lim\limits_i K_{v_i}(z, p)=K(z, p)$ or $=0$ according as p is minimal[5] or not, where $v_i=E[z\in R+B: \delta(z, p) \leqq \frac{1}{i}]$ and that the set of all non minimal points is an F_σ and every

1) The results of the present article were reported at the annual meeting held on 28, May, 1957.

2) Z. Kuramochi: On the behaviour of analytic functions. Osaka Math. J. 7, 1955.

3) O_g, O_{HP}, O_{HB}, O_{HD}, O_{AB} and O_{AD} are the classes of Riemann surfaces on which the Green's function, non constant positive, bounded, Dirichlet bounded, harmonic, bounded analytic and Dirichlet bounded analytic function does not exist respectively.

4) R. S. Martin: Minimal positive harmonic functions. Trans. Amer. Math. Soc. 39, 1941.

5) If positive harmonic function $U(z)$ has no positive function smaller than $U(z)$ except its own multiples, we say that $U(z)$ is minimal. If $K(z, p)$ is minimal, we say p is a minimal point.

positive harmonic function $U(z)$ is representable by a unique mass distribution on $B_1=B-B_0$.[6)]

Let R^∞ be the universal covering surface of R and map R^∞ conformally onto $|\xi|<1$. Let $K(z,p)$ be a minimal function. Then $K(z,p)$ has angular limits almost everywhere on $|\xi|=1$. Then we have easily the following.

Lemma. *Let $K(z,p)$ be a bounded minimal positive harmonic function. Then $K(z,p)$ has angular limits $=M=\sup K(z,p)$ or $=0$ almost everywhere on $|\xi|=1$.*

In fact, let F and F' be sets on $|\xi|=1$ such that $K(z,p)$ has angular limits $\geqq M-\varepsilon$ a.e. (almost everywhere) on F and has angular limits between $M-2\varepsilon$ and ε a.e. on F' for a positive number ε $(0<\varepsilon<\frac{M}{3})$. Then F is a set of positive measure, since $K(z,p)$ is representable by Poisson's integral. Now F' is a set of measure zero, because if it were not so, construct a harmonic function $U(\xi)$ such that $U(\xi)$ has the same angular limits as $K(z,p)$ a.e. on F and 0 a.e. on CF (complementary set of F). Then $U(\xi)$ is a function in R and is not a multiple of $K(z,p)$ and $K(z,p)>U(z)>0$, which implies that $K(z,p)$ is not minimal. This is a contradiction. Hence by letting $\varepsilon\to 0$, $K(z,p)$ has angular limits $=M=\sup K(z,p)$ a.e. on F and 0 a.e. on CF.

Theorem 1. *The set of bounded minimal functions is enumerable.*

Let $K(z,p_i)$ $(i=1,2,\cdots)$ be a bounded minimal function such that $K(z,p_i)$ has angular limits $=M_i$ a.e. on E_i and zero a.e. on CE_i on $|\xi|=1$. Suppose $\text{mes}\,(E_i\cap E_j)\neq 0$ for $i\neq j$. Let $U(z)$ be a harmonic function such that $U(z)$ has angular limits $=\min\,(M_i, M_j)$ on $E_i\cap E_j$ and zero on $C(E_i\cap E_j)$. Then $0<U(z)<K(z,p_i)$, $0<U(z)<K(z,p_j)$ and $U(z)$ is not a multiple of $K(z,p_i)$ or of $K(z,p_j)$. Hence $K(z,p_i)$ or $K(z,p_j)$ is not minimal, whence $\text{mes}\,(E_i\cap E_j)=0$. On the other had, $\text{mes}\,E_i>0$ and $\sum \text{mes}\,E_i\leqq 2\pi$. Hence we have the theorem. In the following we call E_i the *image of* point p_i.

Harmonic measure of a set with respect to Martin's topology.

Let F be a closed set. Put $F_n=E[z\in R+B:\ \delta(z,F)\leqq\frac{1}{n}]$. Let $U_{n,m}(z)$ be a harmonic function in R_m-F_n such that $U_{n,m}(z)=0$ on ∂R_m-F_n and $U_{n,m}(z)=1$ on $R_m\cap F_n$.

Put $U(z)=\lim\limits_m\lim\limits_n U_{n,m}(z)$ and call it the *harmonic measure of the closed set F*. We define the harmonic measure of a Borel set as usual.

6) See 4).

Martin proved that the set of non minimal points is an F_σ of harmonic measure zero.

Lemma 1. *Let R be a Riemann surface which has an enumerably infinite number of bounded minimal functions $K(z, p_i)$ $(i=1, 2, \cdots)$ and a set* (clearly G_δ set) *of Martin's boundary points of harmonic measure zero. Then* $mes \sum E_i = 2\pi$.

Suppose $\sum \text{mes}\, E_i < 2\pi$. Then we can construct a bounded positive harmonic function $U(z)$ in R such that $U(z)$ has angular limits $=0$ a.e. on $\sum E_i$ and $=1$ a.e. on $C(\sum E_i)$. Since $U(z)$ is positive, $U(z)$ is represented by a unique mass distribution μ as follows: $1 > U(z) = \int_{B_1} K(z, p) d\mu(p)$, where B_1 is the set of minimal points. Now the mass distribution μ has no mass at every p_i, since if $U(z)$ has a positive mass μ_0 at p_i, $U(z)$ must have angular limits $\mu_0 c_i M_i$ (c_i is a constant) a.e. on E_i. Since μ is Borel measurable, there exists a closed set F in $B - \bigcup p_i$ such that $U(z) \geqq U'(z) = \int_F K(z, p) d\mu'(p) > 0$, where μ' is the restriction of μ on F. Let H_n be the set of bounded minimal points whose images satisfy $\frac{1}{n} > \text{mes}\, E_i \geqq \frac{1}{n+1}$ $(n=1, 2, \cdots)$. Then the number of points in H_n is finite and $\bigcup H_n = \bigcup p_i$. Put $J_n = E[z \in R+B:\ \delta(z, H_n) \geqq \frac{1}{2}\delta(F, H_n)]$. Then $(B \bigcap^{\infty} J_n) \supset F$.

Denote by $U'_{k,m}(z)$ the lower envelope of positive superharmonic functions in R larger than $U'(z)$ in $(R-R_m) \bigcap_{n=1}^{k} J_n$. Then since $(\bigcap^{k} J_n) \supset F$ and μ' is contained in F, $U'_{k,m}(z) = U'(z)$ for every m and k. Hence $U'(z) = \lim_m U'_{k,m}(z)$. On the other hand, clearly $U'(z) = U_k(z) = \lim_i \lim_m U_{k,m,m+i}(z)$, where $U_{k,m,m+i}(z)$ is a harmonic function in $R_{m+i} - ((R-R_m) \bigcap (\bigcap^{k} J_n)$ such that $U_{k,m,m+i}(z) = U'(z)$ on $\partial(\bigcap^{k} J_n \bigcap (R-R_m))$ and $U_{k,m,m+i}(z) = 0$ on $\partial R_{m+i} - (\bigcap^{k} J \bigcap (R-R_m))$. Let $\omega_{k,m,m+i}(z)$ be a harmonic function in $R_{m+i} - ((R-R_m) \bigcap (\bigcap^{k} J_n))$ such that $\omega_{k,m,m+i}(z) = 1$ on $(R-R_n) \bigcap (\bigcap^{k} J_n)$ and $\omega_{k,m,m+i}(z) = 0$ on $\partial R_{m+i} - ((R-R_m) \bigcap (\bigcap^{k} J_n))$. Then $\lim_m \lim_i \omega_{k,m,m+i}(z) = \omega_k(z)$ is smaller than the harmonic measure of an open set $B - \bigcup^{k} H_n$, because the closed set $(\bigcap^{k} J_n \bigcap B) \subset B - \bigcup^{k} H_n$. Hence the assumption that $B - \bigcup p_i$ is of harmonic measure zero implies $\lim_k \omega_k(z) = 0$. On the other hand, by $\sup U'(z) \leqq 1$, $\omega_k(z) \geqq U_k'(z) > U'(z) > 0$ for every k. Hence $\lim_k \omega_k(z) > 0$. This is a contradiction. Thus $\sum \text{mes}\, E_i = 2\pi$.

