Osaka Mathamatical Journal
Vol. 5, No. 1, June 1953

An Existence Proof of the Generalized Green Function

By Shigeharu HARADA

§1. We denote by $L(u)$ the differential operator

$$L(u)=\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu$$

where $p(x)$ and $q(x)$ are continuous on the interval $\langle a, b\rangle$ and $p(x)>0$. The purpose of this note is to construct concretely the generalized Green function of the differential equation

$$L(u)=0 \tag{1.1}$$

with the boundary conditions

$$\begin{cases} U(u)=u(a)-\gamma u(b)=0 \\ V(u)=p(a)\,u'(a)-\dfrac{p(b)}{\gamma}u'(b)=0\,, \end{cases} \tag{1.2}$$

or

$$\begin{cases} U(u)=u(a)-\gamma p(b)\,u'(b)=0 \\ V(u)=p(a)\,u'(a)+\dfrac{1}{\gamma}u(b)=0 \end{cases} \tag{1.3}$$

where γ is a non-zero real number.

As is well known, in the case there is no solution $u_0(x)\not\equiv 0$ of (1.1) satisfying the boundary conditions (1.2) (or (1.3)), the Green function $G(x, \xi)$ of (1.1) with the boundary conditions (1.2) (or (1.3)) exists and its concrete construction is given in the literatures.[1)]

In the case there exists such a solution $u_0(x)$ of (1.1) with the boundary bonditions (1.2) (or (1.3)), the generalized Green function is also defined in these literatures. But, as far as the author knows, its existence proof cannot be found there. Thus the author assumes that

1) Cf. D. Hilbert: Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, Leipzig und Berlin, 1912.

R. Courant and D. Hilbert: Methoden der mathematischen Physik, I, Berlin, 1931.

A. Kneser: Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik, Braunschweig, 2 te Auflage, 1922.

the following concrete construction of the generalized Green function will be of some use.

The author expresses his hearty thanks to Professor K. Yosida who suggested the problem and gave in essentials the fundamental formula (2.6).

§2. In the case there exists a solution $u_0(x)\not\equiv 0$ of (1.1) with the boundary conditions (1.2) (or (1.3)), the function $G(x, \xi)$ is called *the generalized Green function*[2] when it has the following property; for any $\varphi(x)$ such that $\int_a^b \varphi(x)u_0(x)dx = 0$, the solution $u(x)$ of $L(u) = -\varphi$ with the boundary conditions (1.2) (or (1.3)) is expressed by

$$u(x) = \int_a^b G(x, \xi)\varphi(\xi)d\xi .$$

It is well known that the necessary and sufficient conditions for $G(x, \xi)$ to be the generalized Green function are as follows:

(2.1) $G(x, \xi)$ is continuous in x and ξ and its first partial derivative relative to x is bounded for $x\neq\xi$,

(2.2) for any fixed ξ, $G(x, \xi)$ satisfies the boundary conditions (1.2) (or (1.3)) as the function of x,

(2.3) for any $x\neq\xi$, $G(x, \xi)$ satisfies $L(G(x, \xi)) = \alpha u_0(x)u_0(\xi)$ as the function of x,

$$G_x(\xi+0, \xi)-G_x(\xi-0, \xi) = -1/p(\xi), \tag{2.4}$$

and

$$\int_a^b G(x, \xi)u_0(x)dx = 0 . \tag{2.5}$$

Let $u_1(x)$ be a solution of (1.1) which is linearly independent of $u_0(x)$. Then the pair $u_0(x)$, $u_1(x)$ is a fundamental system of solutions of (1.1), and at least one of the boundary conditions (1.2) fails to be satisfied by $u_1(x)$, say $U(u_1)\neq 0$. Therefore we may assume that there exists a solution $y(x)$ of $L(y) = u_0(x)$ such that $U(y) = 0$ and $V(y)\neq 0$.[3]

2) We use simply "the (generalized) Green function" instead of the (generalized) Green function of the differential equation (1.1) with the boundary conditions (1.2) (or (1.3)).

3) Let $y_0(x)$ be a solution of $L(y_0)=u(x)$. Assume that $y_0(x)$ satisfies both $U(y_0)=0$ and $V(y_0) = 0$. Applying the Green formula to $y_0(x)$ and $u_0(x)$, we get

$$\int_a^b (u_0(x)L(y_0) + y_0(x)L(u_0))\,dx = \int_a^b u_0^2(x)dx = 0 .$$

This contradicts the fact $u_0(x) \not\equiv 0$. Therefore any solution $y_0(x)$ of $L(y_0) = u_0(x)$ does not satisfy at least one of the boundary conditions (1.2). Since, for any c, $y_0(x)+cu_1(x)$ is also a solution of $L(y)=u_0(x)$, the existence of such a solution $y(x)$ is obvious.

Put

$$(2.6)\qquad G(x,\xi)=\begin{cases} k_1u_0(\xi)y(x)+c_0(\xi)u_0(x)+\sqrt{k_1k_2}c_1(\xi)u_1(x) \\ \qquad +k_1k_2\{u_0(\xi)u_1(x)-u_0(x)u_1(\xi)\} & \text{for}\quad x\geqslant\xi \\ k_1u_0(\xi)y(x)+c_0(\xi)u_0(x)+\sqrt{k_1k_2}c_1(\xi)u_1(x) & \text{for}\quad x<\xi, \end{cases}$$

and we shall show that we can determine these coefficients k_1, k_2, $c_0(\xi)$ and $c_1(\xi)$ by the properties (2.1)—(2.5) of the generallized Green function.

The following lemma is needed in the sequel.

