Takahashi, F.
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ASYMPTOTIC BEHAVIOR OF LEAST ENERGY SOLUTIONS TO A FOUR-DIMENSIONAL BIHARMONIC SEMILINEAR PROBLEM

FUTOSHI TAKAHASHI

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Abstract

In this paper, we study the following fourth order elliptic problem (E_p):

$$(E_p)\begin{cases} \Delta^2 u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^4$, $\Delta^2 = \Delta\Delta$ is a biharmonic operator and $p > 1$ is any positive number.

We investigate the asymptotic behavior as $p \to \infty$ of the least energy solutions to (E_p). Combining the arguments of Ren-Wei [8] and Wei [10], we show that the least energy solutions remain bounded uniformly in p, and on convex bounded domains, they have one or two "peaks" away form the boundary. If it happens that the only one peak point appears, we further prove that the peak point must be a critical point of the Robin function of Δ^2 under the Navier boundary condition.

1. Introduction.

In this paper, we study the following fourth order elliptic problem (E_p):

$$(E_p)\begin{cases} \Delta^2 u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^4$, $\Delta^2 = \Delta\Delta$ is a biharmonic operator and $p > 1$ is any positive number. Boundary condition imposed in (E_p) is sometimes called the Navier boundary condition.

One of motivation to study such a problem involving biharmonic operator in $\mathbf{R}^4$ comes from the recent development in conformal geometry on four-manifold. See [3].

On the other hand, problem (E_p) may be regarded as a natural extension to a higher dimensional case, of the two-dimensional problem treated by Ren and Wei [8], [9].

Ren and Wei considered the least energy soluton u_p of the following semilinear

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problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where Ω is a bounded smooth domain $\subset \mathbf{R}^2$. They studied the asymptotic behavior of u_p as the nonlinear exponent $p \to \infty$, and they proved that the least energy solutions remain bounded in L^∞-norm regardless of p. On the shape of solutions, they showed that the least energy solutions must develop one "peak" in the interior of Ω, that is the graph of u_p is becoming like a single spike as $p \to \infty$. Moreover they showed that this peak point must be a critical point of the Robin function of the domain.

Now, in this paper we investigate that whether the analogues of the results of Ren and Wei would hold to the higher dimensional fourth order problem (E_p).

Since the complete structure of solution set of the simple-looking problem (E_p) is widely open, so we restrict our attention, as Ren and Wei did, to the least energy solution constructed as follows.

Let us consider the constrained minimization problem:

$$C_p^2 := \inf \left\{ \int_\Omega |\Delta u|^2 \, dx : u \in H^2 \cap H_0^1(\Omega), \ \|u\|_{p+1} = 1 \right\}. \tag{1.1}$$

Since the Sobolev imbedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is compact for any $p > 1$, we have at least one minimizer $\underline{u}_p$ for the problem (1.1), where $\underline{u}_p \in H^2 \cap H_0^1(\Omega)$, $\|\underline{u}_p\|_{p+1} = 1$.

Without losing generality, we may assume $\underline{u}_p > 0$. Indeed, let v solves

$$\begin{cases} -\Delta v = |\Delta \underline{u}_p| & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Then the maximum principle implies $v \geq |\underline{u}_p|$, so we have

$$C_p^2 = \frac{\int_\Omega |\Delta \underline{u}_p|^2 \, dx}{\left(\int_\Omega |\underline{u}_p|^{p+1} \, dx\right)^{2/(p+1)}} \geq \frac{\int_\Omega |\Delta v|^2 \, dx}{\left(\int_\Omega v^{p+1} \, dx\right)^{2/(p+1)}},$$

that is, the positive function v also minimizes C_p^2.

Set

$$u_p := C_p^{2/(p-1)} \underline{u}_p, \tag{1.2}$$

then u_p solves (E_p) and $C_p = \|\Delta u_p\|_{L^2} / \|u_p\|_{L^{p+1}}$. Standard regularity argument implies that any weak solution $u \in H^2 \cap H_0^1(\Omega)$ satisfies $\Delta u \in W^{2,\sigma} \cap W_0^{1,\sigma}(\Omega)$ $(\forall \sigma > 1)$ (for example, see [4]). Therefore u_p is smooth and $u_p = \Delta u_p = 0$ on $\partial\Omega$.