Class HBN.[7)]

Theorem 2. *A Riemann surface $R \in HBN$, if and only if R has $N-1$ bounded minimal functions $K(z, p_i)$ $(i=1, 2, \cdots, N-1)$ and a set of boundary points of harmonic measure zero.*

Suppose that R has $N-1$ number of bounded minimal functions $K(z, p_i)$ and a set of boundary points of harmonic measure zero. Let $U(z)$ be a bounded harmonic function in R. Then $U(z)$ has angular limits$=$constant a.e. on the image E_i of bounded minimal point p_i, because if there exist two subset E_i' and E_i'' of E_i such that both E_i' and E_i'' are of positive measure and $U(z)$ has angular limits $< L-\varepsilon$ and $> L+\varepsilon$ on E_i' and E_i'' respectively for a positive number $\varepsilon > 0$, we can prove that $K(z, p_i)$ is not minimal as before. Hence every bounded harmonic function $U(z)$ has a constant a.e. on E_i. Hence by $\sum \mathrm{mes}\, E_i = 2\pi$, every $U(z)$ is a linear form of $K(z, p_i)$ $(i=1, 2, \cdots, N-1)$. On the other hand, $K(z, p_i)$ and a constant are linearly independent. Hence $R \in$HBN. Next suppose $R \in$HBN. Then we construct by linear transformations a system of $N-1$ independent harmonic functions $U_i(z)$ which have angular limits$=1$ a.e. on E_i and $=0$ a.e. on CE_i on $|\xi|=1$. As above, we see easily that every $U_i(z)$ is a multiple of a bounded minimal function. Thus we have the theorem.

Let G be a non compact domain in R and let $U(z)$ be a positive harmonic function in G vanishing on ∂G. Put $U_{ex}(z) = \lim_n U_n(z)$, where $U_n(z)$ is the upper envelope of subharmonic functions in R smaller than $U(z)$ in $G \cap (R-R_n)$. Let $V(z)$ be a positive harmonic function in R. Put $T_{inex}(z) = \lim_n V_n(z)$, where $V_n(z)$ is the lower envelope of superharmonic functions larger than $V(z)$ in $G \cap (R-R_n)$ and vanish on $\partial G \cap R_n$. Then we proved

Lemma 2.[8)] *Let $U(z)$ be a positive harmonic function in G vanishing on ∂G.*

$$\text{If} \quad U_{ex}(z) < \infty\,, \quad \text{then} \quad U(z) = {}_{inex}(U_{ex}(z))\,.$$

Theorem 3. *Let G be a non compact domain and let $K(z, p)$ be a bounded minimal function. If $K'(z, p) = K_{inex}(z, p) > 0$, then there exists no analytic function of bounded type*[9)] *in G.*

Suppose $K'(z, p) \leqq M$. Let $\omega_n(z)$ be a harmonic function in $G \cap R_n$ such that $\omega_n(z)=0$ on $\partial G \cap R_n$ and $\omega_n(z)=1$ on $\partial R_n \cap G$. Then $\omega_n(z) \geqq$

7) HBN is the class of Riemann surfaces on which N number of linearly independent bounded harmonic functions exist.

8) Z. Kuramochi: Relations between harmonic dimensions, Proc. Japan Acad. 30, 1954.

9) Z. Kuramochi: Dirichlet problem on Riemann surfaces, I, Proc. Japan Acad. 30, 1954.

$\frac{K'(z, p)}{M}>0$. Map the universal covering surface G^∞ of G conformally onto $|\xi|<1$. Then $K(z, p)$ and $\omega(z)=\lim_n \omega_n(z)$ have angular limits a.e. on $|\xi|=1$. We call the set on which $\omega(z)$ has angular limits $=1$ almost everywhere the image I of the ideal boundary of G. Clearly I is a set of positive measure. Since $M>K'(z, p)>0$, there exist two constant M and δ and a set E such that $K'(z, p)$ has angular limits between M and $M-\delta$ a.e. on $E(\subset I)$ of positive measure. If there exist two sets E_1 and E_2 of positive measure such that $K'(z, p)$ has angular limits between M and $M-\delta$ a.e. on E_1 and between $M-2\delta$ and $M-3\delta$ on E_2, We can define a harmonic function $U(z)$ in G such that $U(z)=0$ on ∂G and $U(z)$ has the same angular limits as $K'(z, p)$ on E_1 and zero on CE_1 Then $U(z)<K'(z, p)$ and $U(z)$ is not a multiple of $K'(z, p)$. Hence by Lemma 2, $U'(z)=U_{ex}(z)<K'_{ex}(z, p)=K(z, p)$ and $U'(z)$ is not a multiple of $K(z, p)$, whence $K(z, p)$ is not minimal. This is a contradiction. Hence $K'(z, p)$ has angular limits $=0$ or $=M$ a.e. on I. Let $A(z)=W$ be an analytic function of bounded type. Then $A(z)$ has angular limits a.e. on I. Let $\{\mathfrak{S}_n\}$ be a sequence of triangulations of the w-lane such that $\mathfrak{S}_{n+1}$ is a subdivision of $\mathfrak{S}_n$ and becomes as fine as please, when $n\to\infty$. Denote by $\{\Delta_n^i\}$ $(i=1, 2, \cdots)$ the triangles of $\mathfrak{S}_n$. The subset where $A(z)$ has angular limits contained in $\overline{\Delta}_n^i$ will be denoted by E_n^i. Then every E_n^i is lineary measurable. There exist at least two E_n^i, $E_{n'}^{i'}$ such that $E_n^i \cap E_{n'}^{i'}=0$ in I and both $\operatorname{mes} E_n^i>0$ and $\operatorname{mes} E_{n'}^{i'}>0$. On the contrary, suppose for every n there exists $i(n)$ such that $\operatorname{mes} E_n^i=\operatorname{mes} I$. $A(z)$ must be a constant contained in $\cap \overline{\Delta}_n^i$. Then we can construct a harmonic function $U(z)$ in G such that $U(z)=0$ on ∂G and has angular limits $=1$ on E_n^i and 0 on $E_{n'}^{i'}$ almost everywhere. This $U(z)$ is not a multiple of $K'(z, p)$. Hence as above $K(z, p)$ is not minimal. This is a contradiction. Thus we have the theorem.

Theorem 4. *Let $v(p)$ be a neighbourhood of a bounded minimal point. Then there exists no analytic function of bounded type in $v(p)$.*

Let $U_n(z)$ be a harmonic function in $R_n \cap v(p)$ such that $U_n(z)=K(z, p)$ on $\partial v(p) \cap R_n$ and $U_n(z)=0$ on $\partial R_n \cap v(p)$. Put $V(z)=U(z)=\lim_n U_n(z)$ in $v(p)$ and $V(z)=K(z, p)$ in $R-v(p)$. Then $V(z)={}_{R+B-v(p)}K(z, p)$. Suppose ${}_{R+B-v(p)}K(z, p)=K(z, p)$. Then $K(z, p)=\int_{R+B-v(p)} K(z, p_\omega)d\mu(p_\omega)$. Since $K(z, p)$ is harmonic in R, $K(z, p)=\int_{B-v(p)} K(z, p_\omega)d\mu(p_\omega)$. If μ is a point mass , $K(z, p)={}_{B-v(p)}K(z, p)=K(z, q) : q \notin v(p)$, which implies $p=q\notin v(p)$. This is a contradiction. Hence μ is not a point mass. We can find a closed set F_n with diameter $<\frac{1}{n_0}$ in $Cv(p)$[10] such that μ on F_n represents a function $U'(z)$ which is not a multiple of $K(z, p)$. Because every μ_n on F_n represents a function $U_n(z)=c_nK(z, p)$. Let $n\to\infty$, then $\bigcap_n F_n=q$ in $cv(p)$. Then $\lim_n \frac{\mu_n}{\text{total mass of }\mu_n}$ represents a function $K(z, q)$ which equales to $K(z, p)$. This implies also $p=q\notin v(p)$. Put $U'(z)=\int_{F_{n_0}} K(z, p_\alpha)d\mu(p_\alpha)$. Then $U'(z)<K(z, p)$ and $U'(z)$ is not a multiple of $K(z, p)$. Hence $K(z, p)$ is not minimal. Thus $K(z, p)-\underset{R+B-v(p)}{K}(z, p)=K'(z, p)>0$. Thus we have the theorem by Theorem 3.