Lemma 1. *For $u_0(x)$ and $u_1(x)$, the following relations hold:*

$$(2.6)\qquad \gamma u_0(b)V(u_1)-\frac{p(b)}{\gamma}u_0'(b)U(u_1)=0$$

and

$$(2.7)\qquad \gamma u_1(b)V(u_1)-\frac{p(b)}{\gamma}u_1'(b)U(u_1)\neq 0\,.$$

Proof. Applying the Green formula to $u_0(x)$ and $u_1(x)$, we get

$$\Big[p(x)\{u_0(x)u_1'(x)-u_0'(x)u_1(x)\}\Big]_a^b=0\,.$$

Since $u_0(x)$ satisfies $U(u_0)=0$ and $V(u_0)=0$, we obtain

$$p(a)\gamma u_0(b)u_1'(a)-\frac{p(b)}{\gamma}u_0'(b)u_1(a)=p(b)u_0(b)u_1'(b)-p(b)u_0'(b)u_1(b)\,,$$

therefore

$$\gamma u_0(b)V(u_1)-\frac{p(b)}{\gamma}u_0'(b)U(u_1)=0\,.$$

Assume that (2.7) does not hold, i.e.

$$(2.8)\qquad \gamma u_1(b)V(u_1)-\frac{p(b)}{\gamma}u_1'(b)U(u_1)=0\,.$$

Since $V^2(u_1)+U^2(u_1)\neq 0$, we get from (2.6) and (2.8)

$$\begin{vmatrix} u_0(b) & u_0'(b) \\ u_1(b) & u_1'(b) \end{vmatrix}=0\,.$$

This is a contradiction to the fact that $u_0(x)$ and $u_1(x)$ are linearly independent. Thus the proof is complete.

First we shall determine k_1. By (2.3) we get

$$(2.9) \qquad \sqrt{k_1}k_2c_1(\xi)U(u_1)-\gamma k_1k_2(u_0(\xi)u_1(b)-u_0(b)u_1(\xi))=0$$

and

$$(2.10) \quad k_1u_0(\xi)V(y)+\sqrt{k_1}k_2c_1(\xi)V(u_1)-\frac{k_1k_2}{\gamma}p(b)(u_0(\xi)u_1'(b)-u_0'(b)u_1(\xi))=0\,.$$

Eliminating $c_1(\xi)$ in (2.9) and (2.10), we obtain

$$\begin{aligned} k_1u_0(\xi)V(y) &= \frac{k_1k_2}{\gamma}p(b)(u_0(\xi)u_1'(b)-u_0'(b)u_1(\xi)) \\ &\quad -\frac{V(u_1)}{U(u_1)}k_1k_2\gamma(u_0(\xi)u_1(b)-u_0(b)u_1(\xi)) \\ &= k_1k_2\left\{u_0(\xi)\left(\frac{p(b)}{\gamma}u_1'(b)-\gamma\frac{V(u_1)}{U(u_1)}u_1(b)\right)\right. \\ &\quad \left.-u_1(\xi)\left(\frac{p(b)}{\gamma}u_0'(b)-\gamma\frac{V(u_1)}{U(u_1)}u_0(b)\right)\right\}. \end{aligned}$$

Since the coefficient of $u_0(\xi)$ is not zero and that of $u_1(\xi)$ is zero (by (2.6) and (2.7)) we get

$$(2.11) \qquad k_2=\frac{V(y)}{\dfrac{p(b)}{\gamma}u_1'(b)-\gamma\dfrac{V(u_1)}{U(u_1)}u_1(b)} \neq 0\,.$$

Since $G(x, \xi)$ satisfies (2.4), we get

$$(2.12) \qquad k_1k_2(u_0(\xi)u_1'(\xi)-u_0'(\xi)u_1(\xi))=-1/p(\xi)\,.$$

Using (2.11) and (2.12) we can determine k_1, i.e.

$$(2.13) \qquad k_1=-\frac{1}{k_2p(\xi)(u_0(\xi)u_1'(\xi)-u_0'(\xi)u_1(\xi))}\,.$$

Using the relation

$$(2.14) \qquad c_1(\xi)=\frac{1}{U(u_1)}\sqrt{k_1}\,\gamma(u_0(\xi)u_1(b)-u_0(b)u_1(\xi))\,,$$

which is obtained from (2.9), and (2.12), we can determine completely $c_1(\xi)$.

Then we can determine $c_0(\xi)$ by the orthogonal relation (2.5).

It is obvious that by our construction the function $G(x, \xi)$ obtained above satisfies (2.1)—(2.5). Thus the construction of the generalized Green function is completed in the case when the boundary conditions are (1.2).

§ 3. In the case when boundary conditions are (1.3) we can obtain the following lemma in the same way as in the proof of Lemma 1.

Lemma 2. *Let* $u_0(x)$ *be a solution of* (1.1) *which satisfies the boundary conditions* (1.3) *and* $\not\equiv 0$. *Let* $u_1(x)$ *be a solution of* (1.1) *which is linaerly independent of* $u_0(x)$ *and does not satisfy at least one of the conditions of* (1.3). *Then we get*

$$\gamma p(b)u_0'(b)V(u_1)+\frac{1}{\gamma}u_0(b)U(u_1)=0 \tag{3.1}$$

and

$$\gamma p(b)u_1'(b)V(u_1)+\frac{1}{\gamma}u_0(b)U(u_1)\neq 0\,. \tag{3.2}$$

Using this lemma, we can also construct the generalized Green function of (1.1) with the boundary conditions (1.3) in the same way as in §2.

(Received March 6, 1953)