We call u_p the *least energy solution* to (E_p).

Our first result is the same as the one in Ren and Wei.

Theorem 1.1. *Let u_p be a least energy solution to (E_p). Then there exist C_1, C_2 (independent of p), such that*

$$0 < C_1 < \|u_p\|_{L^\infty} < C_2 < \infty$$

for p large enough.

To state further results, we need some definitions. Set

$$w_p := \frac{u_p}{\int_\Omega u_p^p\, dx}. \tag{1.3}$$

For a sequence w_{p_n} of w_p, we define the *blow up set S* of $\{w_{p_n}\}$ as usual:

$$S := \{x \in \overline{\Omega} \colon \exists \text{ a subsequence } w_{p_n'},\ \exists \{x_n\} \subset \Omega \text{ such that } x_n \to x \text{ and } w_{p_n'}(x_n) \to \infty\}.$$

We also define a *peak point P* for u_p to be a point in $\overline{\Omega}$ such that u_p does not vanish in the L^∞ norm in any small neighborhood of P as $p \to \infty$. It turns out later that the set of peak points of $\{u_p\}$ is contained in the blow up set of $\{w_p\}$.

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^4$ be a smooth convex bounded domain. Then for any sequence w_{p_n} of w_p with $p_n \to \infty$, there exists a subsequence (still denoted by w_{p_n}) such that the blow up set S of this subsequence is contained in Ω and has the property $1 \le card(S) \le 2$.*

If $card(S) = 1$ and $S = \{x_0\}$ (one point blow up), then:

$$f_n := \frac{u_{p_n}^{p_n}}{\int_\Omega u_{p_n}^{p_n}\, dx} = \left(\int_\Omega u_{p_n}^{p_n}\, dx\right)^{p_n - 1} w_{p_n}^{p_n} \overset{*}{\rightharpoonup} \delta_{x_0} \tag{1}$$

in the sense of Radon measures of Ω.

(2) *$w_{p_n} \to G_4(\cdot, x_0)$ in $C^4_{loc}(\overline{\Omega} \setminus \{x_0\})$ where $G_4(x, y)$ denotes the Green function of Δ^2 under the Navier boundary condition*:

$$\begin{cases} \Delta_x^2 G_4(x, y) = \delta_y(x), & x \in \Omega, \\ G_4(x, y)|_{x \in \partial\Omega} = \Delta G_4(x, y)|_{x \in \partial\Omega} = 0. \end{cases}$$

(3) *x_0 is a critical point of the Robin function $R_4(x) = H_4(x, x)$, where*

$$H_4(x, y) := G_4(x, y) + \frac{1}{4\sigma_4} \log |x - y|$$

denotes the regular part of G_4 *and* $\sigma_4 = 2\pi^2$ *is the volume of the unit sphere* S^3 *in* $\mathbf{R}^4$.

REMARK 1.3. (1) In Theorem 1.2, the convexity assumption of Ω is needed to derive a uniform boundary estimate of solutions to (E_p) by the use of Method of Moving Planes (MMP).

In Δ case, MMP still works well for non-convex domains through the application of Kelvin transformations, and leads to a uniform boundary estimate. However in our Δ^2 case, Kelvin transformation does not work well simply because Laplacian of a Kelvin transformed function is not 0 on the boundary. It is unclear that actually the boundary blow-up can occur if Ω is not convex.

(2) We conjecture that $\mathrm{card}(S) = 1$ at least for any convex domain. If the following upper bound

$$M := \limsup_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} < e,$$

holds true, then we can prove the conjecture affirmatively.

Further we believe that in this case the blow up point x_0 is not only a critical point, but the maximum point of Robin function. In 2-dimensional Δ case, this is already proved by Flucher and Wei [5].

2. Estimates for C_p

First we recall D. Adams' version of higher order Trudinger-Moser inequality [1]. This is a key ingredient in our analysis.