Theorem 5. *Let R be a Riemann surface such that R has an enumerably infinite number of bounded minimal functions $K(z, p_i)$ $(i=1, 2, \cdots)$ and a set of boundary points of harmonic measure zero. Let G be a non compact domain such that $0<\omega(z)=\lim_n \omega_n(z)$, where $\omega_n(z)$ is a harmonic function in $G\bigcap R_n$ such that $\omega_n(z)=0$ on $\partial G\bigcap R_n$ and $\omega_n(z)=1$ on $\partial R_n\bigcap G$. Then there exists no analytic function of bounded type in G.*

Put $U_i(z)=\frac{K(z, p_i)}{\sup K(z, p_i)}$. Then $U_i(z)$ has angular limits$=1$ on E_i and 0 on CE_i almost everywhere and by Lemma 1, $\sum U_i(z)\equiv 1$. Let $V_n^i(z)$ be a harmonic function in $G\bigcap R_n$ such that $V_n^i(z)=U_i(z)$ on $G\bigcap\partial R_n$ and $V_n^i(z)=0$ on $\partial G\bigcap R_n$. Put $V^i(z)=\lim_n V_n^i(z)$. Then $\sum V^i(z)=\omega(z)>0$. Hence there exists at least one $K(z, p_i)$ such that $K'(z, p_i)>0$. Hence we have the theorem by Theorem 3.

Remark. The condition $\omega(z)>0$ in Theorem 5 cannot be replaced by $\lim_n \lim_i w_{n, n+i}(z)=w(z)>0$, where $w_{n, n+i}(z)$ is harmonic in $R_{n+i}-(G\bigcap(R_{n+i}-R_n))$ such that $w_{n, n+i}(z)=1$ on $\partial(G\bigcap(R_{n+i}-R_n))$ and $w_{n, n+i}(z)=0$ on $\partial R_{n+i}-G$. In fact, we constructed a Riemann surface R[11] with positive boundary $\in O_{HP}\subset O_{HB}\subset$HNB such that R is a covering surface over the W-plane and R is symmetric with respect to the real axis. Let G_U and G_L be the parts of R lying over the upper and lower half plane respectively. Then $G_U+G_L=R$. If we consider the above function with respect to G_U. Then clearly $w(z)>0$. But the function $\frac{1}{W+i}$: $W=W(z)$ is a bounded analytic function on G_U. Hence $w(z)>0$ is not sufficient condition.

10) $Cv(p)$ means the complementary set of $v(p)$.

11) See 2).

Part II

Martin's topology.[12] Let $N(z, p)$ be a harmonic function in $R-R_0$ with one logarithmic singularlity at p such that $N(z, p)=0$ on ∂R_0 and $N(z, p)$ has the minimal Dirichlet[13] integral over $R-R_0$. Then we define as in case of $K(z, p)$ the ideal boundary points. All the ideal boundary points is denoted by B. Put $\bar{R}=R-R_0+B$. Distance between two points p and q is defined as

$$\sup_{z\in R_1-R_0}\left|\frac{N(z, p)}{1+N(z, p)}-\frac{N(z, q)}{1+N(z, p)}\right|=\delta(p, q)\,.$$

Capacity of a closed set F in $\bar{R}$. Put $F_n=E[z\in\bar{R}:\ \delta(z, F)\leqq\frac{1}{n}]$. Let $\omega_n(z)$ be a harmonic function in $\bar{R}-F_n$ such that $\omega_n(z)=0$ on ∂R_0, $\omega(z)=1$ on F_n and $\omega_n(z)$ has the minimal Dirichlet integral (we abbreviate by M.D.I.) over $\bar{R}-F_n$. Then $\omega_n(z)\rightarrow\omega(z)$ in mean as $n\rightarrow\infty$. We call $\omega(z)$ C.P. (the capacitary potential) of F and $\int_{\partial R_0}\frac{\partial\omega(z)}{\partial n}ds$ the capacity of F. We proved that $\omega(z)>0$ implies $\sup_{z\in\bar{R}}\omega(z)=1$.[14] The capacity of a Borel set in $\bar{R}$ is defined as usual.

Capacity of the set of the ideal boundary determined by a non compact domain G.

Let $\omega_n(z)$ be a harmonic function in $\bar{R}-(G\cap(R-R_n))$ such that $\omega_n(z)=0$ on ∂R_0, $\omega_n(z)=1$ on $G\cap(R-R_n)$ and $\omega_n(z)$ has M.D.I. over $\bar{R}-(G\cap(R-R_n))$. Then $\omega_n(z)\rightarrow\omega(z)$ in mean. Then we call $\omega(z)$ C.P. of the boundary $(B\cap G)$ determined by G. Then we proved the following[15]

1) $\omega(z)$ superharmonic in[15] $\bar{R}$ and $\omega(z)>0$ implies $\sup_{z\in G}\omega(z)=1$.

2) The C.P. of the ideal boundary $(B\cap G\cap G_\delta)$ is zero, where $G_\delta=E[z\in R:\ \omega(z)<1-\delta]:\ \delta>0$.

3) There exist regular curves C_ε such that $\int_{C_\varepsilon}\frac{\partial\omega(z)}{\partial n}ds=D(\omega(z))$ for almost all C_ε: $(1>\varepsilon>0)$.

4) $\omega(z)$ has M.D.I. among all functions with value $\omega(z)$ in $R-R_0-G'$ for every $G'\supset G$.

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12) Z. Kuramochi: Mass distributions on the ideal boundaries, II, Osaka Math. Journ, 8, 1956.

13) The Dirichlet integral is taken with respect to $N(z, p)-\log\frac{1}{|z-p|}$ in a neighbourhood of p.

14) Z. Kuramochi: Mass distributions on the ideal boundaries, III in this volume.

15) See 11).

16) Let $U(z)$ be a positively harmonic function satisfying $D(\min(M, U(z))<\infty$, if $U(z)\geqq U_G(z)$ for every compact or non compact domain G, we say $U(z)$ is superharmonic in $\bar{R}$, where $U_G(z)=\lim_{M=\infty}U_G^M(z)$, $U_G^M(z)=\min(M, U(z))$ on ∂G and $U_G^M(z)$ has M.D.I. over G.

Capacitary potential of the ideal boundary determined by G_2 with respect to G_1.

Let $G_1 \supset G_2$ be two non compact domains. Let $\omega_{n,n+i}(z)$ be a harmonic function in $((G_1-G_2)\cap R_{n+i})-(G_2\cap(R_{n+i}-R_n))$ such that $\omega_{n,n+i}(z)=0$ on $(\partial G_1\cap R_{n+i})$, $\omega_{n,n+i}(z)=1$ on $\partial G_2\cap(R_{n+i}-R_n)+(G_2\cap\partial R_n)$ and $\frac{\partial}{\partial n}\omega_{n,n+i}(z)=0$ on $\partial R_{n+i}\cap(G_1-G_2)$. If $D(\omega_{n,n+i}(z))<M$ for constant M and for every i. Then $\omega_{n,n+i}(z)\to\omega_n(z)$ in mean and $\omega_n(z)\to\omega(z)$ in mean also. We call C.P. of the ideal boundary $(B\cap G_2)$ determined by G_2 with respect to G_1. Then we have the same properties 1), 2), 3) and 4) as above.

We proved the following facts in (II)[11]: the value of $N(z, q)$ (minimal or not) at a minimal point ($N(z, p)$ is minimal) is given by

$$N(p, q)=\frac{1}{2\pi}\lim_{m\to M}\int_{C_m} N(z, q)\frac{\partial}{\partial n}N(z, p)ds\,,$$

where $M=\sup N(z, p)$ and C_m is a regular curve such that $C_m=E[z\in R: N(z, p)=m]$ and $\int_{C_m}\frac{\partial}{\partial n}N(z, p)ds=2\pi$. For non minimal point $p(N(z, p)=\int_{B_1}N(z, p_\alpha)d\mu(p_\alpha): p_\alpha\in B_1)$, $N(p, q)$ is given by $\int_{B_1}N(p_\alpha, q)d\mu(p_\alpha)$, where B_1 is the set of minimal points.

1) *$N(z, q)$ is lower semicontinuous in $\bar{R}$ with respect to Martin's topology.*

2) *Let $V_m(p)=E[z\in R: N(z, p)>m]$ and $v_n(p)=E[z\in\bar{R}: \delta(z, p)<\frac{1}{n}]$ and suppose $p\in R+B_1$. Then $N_{V_m(p)}(z, p)=N(z, p)$ for every $m<\sup N(z, p)$ and $N_{v_n(p)}(z, p)=N(z, p)$ for every n.*

3) *For every $V_m(p): p\in R+B_1$, there exists a number n such that*

$$V_m(p)\supset(R\cap v_n(p))\,.$$

4) *If $N(z, p)$ is bounded, p is minimal and $N(z, p)=k\omega(z)$, where $\omega(z)$ is C.P. of p. In this case, $\omega(p)=1$ and $\omega(z)$ is continuous at p by* (3).