Theorem ([1]). *Let* $\Omega \subset \mathbf{R}^{2n}$ *be a bounded domain and define* $X := H^n(\Omega) \cap \{u\colon (-\Delta)^j u \in H_0^1(\Omega),\ j = 0, 1, \dots, [(n-1)/2]\}$. *Then there exists* $C_0 = C_0(n)$ *such that*

$$\int_\Omega \exp\left(\alpha_{2n} \frac{u^2(x)}{\|(-\Delta)^{n/2} u\|^2_{L^2(\Omega)}}\right) dx \le C_0|\Omega|, \quad \forall u \in X.$$

Here, $\alpha_{2n} = 2^{2n-1} n!\,(n-1)!\,\sigma_{2n}$, σ_{2n} *is the volume of* S^{2n-1} *and* $|\Omega|$ *is the Lebesgue measure of* Ω.

In $\mathbf{R}^4$ $(n = 2)$, *we have*

$$\int_\Omega \exp\left(\alpha_4 \frac{u^2(x)}{\|\Delta u\|^2_{L^2(\Omega)}}\right) dx \le C_0|\Omega|, \quad \forall u \in H^2 \cap H_0^1(\Omega)$$

where $\alpha_4 = 16\sigma_4 = 32\pi^2$.

Using this, we derive the following refined Sobolev imbedding. Though the proof is the same as in [8], we state here for reader's convenience.

Lemma 2.1 (refinement of the Sobolev imbedding). *For any $t \geq 2$, there exists $D_t > 0$ such that for any $u \in H^2 \cap H_0^1(\Omega)$,*

$$\|u\|_{L^t(\Omega)} \leq D_t t^{1/2} \|\Delta u\|_{L^2(\Omega)}$$

holds true. Furthermore, we have

$$\lim_{t \to \infty} D_t = (32\sigma_4 e)^{-1/2}.$$

Proof. Let $u \in H^2 \cap H_0^1(\Omega)$. Elementary inequality says that $x^s/\Gamma(s+1) \leq e^x$ for $\forall x \geq 0$, $\forall s \geq 0$, where $\Gamma(s)$ is the Γ function.

By Adams' version Trudinger-Moser inequality, we have

$$\begin{aligned}
&\frac{1}{\Gamma((t/2)+1)} \int_\Omega |u|^t \, dx \\
&= \frac{1}{\Gamma((t/2)+1)} \int_\Omega \left(\alpha_4 \frac{u^2}{\|\Delta u\|^2_{L^2(\Omega)}} \right)^{t/2} dx \times \alpha_4^{-t/2} \|\Delta u\|^t_{L^2(\Omega)} \\
&\leq \int_\Omega \exp \left(\alpha_4 \frac{u^2(x)}{\|\Delta u\|^2_{L^2(\Omega)}} \right) dx \times \alpha_4^{-t/2} \|\Delta u\|^t_{L^2(\Omega)} \\
&\leq C_0 |\Omega| \alpha_4^{-t/2} \|\Delta u\|^t_{L^2(\Omega)}.
\end{aligned}$$

Set

$$D_t := \left(\Gamma \left(\frac{t}{2} + 1 \right) \right)^{1/t} C_0^{1/t} |\Omega|^{1/t} \alpha_4^{-1/2} t^{-1/2}.$$

Then we have

$$\|u\|_{L^t(\Omega)} \leq D_t t^{1/2} \|\Delta u\|_{L^2(\Omega)}.$$

Stirling's formula says that $(\Gamma((t/2)+1))^{1/t} \sim (t/(2e))^{1/2}$ as $t \to \infty$. So we have

$$\lim_{t \to \infty} D_t = \left(\frac{1}{2\alpha_4 e} \right)^{1/2},$$

which is a desired result. □

Recall C_p defined in (1.1). Using the above Lemma and energy comparison, we get the following.

Lemma 2.2 (asymptotics for C_p). *We have*

$$\lim_{p \to \infty} p^{1/2} C_p = (32\sigma_4 e)^{1/2}.$$

Proof. Lower bound $(32\sigma_4 e)^{1/2} \le \liminf_{p\to\infty} p^{1/2} C_p$ is a direct consequence of Lemma 2.1 and the fact $C_p = \|\Delta u_p\|_{L^2}/\|u_p\|_{L^{p+1}}$ for least energy solutions u_p.

Therefore we must prove only the upper bound: $\limsup_{p\to\infty} p^{1/2} C_p \le (32\sigma_4 e)^{1/2}$. We will do this by constructing a suitable test function for the value C_p.

We may assume $0 \in \Omega$ and $B_L(0) \subset \Omega$. For $0 < l < L$, consider Moser function

$$m_l(|x|) := \begin{cases} \dfrac{A}{\sigma}, & 0 \le |x| \le l, \\ \left(\dfrac{1}{\sigma A}\right) \log \dfrac{L}{|x|}, & l \le |x| \le L, \\ 0 & \Omega \cap \{L \le |x|\} \end{cases}$$

where $A^2 = \log(L/l)$ and $\sigma = \sqrt{4\sigma_4} = 2\sqrt{2}\pi$.

Note that the Moser function $m_l \in H_0^1(\Omega)$ but not in $H^2(\Omega)$. So we cannot test the value C_p by m_l directly.

Define

$$g_l(|x|) := \begin{cases} 0, & 0 \le |x| \le l, \\ \left(\dfrac{1}{\sigma A}\right) \dfrac{-2}{|x|^2}, & l \le |x| \le L, \\ 0, & \Omega \cap \{L \le |x|\}. \end{cases}$$

Then $g_l \in L^2(\Omega)$ and $\|g_l\|_{L^2(\Omega)} = 1$ by our choice of σ.

Take the unique solution $a \in H_0^1(\Omega)$, $\tilde{a} \in H_0^1(B_L)$ to the problem

$$\begin{cases} -\Delta a = |g_l| & \text{in } \Omega, \\ a|_{\partial\Omega} = 0 \end{cases}$$

and

$$\begin{cases} -\Delta \tilde{a} = |g_l| & \text{in } B_L, \\ \tilde{a}|_{\partial B_L} = 0 \end{cases}$$

respectively.

By elliptic regularity, we have $a \in H^2 \cap H_0^1(\Omega)$. On the other hand, if extended by 0, $\tilde{a}$ can be regarded as in $H_0^1(\Omega)$, but not in $H^2 \cap H_0^1(\Omega)$. Also note that $\tilde{a}$ is a radial function which is explicitly calculate by the formula

$$\tilde{a}(r) = \tilde{a}(0) + \int_{t=0}^{t=r} t^{-3} \left(\int_{s=0}^{s=t} -s^3 |g_l(s)|\, ds \right) dt$$

for $0 \le r = |x| \le L$. Indeed, by definition of g_l, we calculate

$$\tilde{a}(r) = \tilde{a}(0), \quad (0 \le r \le l)$$

$$\tilde{a}(r) = \tilde{a}(0) + \left(\frac{1}{\sigma A}\right)\left(\frac{1}{2} - \frac{1}{2}\left(\frac{l^2}{r^2}\right) + \log\left(\frac{l}{r}\right)\right). \quad (l \le r \le L) \tag{2.1}$$

Here, $\tilde{a}(0)$ is determined by the boundary condition $\tilde{a}(L) = 0$:

$$\tilde{a}(0) = \left(\frac{1}{\sigma A}\right)\left(\frac{1}{2}\left(\frac{l^2}{L^2}\right) - \frac{1}{2} + \log\left(\frac{L}{l}\right)\right). \tag{2.2}$$

Maximum principle implies that $a \in H^2 \cap H_0^1(\Omega)$ satisfies $a > \tilde{a}$ in Ω. Therefore $\|a\|_{L^{p+1}(\Omega)} \ge \|\tilde{a}\|_{L^{p+1}(\Omega)} \ge \|\tilde{a}\|_{L^{p+1}(B_l(0))}$. Also $\|\Delta a\|_{L^2(\Omega)} = \|g_l\|_{L^2(\Omega)} = 1$.