5)[17] *Let $\omega(z)$ be the function in* (4). *Then $\omega(z)<1$ for $z\in\bar{R}-p$.*

Lemma 3. *Let $N(z, p)$ and $N(z, q)$ $(p\neq q)$ be bounded minimal functions. Let $\omega(z)$ be C.P. of p and put $G_{1-\delta}=E[z\in R: \omega(z)>1-\delta]$. Let $\omega^{1-\delta}(z)$ be C.P. of $(G_{1-\delta}\cap q)$. Then there exists a constant δ_0 such that*

$$\omega^{1-\delta}(z)=0\qquad \textit{for}\quad \delta<\delta_0\,.$$

Let F be a closed set in $\bar{R}$ and let $U(z)$ be a positive superharmonic function in $\bar{R}$ vanishing on ∂R_0. Let $U_{F_n}(z)$ be the lower envelope of superharmonic functions in $\bar{R}$ larger than $U(z)$ in $F_n=E[z\in\bar{R}: \delta(z, F)$

17) As for 5), see Theorem 4 in 14).

$\leqq \frac{1}{n}]$. Then we proved[18] that $U_{F_n}(z) \downarrow U_F(z)$ and $U_F(z)$ is given by $\int_F N(z, p)d\mu(p)$. Since $(v_n(q) \cap G_{1-\delta}) \subset v_n(q)$, $\omega^{1-\delta}(z) = K\omega'(z)$, where $\omega'(z)$ is C.P. of q. But $\sup \omega^{1-\delta}(z) = 1 = \sup \omega'(z)$ implies $K=1$. Hence

$$\omega(z) \geqq (1-\delta)\omega^{1-\delta}(z) = (1-\delta)\omega'(z)\,.$$

Assume $\omega^{1-\delta}(z) > 0$ for $\delta > 0$. Then by letting $\delta \to 0$, $\omega(z) \geqq \omega'(z)$ and $\omega(q) \geqq \omega'(q) = 1$. This contradicts to the property (5). Hence we have lemma.

Lemma 4. *Let $R-R_0$ be a Riemann surface with a finite number of bounded minimal functions $N(z, p_i)$ $(i=1, 2, \cdots, k)$ and a set of the ideal boundary points (an open set) of capacity zero. We map the universal covering surface $(R-R_0)^\infty$ of $(R-R_0)$ onto $|\xi|<1$. Consider $\omega_i(z)$, C.P. of p_i in $|\xi|<1$. Then $\omega_i(z)$ has angular limits a.e. on $|\xi|=1$. Denote by E_i the set on which $\omega_i(z)$ has angular limits $=1$ almost everywhere. Then*

$$mes \sum E_i = 2\pi - r_0\,,$$

where r_0 is the measure of the image of ∂R_0.

Suppose $\text{mes} \sum E_i < 2\pi - r_0$. Then there exists a set H on $|\xi|=1$ of positive measure in the complementary set of the sum $\sum E_i$ and the image of ∂R_0 and a constant δ such that every $\omega_i(z)$ has angular limits $< 1-\delta$ a.e. on H. Then there exists a closed set $H' \subset H$ such that $\text{mes}(H-H') < \varepsilon$ and $\omega_i(z) < 1-\delta+\varepsilon$ $(i=1, 2, \cdots, k)$ in the intersection of $(|\xi|>1-\varepsilon)$ and the angular domain D containing endparts $A(\theta) = \arg|\xi-\xi_0| < \frac{\pi}{2}-\varepsilon$, $\xi_0 = e^{i\theta} \in H'$ for any given positive number $\varepsilon > 0$. Let D' be one component of D and let $U(\xi)$ be a harmonic function in D' such that $U(\xi)=1$ on $H' \cap \partial D'$ and $U(\xi)=0$ on $\partial D' - H'$. Then $U(\xi) > 0$. On the other hand, there exists a constant α by the property (4) such that $G^i_{1-\alpha} = E[z \in \bar{R} : \omega_i(z) > 1-\alpha]$ is open by the lower semi-continuity of $\omega_i(z)$ and $G^i_{1-\alpha} \ni p_i$.

Construct a harmonic function $W_{n,n+i}(z)$ in $R_{n+i} - R_0 - (\sum_i G^i_{1-\alpha})$ such that $W_{n,n+i}(z)=1$ on $(R_{n+i}-R_n) \cap \partial(\sum G^i_{1-\alpha})$ and $W_{n,n+i}(z)=0$ on $\partial R_0 + \partial R_{n+i} - \sum G^i_{1-\alpha}$. Then $W_{n,n+i}(z) \uparrow W_n(z)$ and $W_n(z) \downarrow W(z) < \omega(z)$, where $\omega(z)$ is C.P. of a closed set $(B - \sum G^i_{1-\alpha})$. But the fact the open set $B - \sum p_i$ is capacity zero means that every closed set contained in $B - \sum p_i$ is of capacity zero. Hence $0 = \omega(z) \geqq W(z) = 0$. Now since $\omega_i(z) < 1-\delta+\varepsilon$ in $D' \cap E[|\xi|>1-\varepsilon]$, the image of $\sum G^i_{1-\alpha}$ does not

18) See 11).

intersect $D'\cap E[|\xi|>1-\varepsilon]$ for $\alpha<\frac{\delta}{2}$. Hence $0=W(z)>U(\xi)>0$. This is a contradiction. Thus $\text{mes}\sum E_i=2\pi-r_0$.

Lemma 5. *Let $R-R_0$ be a Riemann surface in Lemma 4. Then every $\omega_i(z)$ has angular limits$=L^i_j$ a.e. on E_j $(j=1,2,\cdots,k)$ and mes $(E_i\cap E_j)=0$ for $i\neq j$ and mes $E_i>0$ for every i.*

$\omega_i(z)$ has angular limits$=1$ a.e. on E_i. Suppose there exist two set $E'(\subset E_i)$ and $E''(\subset E_i)$ of positive measure on which $\omega_i(z)$ has angular limits$>L$ and $<L-\delta$ a.e. on E' and E'' respectively for numbers L and $\delta>0$. Put $H^i_{L-\delta'}=E[z\in R:\ \omega_i(z)>L-\delta'$ $(0<\delta'<\frac{\delta}{2})$. Then since mes $E'>0$, $H^i_{L-\delta'}\cap G^j_{1-\alpha}(=E[z\in R:\ \omega_i(z)>1-\alpha)$ determines a set of the boundary of positive harmonic measure. On the other hand, by Lemma 3, there exist δ_0 and n_0 such that C.P. of $(B\cap G^j_{1-\delta}\cap(\sum_{k\neq j}v_n(p_k))<\varepsilon$ for $\delta<\delta_0$ and $n>n_0$ for any given $\varepsilon>0$, whence the harmonic measure of $(B\cap G^j_{1-\delta}\cap(\sum_{k\neq j}v_n(p_k))<\varepsilon$. Hence by $\text{Cap}(B)=\text{Cap}(\sum p_i)$, harmonic measure of $(H^i_{L-\delta'}\cap B\cap G^j_{1-\delta}\cap p_j)\geqq W(z)-\varepsilon$, where $W(z)$ is the harmonic measure of $(H^i_{L-\delta'}\cap B\cap G^j_{1-\delta})$. Let $\varepsilon\to 0$. Then the harmonic measure of $(H^i_1\cap B\cap p_j)\geqq W(z)$.

Let $\omega_{H^i_{L-\delta'}\cap v_n(p_j)}(z)$ be the lower envelope of superharmonic functions in $\bar{R}$ larger than $\omega_i(z)$ in $H^i_{L-\delta'}\cap v_n(p_j)\cap(R-R_m)$. Let $m\to\infty$ and then $n\to\infty$. Then $\omega_{H^i_{L-\delta'}\cap v_n(p_j)}(z)\downarrow\omega^*(z)=\int_{p_j}N(z,p)d\mu(p)=k\omega_j(z)$. Now $\omega^*(z)>(L-\delta')W(z)$. Hence by mes $E'>0$, $W(z)$ has angular limits$=1$ a.e. on E'. But $\omega_j(z)$ has angular limits$=1$ a.e. on E', which implies $k\geqq L-\delta'$. Hence by $\omega_i(z)\geqq\omega^*(z)\geqq(L-\delta')\omega_j(z)$, $\omega_i(z)$ has angular limits $\geqq L-\delta'$ a.e. on $E_j(\supset E'')$. This contradicts to the assumption. Hence $\omega_i(z)$ has angular limits$=L^i_j$ a.e. on E_j.

Next suppose mes $(E_i\cap E_j)>0$. Then both $\omega_i(z)$ and $\omega_j(z)$ have angular limits$=1$ a.e. on E_i+E_j, whence mes $(E_i-E_j)=0$ and $\omega_i(z)\equiv\omega_j(z)$. This contradicts to $p_i\neq p_j$. Hence mes $(E_i\cap E_j)=0$.