Now, we test the value C_p with $a \in H^2 \cap H_0^1(\Omega)$. Then

$$C_p \le \frac{\|\Delta a\|_{L^2(\Omega)}}{\|a\|_{L^{p+1}(\Omega)}} \le \frac{1}{\|\tilde{a}\|_{L^{p+1}(B_l(0))}} = |\tilde{a}(0)|^{-1}|B_l(0)|^{-1/(p+1)}.$$

Setting $l = L\exp(-(p+1)/8)$, multiplying $p^{1/2}$ and letting $p \to \infty$ in the above inequality, we have, by using (2.2), $\limsup_{p\to\infty} p^{1/2}C_p \le (32\sigma_4 e)^{1/2}$. This proves Lemma. □

Since

$$C_p = \frac{\|\Delta u_p\|_{L^2}}{\|u_p\|_{L^{p+1}}}$$

and

$$\int_\Omega |\Delta u_p|^2\,dx = \int_\Omega u_p^{p+1}\,dx,$$

we have

$$C_p^2 = \|u_p\|_{p+1}^{p-1}, \quad p\|u_p\|_{p+1}^{p+1} = pC_p^2 C_p^{4/(p-1)}.$$

Therefore Lemma 2.2 implies

Corollary 2.3.

$$\lim_{p\to\infty} p\int_\Omega u_p^{p+1}\,dx = 32\sigma_4 e, \quad \lim_{p\to\infty} p\int_\Omega |\Delta u_p|^2\,dx = 32\sigma_4 e.$$

3. Proof of Theorem 1.1

First, we show that a uniform lower bound of L^∞ norm exists for any solution u to (E_p).

Indeed, let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ acting on $H_0^1(\Omega)$ and $\phi_1 > 0$ be a first eigenfunction corresponding to λ_1.

Then, Green's formula implies

$$0 = \int_\Omega (u\Delta^2\phi_1 - \phi_1\Delta^2 u)\,dx = \int_\Omega (u\lambda_1^2\phi_1 - \phi_1 u^p)\,dx,$$

that is,

$$\int_\Omega u\phi_1(\lambda_1^2 - u^{p-1})\,dx = 0.$$

Hence $\lambda_1^2 - u^{p-1}$ must change the sign on Ω, which leads to

$$\|u\|_{L^\infty(\Omega)}^{p-1} \geq \lambda_1^2.$$

To obtain a uniform upper bound of $\|u_p\|_{L^\infty(\Omega)}$, we use an argument with the coarea formula and the isoperimetric inequality in $\mathbf{R}^4$.

Set

$$\gamma_p := \max_{x\in\overline{\Omega}} u_p(x),\quad \Omega_t := \{x \in \Omega : t < u_p(x)\},\quad \mathcal{A} := \left\{x \in \Omega : \frac{\gamma_p}{2} < u_p(x)\right\}.$$

By Lemma 2.1 and Corollary 2.3, we have

$$\left(\int_\Omega u_p^{4p}\,dx\right)^{1/(4p)} \leq D_{4p}(4p)^{1/2}\|\Delta u_p\|_{L^2(\Omega)} \leq M$$

where M is independent of p if p large. From this and the Chevyshev inequality we have

$$\left(\frac{\gamma_p}{2}\right)^{4p} |\mathcal{A}| \leq M^{4p}. \tag{3.1}$$

On the other hand, denote $v_p = -\Delta u_p$. Integration by parts leads to

$$\int_{\partial\Omega_t} |\nabla u_p|\,d\sigma = \int_{\Omega_t} v_p\,dx.$$

Coarea formula says

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{d\sigma}{|\nabla u_p|}.$$

Then Schwartz inequality implies

$$-\frac{d}{dt}|\Omega_t| \int_{\Omega_t} v_p\,dx = \int_{\partial\Omega_t} |\nabla u_p|\,d\sigma \int_{\partial\Omega_t} \frac{d\sigma}{|\nabla u_p|} \geq |\partial\Omega_t|^2.$$

Note that the isoperimetric inequality in $\mathbf{R}^4$

$$|\partial E| \geq 4\omega_4^{1/4}|E|^{3/4}$$

where $E \subset \mathbf{R}^4$ is a Caccioppoli set and $\omega_4 = |B^4| = |S^3|/4 = \pi^2/2$ is four dimensional volume of the unit ball. Applying $E = \Omega_t$, we have

$$-\frac{d}{dt}|\Omega_t| \int_{\Omega_t} v_p \, dx \geq 16\omega_4^{1/2}|\Omega_t|^{3/2}.$$