Every $\omega_i(z)$ has L^i_j (<1) a.e. on E_j. On the other hand, $\omega_i(z)$ is representable by Poisson's integral. Assume mes $(E_i)=0$. Then $\omega_i(z)\leqq\max_{j\neq i}(L^i_j)<1$. This contradicts to $\sup\omega_i(z)=1$. Hence mes $(E_i)>0$ for every i. Thus we have Lemma 5.

Lemma 6. *Let $R-R_0$ be a Riemann surface in Lemma 4 and let $U(z)$ be a Dirichlet bounded harmonic function vanishing on ∂R_0. Then $U(z)$ has angular limits$=$constant a.e. on E_i for every i.*

Suppose there exist two sets $E'(\subset E_i)$ and $E''(CE_i)$ of positive measure such that $U(z)$ has angular limits$>L+\delta$ and $<L-\delta$ a.e. on E' and E''

respectively for constants L and $\delta>0$. Put $g=E[z\in R: U(z)<L-\delta+\varepsilon']$ and $G^i_{1-\varepsilon'}=E[z\in R: \omega_i(z)>1-\varepsilon']$ $(0<\varepsilon'<\frac{\delta}{2})$. Since $U(z)$ has angular limits$>L+\delta$ on E' and $\omega_i(z)$ has angular limits$=1$ a.e. on E_i, there exists a closed set $E^*(\subset E')$ of positive measure such that both $U(z)$ and $\omega_i(z)$ converge uniformly in angular domain. Let D be an angular domain containing endparts $A(\theta)=\arg|\xi-\xi_0|<\frac{\pi}{2}-\varepsilon'$: $\xi_0\in E^*$. Then there exists a number ε_0 such that $D_{\varepsilon_0}=D\cap E[|\xi|>1-\varepsilon_0]$ is contained in the image of $(G^i_{1-\varepsilon'}\cap g)$ and the image of $g'=E[z\in R: U(z)>L+\delta]$ does not intersect D_{ε_0}. Hence the harmonic measure $W'(z)$ of $(B\cap G^i_{1-\varepsilon'}\cap g)$ with respect to $Cg'=E[z\in R: U(z)\leqq L+\delta]$ is positive, where $W'(z)=\lim\limits_n\lim\limits_i W'_{n,n+i}(z)$ and $W'_{n,n+i}(z)$ is harmonic in $R_{n+i}\cap(Cg'-(R-R_n)\cap g\cap G^i_{1-\varepsilon'}))$ such that $W'_{n,n+i}(z)=0$ on $\partial g'+\partial R_{n+i}-(g\cap G^i_{1-\varepsilon'})$ and $W'_{n,n+i}(z)=1$ on $\partial((R_{n+i}-R_n)\cap(g\cap G^i_{1-\varepsilon'}))$. Now let $\omega^*(z)$ be C.P. of $(g\cap B\cap G^i_{1-\varepsilon'})$ with respect to Cg'. Then $\omega^*(z)>W'(z)$ and by the Dirichlet principle

$$D(\omega^*(z))\leqq\frac{1}{4\delta^2}D(U(z))<\infty\,.$$

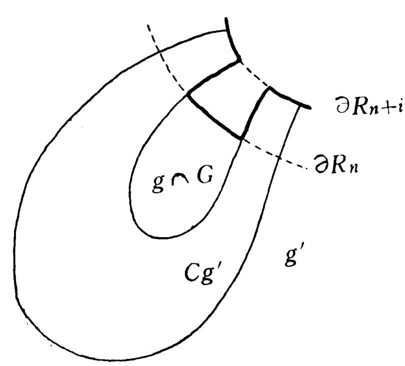

Fig. 1

Next by Lemma 3, there exists ε such that Cap $(G^i_{1-\varepsilon}\cap\sum\limits_{k\neq i}p_k)=0$ and Cap $(G^i_{1-\varepsilon}\cap\sum\limits_{k\neq i}p_k)=0$ with respect to Cg', whence Cap $(g\cap B)$ with respect to $Cg'=$ Cap $(g\cap p_i\cap G^i_{1-\varepsilon})$ with respect to $Cg'=$Cap $(g\cap p_i)$ with respect to Cg'. Hence

$$0<W'(z)<\omega^*(z)=\omega(z)\,.$$

where $\omega(z)$ is C.P. of $(g\cap p_i)$ with respect to Cg'.

Let $\omega^m_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-(v_m(p_i)\cap Cg'\cap(R_{n+i}-R_n))$ such that $\omega^m_{n,n+i}(z)=0$ on ∂R_0, $\omega^m_{n,n+i}(z)=1$ on $\partial(v_m(p_i)\cap g'\cap(R_{n+i}-R_n))$ and $\frac{\partial\omega^m_{n,n+i}}{\partial n}(z)=0$ on $\partial R_{n+i}-(g'\cap v_m(p_i))$. Then $\omega^m_{n,n+i}(z)\to\omega^m_n(z)$, $\omega^m_n(z)\to\omega^m(z)$ and $\omega^m(z)\to\omega^{**}(z)$ in mean, i.e. $\omega^{**}(z)$ is C.P. of $(B\cap p_i\cap g')$. But $\omega^{**}(z)$ has mass only at $\bigcap\limits_{m>0}v_m(p_i)=p_i$. Hence $\omega^{**}(z)=K\omega_i(z)$. But as above we see by mes $E''>0$ and by $\sup\omega^{**}(z)=1$, we have

$$\omega_i(z)=\omega^{**}(z)>\omega^*(z)>0\,. \qquad (1)$$

Let C_{δ_i} $(i=1,2)$ be regular curves of $\omega^*(z)$ and let $\omega^*_{n+j}(z)$ be a harmonic function in $(R_{n+j}-R_0)\cap E[z\in R:\ \delta_1<\omega^*(z)<\delta_2]$ with values δ_i on C_{δ_i} and $\frac{\partial\omega^*_{n,n+j}}{\partial n}(z)=0$ on $\partial R_{n+i}\cap E[z\in R:\ \delta_1<\omega^*(z)<\delta_2]$. Then by property (4), $\omega^*_{n+j}(z)\to\omega^*_n(z)$ and $\omega^*_n(z)\to\omega^*(z)$ in mean.

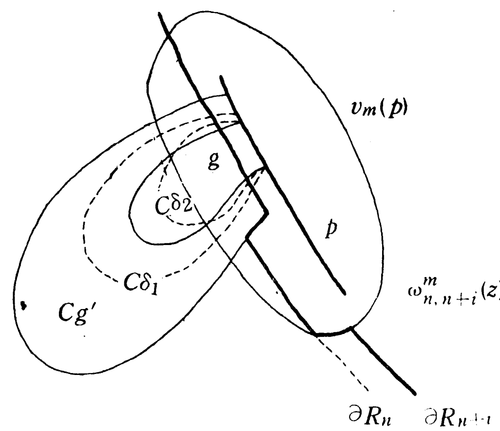

Fig. 2

Apply the Green's formula

$$\int_{C_{\delta_1}+C_{\delta_2}}\omega^m_{n,n+j}(z)\frac{\partial}{\partial n}\omega^*_{n+j}(z)ds=\int_{C_{\delta_1}+C_{\delta_2}}\omega^*_{n+j}(z)\frac{\partial}{\partial n}\omega^m_{n,n+j}(z)ds\,. \tag{2}$$

But $\int_{C_{\delta_i}\cap R_{n+j}}\frac{\partial}{\partial n}\omega^m_{n,n+j}(z)ds=0$, whence $\int_{C_{\delta_1}\cap R_{n+j}}\omega^m_{n,n+j}(z)\frac{\partial}{\partial n}\omega^*_{n,n+j}(z)ds=$ $\int_{C_{\delta_2}\cap R_{n+j}}\omega^m_{n,n+j}(z)\frac{\partial}{\partial n}\omega^*_{n,n+j}(z)\,ds$. By the regularity of C_{δ_i} and by letting $j\to\infty$, $n\to\infty$ and $m\to\infty$. Then by (2)

$$\int_{C_{\delta_1}}\omega_i(z)\frac{\partial\omega^*}{\partial n}(z)ds=\int_{C_{\delta_2}}\omega_i(z)\frac{\partial}{\partial n}\omega^*(z)ds\,. \tag{3}$$

Since $(C_{\delta_1}\cap R)>0$ and $\omega_i(z)<1$ in R, $\int_{C_{\delta_1}}\omega_i(z)\frac{\partial}{\partial n}\omega^*(z)ds<\int_{C_{\delta_1}}\frac{\partial}{\partial n}\omega^*(z)ds$ $-\delta_0$ for a number $\delta_0>0$. On the other hand, $\omega^*(z)<\omega_i(z)$ implies $\omega_i(z)$ $\uparrow 1$ on C_{δ_2} as $\delta_2\uparrow 1$. But the right hand of (3) means that there exists at least one point z' on C_{δ_2} such that $\omega_i(z')<1-\delta_0$ on every regular curve C_{δ_2}. This contradict to $\omega_i(z)>\omega^*(z)\uparrow 1$ on C_{δ_2} as $\delta_2\uparrow 1$. Hence we have the lemma.