Define $r(t)$ such that $|\Omega_t| = \omega_4 r^4(t)$. Then

$$\frac{d}{dt}|\Omega_t| = 4\omega_4 r^3(t) r'(t).$$

Note that $r'(t) < 0$. Hence we have:

$$\frac{1}{4\omega_4 r^3(t)} \int_{\Omega_t} v_p \, dx \geq -\frac{1}{r'(t)},$$

$$-\frac{dt}{dr} \leq \frac{1}{4\omega_4 r^3(t)} \int_{\Omega_t} v_p \, dx \leq \frac{1}{4\omega_4 r^3(t)} \Big(\sup_\Omega v_p\Big)|\Omega_t| = \Big(\sup_\Omega v_p\Big)\frac{r}{4}.$$

Integrating the last inequality from $r = 0$ to $r = r_0$, we have

$$t(0) - t(r_0) \leq \frac{1}{8}\Big(\sup_\Omega v_p\Big) r_0^2.$$

Choose r_0 such that $t(r_0) = \gamma_p/2$, that is, $|\mathcal{A}| = |\Omega_{\gamma_p/2}| = \omega_4 r_0^4$. Then the above inequality implies

$$\gamma_p \leq \frac{1}{4}\Big(\sup_\Omega v_p\Big) r_0^2.$$

As $v_p = -\Delta u_p$ satisfies

$$\begin{cases} -\Delta v_p = u_p^p & \text{in } \Omega, \\ v_p|_{\partial\Omega} = 0, \end{cases}$$

we know, by elliptic estimate

$$\sup_\Omega v_p \leq C \sup_\Omega u_p^p \leq C\gamma_p^p$$

where $C = C(\Omega)$ (see [6] Theorem 3.7). Thus we have

$$\gamma_p \leq \frac{C}{4}\gamma_p^p r_0^2 = \frac{C}{4}\gamma_p^p \left(\frac{1}{\omega_4}|\mathcal{A}|\right)^{1/2}. \tag{3.2}$$

By (3.1) and (3.2), we have

$$\gamma_p^2 \leq \left(\frac{C}{4}\right)^2 \left(\frac{1}{\omega_4}\right) \gamma_p^{2p} \left(\frac{2M}{\gamma_p}\right)^{4p}$$

which implies

$$\gamma_p \le \left(\frac{C^2}{16\omega_4}\right)^{1/(2p+2)} (2M)^{2p/(p+1)}.$$

Therefore we conclude that there exists $C > 0$ (independent of p) such that $\gamma_p \le C$ for p large. □

From Theorem 1.1, we have the following consequence.

Corollary 3.1. *There exist $C_1, C_2 > 0$ independent of p large such that*

$$\frac{C_1}{p} \le \int_\Omega u_p^p \, dx \le \frac{C_2}{p}$$

holds true.

Proof. By Corollary 2.3 and Theorem 1.1, we have

$$C \le p \int_\Omega u_p^{p+1} \, dx \le \|u_p\|_{L^\infty(\Omega)} \times p \int_\Omega u_p^p \, dx \le C' p \int_\Omega u_p^p \, dx$$

where C, C' is a positive constant independent of p.

On the other hand, Hölder inequality implies

$$p \int_\Omega u_p^p \, dx \le \left(p \int_\Omega u_p^{p+1} \, dx\right)^{p/(p+1)} \times p^{1/(p+1)} |\Omega|^{1/(p+1)}.$$

Note that as $p \to \infty$, RHS of the above inequality is bounded from Corollary 2.3. This proves the conclusion. □

4. Proof of Theorem 1.2

Set

$$w_p := \frac{u_p}{\int_\Omega u_p^p \, dx} = \frac{u_p}{\lambda_p}, \quad \lambda_p := \int_\Omega u_p^p \, dx,$$

and

$$f_p(x) := \frac{u_p^p(x)}{\int_\Omega u_p^p \, dx}.$$

By our assumption of the convexity of Ω and the Method of Moving Plane, it is standard by now to derive the uniform boundary estimate of $\{w_p\}$ which leads to the fact that the blow up set of $\{w_p\}$ is contained in the interior of Ω; so we omit the proof of this fact.