Theorem 6. *Let R be a Riemann surface with positive boundary. Then $R\in HND$[19] if and only if, the ideal boundary points of $R-R_0$ consists*

19) HDN is the class of Riemann surfaces on which N number of linearly independent Dirichlet bounded harmoni^c functions exist.

of N number of bounded minimal points p_i ($N(z, p_i)$ is bounded minimal) and a set of capacity zero.

Let $U(z)$ be a Dirichlet bounded harmonic function in R. Let $U_n(z)$ be a harmonic function in $R_n - R_0$ such that $U_n(z) = U(z)$ on ∂R_0 and $U_n(z) = 0$ on ∂R_n. Then clearly $U_n(z)$ converges to a function $U'(z)$ and $D(U(z) - U'(z)) < \infty$. Put $U^*(z) = U(z) - U'(z)$. Then $U^*(z)$ is uniquely determined by $U(z)$. Hence we have only to consider Dirichlet bounded harmonic functions in $R - R_0$ vanishing on ∂R_0 instead of Dirichlet bounded harmonic functions in R.

Suppose that $R - R_0$ is a Riemann surface in Lemma 4. Let $U(z)$ be a Dirichlet bounded harmonic function vanishing on ∂R_0. Then $U(z)$ is represented by Poisson's integral[20] and by lemmata 4, 5, 6 $U(z)$ has angular limits $= a_i$ a.e. on E_i. Hence $U(z)$ is a linear form of $N(z, p_i)$. Cleary $N(z, p_i)$ are linearly independent, hence such a Riemann surface $\in$ HND. Next suppose $R \in$ HND. If the capacity of the set of boundary points of capacity zero is positive, we can easily construct a infinite number of Dirichlet bounded harmonic functions which are linearly independent. Hence the capacity of the above set is zero. We see easily that there are exact N number of bounded minimal functions $N(z, p_i)$ in $R - R_0$. Thus we have the theorem.

In another article contained in this volume[21], we proved that every minimal function $N(z, p) = U(z, p) + V(z, p)$, where $U(z, p)$ is representable by Poisson's integral and $V(z, p)$ is a generalized Green's function. Let p be a minimal point of capacity zero and suppose $\overline{\lim_{z \to p}} G(z, q) = 0$ for the Green's funtion $G(z, q)$ (we say that p is regular for the Green's function) and $\sup N(z, p) = \infty$. Then $V(z, p) = 0$. In this case, let $U_n(z)$ be a harmonic function in $R_n - R_0$ such that $U_n(z) = \min(M, N(z, p))$ on $\partial R_0 + \partial R_n$. Then clearly, $U_n(z) \to U(z) > 0$ and by the Dirichlet principle $D(U(z)) \leqq 2\pi M$. Put $U(z, M) = \dfrac{U(z)}{\sup U(z)}$ and $D(U(z, M)) = A_M$. Then we see easily $A_M \downarrow 0$ as $M \uparrow \infty$. Let M_i $(i = 1, 2, \cdots)$ be a sequence such that $A_{M_i} \downarrow 0$. Since $\sup U(z, M_i) = 1$, each $U(z, M_i)$ are linearly independent. Hence we have the following

Theorem 7. *$R \in HND$ has no minimal point p of capacity zero ($N(z, p)$ is minimal and $\sup N(z, p) = \infty$) such that $\overline{\lim_{z \to p}} G(z, q) = 0$.*

Corollary. *If R has only regular boundary points for the Green's*

20) Z. Kuramochi: On the existence of harmonic functions on Riemann surfaces. Osaka Math. Journ. 7, 1955.

21) Z. Kuramochi: On harmonic functions representable by Poisson's integral.

function, $R \in HND$ if and only if, the set of minimal functions consists of exact N number of bounded minimal functions.

Theorem 8. *Let G be a non compact domain and let $N(z, p)$ $(=\kappa\omega(z))$, where $\omega(z)$ is C.P. of $p)$ be a bounded minimal function and let $U(z)$ be a harmonic function in G such that $U(z)=N(z, p)$ on ∂G and $U(z)$ has M.D.I. over G. If $N(z, p)>U(z)$, then there exists no Dirichlet bounded analytic function in G.*

Lemma 7. *Let G and $N(z, p)$ be as above. Then there exists a non compact domain g such that $(B \cap g \cap v_n(p))$ is positive capacity with respect to G.*

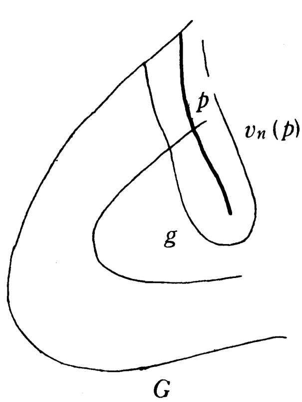

Fig. 3

Since $U(z)$ has M.D.I. among all functions with value $N(z, p)$ on ∂G, $U(z)=U(G', z)$, where $U(G', z)=U(z)$ on $\partial G+\partial G'$ and $U(G', z)$ has M.D.I. over $G-G'$ for every domain $G'\subset G$. Since $N(z, p)=N_{v_n(p)}(z, p)$, $N(z, p)=N(v_n(p), (z, p)$, where $N(v_n(p), z, p)=N(z, p)$ on $\partial G-v_n(p)+(\partial v_n(p)\cap G)$ and has M.D.I. over $G-v_n(p)$. Put $V(z)=N(z, p)-U(z)$ and $g=E[z\in G: V(z)\geqq\frac{M}{2}]$, where $M=\sup V(z)$. Hence

$$\begin{aligned}V(z)&=V(v_n(p)\cap G, z)\leqq V(v_n(p)\cap g, z)\\&\leqq V(v_n(p)\cap Cg, z)\leqq V(v_n(p)\cap Cg, z)\\&\quad+M\omega'(v_n(p)\cap g, z),\end{aligned}$$

where $V(S, z)$ is the function in $G-S$ such that $V(S, z)=V(z)$ on $\partial S+\partial G$ and $V(S, z)$ has M.D.I. over $G-S$ and $\omega'(v_n(p)\cap g, z)$ is the C.P. of $(g\cap v_n(p))$ with respect to G.

Clearly $D(\omega'(v_n(p)\cap g, z))\leqq\frac{4}{M^2}D(U(z))<\infty$. If $\omega'(v_n(p)\cap g, z)\downarrow 0$, as $n\to\infty$. Then sup $V(z)\leqq\frac{M}{2}$. This is a contradiction. Hence $(g\cap p)$ is a set of positive capacity with respect to G.

Let $\omega(g\cap p, z)$ be C.P. of $(g\cap p)$ with respect to $R-R_0$. Then

$$\omega(g\cap p, z)\geqq\omega'(g\cap p, z)>0,$$

whence $\sup\omega(g\cap p, z)=1$, but $\omega(g\cap p, z)$ has no mass except at p, whence $\omega(g\cap p, z)=\omega(z)$, where $\omega(z)$ is C.P. of p.

Lemma 8. *Let G and g be domains in Lemma 7. If a Dirichlet bounded analytic function $W(z)$ exists in G. Then we can find a non compact domain g' such that C.P. of $(g\cap g'\cap p)$ with respect to G is positive and as small as we please.*

Suppose a Dirichlet bounded analytic function $W(z)$ in G. Let $\{\mathfrak{S}_n\}$ be a sequence of triangulation of the w-plane such that whose every triangle $\{\Delta_n^i\}$ has a diameter $<\dfrac{1}{n}$ and $\mathfrak{S}_{n+1}$ is a subdivision of $\mathfrak{S}_n$ and becomes as fine as we please as $n\to\infty$. The part of G whose image lying on Δ_n^i consists of at most enumerably infinite number of component G_n^{ij} $(j=1,2,\cdots)$ compact or not. Then $G=\sum\limits_{n,i,j} G_n^{ij}$. Let $\omega_n'^{ij}(z)$ be C.P. of $(G_n^{ij}\frown p\frown g)$ with respect to G. Then $\sum \omega_n'^{ij}(z)\geqq\omega'(g\frown p, z)>0$, whence there exists at least one component G_n^{ij} such that $\omega_n'^{ij}(z)>0$. Suppose $\omega_{n_0}'^{i_0j_0}(z)>0$. Let L be a compact and bounded arc on ∂G such that the projection of L has a positive distance δ from $\Delta_{n_0}^{i_0j_0}$ [22] ($G_{n_0}^{i_0j_0}$ lies no $\Delta_{n_0}^{i_0j_0}$). As the way above mentioned we can find two sequences

$$G_{n_0}^{i_0j_0}\supset G_1\supset G_2\supset G_3,\cdots$$
$$\Delta_{n_0}^{i_0}\supset\Delta_1\supset\Delta_2\supset\Delta_3,\cdots$$

such that G_s lies on Δ_s, diameter of $\Delta_s<\dfrac{1}{s}$ and C.P. $\omega_s'(z)$ of $(G_s\frown p\frown g)$ with respect to G is positive.

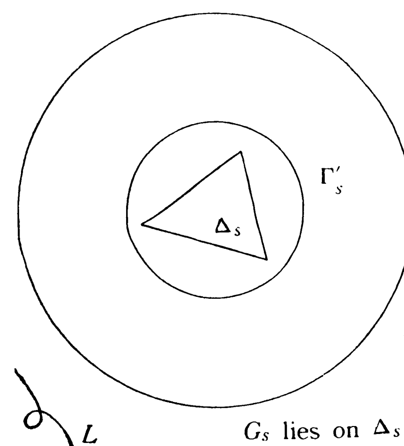

Fig. 4

Let Γ_s and Γ_s' be two concentric circles such that the radius of $\Gamma_s=\dfrac{\delta}{4}$, radius of $\Gamma_s'<\dfrac{1}{s}$ and $\partial\Gamma_s'$ encloses Δ_s. Let $V_s(w)$ be a harmonic function in $\Gamma_s-\Gamma_s'$ such that $V_s(w)=1$ on $\partial\Gamma_s'$ and $V_s(w)=0$ on $\partial\Gamma_s$. Then we see $K_s=\max\left(\left|\dfrac{\partial V(w)}{\partial u}\right|^2+\left|\dfrac{\partial V(w)}{\partial v}\right|^2\right)(w=u+iv)$ $\to 0$ as $s\to\infty$. Suppose $D(A(z))<A$. Then the area of the image of G by $w=A(z)<A$. Let $\tilde{V}_s(w)$ be a continuous function in the whole w-plane such that $\tilde{V}_s(w)\equiv 1$ in Γ_s', $\tilde{V}_s(w)$ is harmonic in $\Gamma_s-\Gamma_s'$ and $\tilde{V}_s(w)$ $\equiv 0$ outside of Γ_s. Let $\tilde{V}_s(z)$ be a continuous function in G such that $\tilde{V}_s(z)\equiv\tilde{V}_s(w(z))$. Then since the image of L lies outside of Γ_s, $\tilde{V}_s(z)=0$ on L and $\tilde{V}_s(z)=1$ in G_s. Then

$$D_G[\tilde{V}_s(z)]\leqq AK_s.$$

Let $\omega_{s,l,m}(z)$ be a harmonic function in $(G\frown R_m)-(v_l(p)\frown g\frown G_s)$ such

22) If we take sufficienty small Δ, we can find L as above mentioned.

that $\omega_{s,m,l}(z)=0$ on L, $\omega_{s,l,m}(z)=1$ on $\partial(v_l(p)\frown g\frown G_s)$ and $\frac{\partial}{\partial n}\omega_{s,m,l}(z)$ $=0$ on $\partial R_m-(g\frown v_l(p)\frown G_s)$. Then

$$D[\omega_{s,m,l}(z)]\leqq D[\tilde{V}_s(z)]\leqq AK_s.$$

Clearly $\omega_{s,m,l}(z)\geqq\omega_s'(z)>0$. Let $m\to\infty$ and then $l\to\infty$, then $\omega_{s,m,l}(z)\to$ $\omega_s^*(z)>\omega_s'(z)>0$ and $D(\omega_s^*(z))=\int_L\frac{\partial\omega_s^*(z)}{\partial n}ds\leqq AK_s$. Let $s\to\infty$, then $\int_L\frac{\partial\omega_s^*(z)}{\partial n}ds\to 0$ and further $\max_{z\in L}\left|\frac{\partial\omega_s^*(z)}{\partial n}\right|\to 0$ $(<\min_{z\in L}\frac{\partial\omega'(z)}{\partial n})$. Hence there exists a point z_0 in a neighbourhood of L and a number s_0 such that

$$\omega_s^*(z_0)>\omega'(z_0)\qquad\text{for}\quad s\geqq s_0,$$

where $\omega'(z)$ is C.P. of $(g\frown p)$ with respect to G.

Let $\omega_s''(z)$ be C.P. of $(p\frown g\frown G_s)$ with respect to G. Then by the Dirichlet principle

$$D(\omega'(z))\geqq D(\omega_s''(z))\geqq D(\omega_s^*(z)),$$

On the other hand, clear $\omega_s^*(z)\geqq\omega_s''(z)$. Hence $\omega'(z_0)>\omega_s^*(z_0)>\omega_s''(z_0)$, whence

$$0<\omega_s''(z)<\omega'(z).$$

Take G_s as g' in the lemma, then we have the lemma. In the sequel we denote $\omega_s''(z)$ by $\omega''(z)$ for simplicity.

Proof of the theorem.

Put $D=E[z\in R:\ \omega'(z)-\omega''(z)\geqq\frac{M}{3}]$ and $D'=E[z\in R:\ \omega'(z)-\omega''(z)$ $\geqq\frac{2}{3}M)$ $(M=\sup(\omega'(z)-\omega''(z))$. Since $\omega'(z)=1$ in $(g\frown p)$ and $\omega''(z)=1$ in $(g'\frown p)$ except at most capacity zero with respect to G by property (2), Cap $(D\frown p)=0$ with respect to G.

Whence Cap $((g'-D)\frown p)>0$ with respect to G. (4)

$\omega'(z)$ and $\omega''(z)$ are C.P.s of $(p\frown g)$ and $(p\frown g')$ respectively. Then by property (4) $\omega'(z)$ and $\omega''(z)$ have M.D.I. over $D-(g\frown v_n(p))$ among all functions with values $\omega'(z)$ and $\omega''(z)$ on $\partial D+\partial(g\frown v_n(p)\frown D)$ respectively. Hence $\omega'(z)-\omega''(z)$ has also M.D.I. over $D-(g\frown v_n(p))$ among all functions with value $\omega'(z)-\omega''(z)$ on $\partial D+\partial(D\frown g\frown v_n(p))$. Let $V_n(z)$ be a harmonic function in D such that $V_n(z)=\min(\omega'(z)-\omega''(z),\ \frac{M}{3})$ on $\partial D+$ $\partial(g\frown v_n(p))$ and $V_n(z)$ has M.D.I. over $D-(D'\frown v_n(p))$. Let $\tilde{V}_n(z)$ be a harmonic function in $D-(D'\frown v_n(p))$ such that $\tilde{V}_n(z)=1$ on $\partial(D'\frown v_n(p))$,

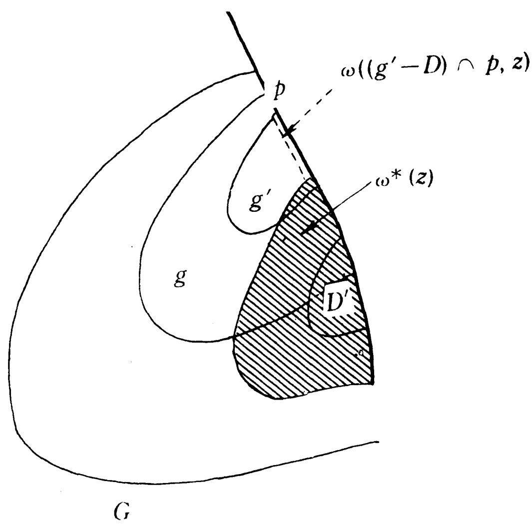

Fig. 5

$\tilde{V}_n(z)=0$ on ∂D and $\tilde{V}_n(z)$ has M.D.I. Then

$$D(\tilde{V}_n(z)) \leqq \frac{4}{9M^2} D(\omega'(z)-\omega''(z))$$

by the maximum principle

$$0 < \omega'(z)-\omega''(z) < V_n(z)+M\tilde{V}_n(z) .$$

Let $n\to\infty$, if $\lim\limits_n \tilde{V}_n(z)=0$, $M=\sup(\omega'(z)-\omega''(z)) \leqq \frac{2M}{3}$. This is contratiction.

Hence C.P. $\omega^*(z)$ of $(D'\bigcap p)>0$ with respect to D. (5)

Let $\omega((g'-D)\bigcap p, z)$ be C.P. of $((g'-D)\bigcap p)$ in G. Then by (4) $\omega((g'-D)\bigcap p, z)>0$ and since $\sup \omega((g'-D)\bigcap p, z)=1$ and $(g'\bigcap p-D) \subset p$,

$$\omega((g'-D)\bigcap p, z) = \omega(z) ,$$

where $\omega(z)$ is C.P. of p.

On the other hand, $\omega^*(z)>0$ and clearly $\omega^*(z)<\omega(z)$.

Hence as in Lemma 6, we can prove that there exists at least a point z_i such that $\omega((g'-D)\bigcap p, z_i)<1-\delta_0(\delta_0>0)$ on every regular curve C_δ of $\omega^*(z)$ as $\delta\uparrow 1$. This contradicts to $\omega((g'-D)\bigcap p, z)\equiv\omega(z) > \omega^*(z)$. Hence we have the theorem.

Theorem 9. *Let $v(p)$ be a neighbourhood of a bounded minimal point p. Then there exists no Dirichlet bounded analytic function in $v(p)$.*

Let $U(z)$ be a harmonic function in $v(p)$ such that $U(z)=N(z, p)$

on $\partial v(p)$ and $U(z)$ has M.D.I. over $v(p)$. Then $U(z)=N_{B-Cv(p)}(z, p)$. Suppose $U(z)=N(z, p)$. Then $U(z)=\int_{B-v(p)} N(z, p)d\mu(p)$. Then we can as in Theorem 4 prove that there exist two positive mass distributions μ_1 and μ_2 in $cv(p)$ such that μ_1 and μ_2 represent functions which are not multiples of $N(z, p)$. Hence $N(z, p)-\int N(z, p)d\mu_1(p)=\int N(z, p)d\mu_2(p)>0$. This contradicts the minimality[23] of $N(z, p)$. Hence $N(z, p)>U(z)$. Hence by Theorem 8, we have Theorem 9.

Theorem 10. *Let G be a non compact domain in $R\in HND$. If there exists a non constant Dirichlet bounded harmonic function $U(z)$ vanishing on ∂G, then there exists no Dirichlet bounded analytic function in G.*

We can suppose that $G\subset R-R_0$. Then by Theorem 6, $R-R_0$ has N number of bounded minimal points. Map the universal covering surface G^∞ onto $|\xi|<1$. Then $U(z)$ is represented by Poisson's integral. Hence there exists a set E of positive measure on $|\xi|=1$ such that $U(z)$ has angular limits $>\delta$ or $<-\delta$ a.e. on E. We can suppose $U(z)>\delta$ on E. Put $G'=E[z\in R: U(z)>\frac{\delta}{2}]$ and let $\omega^*(z)$ be C.P. of $(B\frown G')$ with respect to G. Then

$$D(\omega^*(z))<\frac{4}{\delta^2}D(U(z)) \quad \text{and by } \operatorname{mes} E>0, \quad \omega^*(z)>0.$$

Since by Theorem 6 $\operatorname{Cap}(B-\sum p_i)=0$ and $\operatorname{Cap}(B-\sum p_i)$ with respect to $G=0$, $\operatorname{Cap}(G'\frown\sum p_i)$ with respect to $G=\operatorname{Cap}(B\frown G')$ with respect to G.

Then there exists at least a point p_i such that $\operatorname{Cap}(G'\frown p_i)$ with respect to $G>0$, whence C.P. $\omega(G'\frown p_i, z)$ of $(G'\frown p_i)=\omega_i(z)$ by $\sup \omega(G'\frown p_i, z)=1$. Put $\omega^{**}(z)=\omega(G'\frown p_i, z)$. Next let $V(z)$ be a harmonic function in G such that $V(z)=\omega^{**}(z)$ on ∂G and $V(z)$ has M.D.I. over G. Then as in Theorem 6

$$\int_{C_1+C_2} V(z)\frac{\partial\omega^*}{\partial n}(z)ds=\int_{C_1+C_2}\omega^*(z)\frac{\partial}{\partial n}V(z)ds$$

$$\int_{C_1} V(z)\frac{\partial\omega^*}{\partial n}(z)ds=\int_{C_2} V(z)\frac{\partial\omega^*}{\partial n}(z)ds,$$

where C_1 and C_2 are regular curve of $\omega^*(z)$. Then there at least a point

23) If $U(z)$ has no functions $V(z)$ such that both $V(z)>0$ and $U(z)-V(z)>0$ are harmonic and superharmonic in $\overline{R-R_0}$ except its own multiples, we say that $U(z)$ is a minimal function.

z_i on C_2 such that $V(z_i) < 1-\delta_0$ for a positive number δ_0, whence $V(z) < \omega_i(z)$. Hence there exists at least a point p_i such that $N(z, p_i) - N_{CG}(z, p_i) > 0$, whence by Theorem 8, we have the theorem.

Part III

Suppose an analytic function $w = f(z)$ in R. Let w_0 be a point of the w-plane. Then the part of R on $|w-w_0| < r$ consists of at most enumerably infinite number of components. Such one component is called a connected piece on $|w-w_0| < r$. Then

Theorem 11. *Let R be a Riemann surface in Theorem 5, i.e., there exists at most enumerably infinite number of bounded minimal functions $K(z, p_i)$ and a set of boundary of harmonic measure zero. Then every connected piece C on $|w-w_0| < r$ covers $|w-w_0| < r$ except at most a set of capacity zero.*

Let G be a non compact domain such that $f(G) = C$. Suppose C does not cover a set F (clearly closed) of positive capacity. Then there exists a subset F' of F of positive capacity such that $F' \subset E[|w-w_0| < r' < r]$. Then there exists a positive bounded harmonic function $\omega(w)$ in C vanishing on $|w-w_0| = r$. Put $\omega(z) = \omega(f(z))$. Then $\omega(z)$ is bounded harmonic function in G vanishing on ∂G. Then by Theorem 6, there exists no bounded analytic function in G. But $|f(z)-w_0| < r$ on G. This is a contradiction. Hence we have the theorem.

Theorem 12. *Let $R \in HND$, and let C be a connected piece on $|w-w_0| < r$. If the area of C is finite, C cover $|w-w_0| < r$ except at most a set of capacity zero.*

Suppose C does not cover a set of positive capacity. Then as in Theorem 11, there exists a non constant positive bounded harmonic function $\omega(z)$ in G vanishing on ∂G. We map the universal covering surface G^∞ onto $|\xi| < 1$. Then there exists a set E of positive measure such that $\omega(z)$ has angular limits $> \delta_0 > 0$ on E for a constant δ_0. Now $f(z) = w$ is bounded in G. $f(z)$ has angular limits a.e. on E. Then there exists a number r' and a set $E'(\subset E)$ of positive measure such that $f(z)$ has angular limits in $|w-w_0| < r' < r$ a.e. on E'. Hence there exists a closed set $E'' \subset E'$ of positive measure such that both $\omega(z)$ and $f(z)$ converge uniformly in angular domain. Put $g = E[z \in G : \omega(z) > \frac{\delta}{2}] \cap E[z \in G : |f(z)-w_0| < r']$. Let $V_{n,n+i}(z)$ be a harmonic function in $G \cap (R_{n+i} - ((R_{n+i}-R_n) \cap g))$ such that $V_{n,n+i}(z) = 0$ on $(\partial G \cap R_{n+i}) + \partial R_{n+i} - g$ and $V_{n,n+i}(z) = 1$ on $\partial(R_{n+i}-R_n) \cap g$. Then $\lim_n \lim_i V_{n,n+i}(z) > 0$. Next let

$S(w)$ be a harmonic function in $r'<|w-w_0|<r$ such that $S(w)=0$ on $|w|=r'$ and $S(w)=1$ on $|w|=r$ and $S(w)\equiv 1$ in $|w-w_0|<r'$. Then $\max\left(\left|\frac{S(w)}{\partial u}\right|^2+\left|\frac{S(w)}{\partial v}\right|^2\right)\leqq K$: $w=u+iv$. Let $T(z)$ be a continuous function in G such that $T(z)\equiv S(f(z))$. Then $D(T(z))<KD_G(f(z))$. Hence there exists a harmonic function such that $W(z)>0$, $W(z)=0$ on ∂G and $D(W(z))\leqq D(T(z))\leqq KD(f(z))$ by the Dirichlet principle. Hence by Theorem 10, we have the theorem.

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