At this stage, we recall a useful lemma proved by Wei ([10] Lemma 2.3).

Lemma 4.1 (Brezis-Merle type L^1 estimate for Δ^2). *Let u be a C^4 solution of*

$$\begin{cases} \Delta^2 u = f(x) & \text{in } \Omega \subset \mathbf{R}^4, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases}$$

where $f \in L^1(\Omega)$, $f \geq 0$. Then for any $\varepsilon \in (0, 16\sigma_4)$, there holds

$$\int_\Omega \exp\left(\frac{(16\sigma_4 - \varepsilon)|u(x)|}{\|f\|_{L^1}}\right) dx \leq \frac{16\sigma_4^2}{\varepsilon}|\Omega|.$$

Later we need a variant of ([10] Corollary 2.4) with no boundary condition.

Corollary 4.2. *Let u_n be a sequence of C^4 solutions of $\Delta^2 u = V_n e^u$ in $\Omega \subset \mathbf{R}^4$ with no boundary condition. Assume for some $p \in (1, \infty)$,*
(1) $\|V_n\|_{L^p(\Omega)} \leq C_1$,
(2) $\|u_n^+\|_{L^1(\Omega)} \leq C_2$ *where* $u_n^+ = \max\{u, 0\}$,
(3) $\int_\Omega |V_n| e^{u_n}\, dx \leq \exists \varepsilon_0 < 16\sigma_4/p'$
where p' is a Hölder conjugate exponent of p. Then, $\{u_n^+\}$ is uniformly bounded in $L^\infty_{loc}(\Omega)$.

The proof of Corollary 4.2 is done along the line of the proof of Corollary 4 in [2], so we omit it here.

From now on, let u_n, w_n, λ_n, f_n denote u_{p_n}, w_{p_n}, λ_{p_n}, f_{p_n} respectively. First, we remark that the blow up set S of the sequence $\{w_n\}$ satisfies $S \neq \phi$. Indeed,

$$\sup_{x \in \overline{\Omega}} w_n(x) \geq \frac{C}{\lambda_n} \to \infty$$

by Theorem 1.1 and the fact that $\lambda_n = O(1/p_n) \to 0$ as $n \to \infty$ by Corollary 3.1.

This also shows that the set of peak points of $\{u_n\}$ is contained in the blow up set of $\{w_n\}$.

Note that

$$f_n(x) = \frac{u_n^{p_n}(x)}{\int_\Omega u_n^{p_n}\, dx} \in L^1(\Omega), \quad f_n \geq 0, \quad \int_\Omega f_n\, dx = 1.$$

Then there exists a subsequence (still denoted by u_n) such that

$$f_n \overset{*}{\rightharpoonup} \mu, \quad \mu(\Omega) \leq 1$$

in the sense of Radon measures of Ω.

Now we define an important quantity

$$L_0 := \frac{\limsup_{p\to\infty}\left(p\int_\Omega u_p^p\,dx\right)}{e}. \tag{4.1}$$

From a little bit more precise estimate in the proof of Theorem 1.1 and Hölder inequality, we know that L_0 satisfies $1 < L_0 \le 32\sigma_4$.

For any $\delta > 0$, we call a point $x_0 \in \Omega$ a *δ-regular point of $\{u_n\}$* if there exists $\varphi \in C_0(\Omega)$, $0 \le \varphi \le 1$ with $\varphi \equiv 1$ near x_0 such that

$$\int_\Omega \varphi\,d\mu < \frac{16\sigma_4}{L_0 + 2\delta}.$$

Further we define for $\delta > 0$, *δ-irregular set of a sequence $\{u_n\}$* as

$$\Sigma(\delta) := \{x_0 \in \Omega : x_0 \text{ is not a } \delta\text{-regular point}\}.$$

Note that

$$x_0 \in \Sigma(\delta) \Rightarrow \mu(x_0) \ge \frac{16\sigma_4}{L_0 + 2\delta}.$$

The next result is a key lemma in the proof of Theorem 1.2.

Lemma 4.3 (smallness of μ implies boundedness). *Let x_0 be a δ-regular point of a sequence $\{u_n\}$, then $\{w_n\}$ is bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0 > 0$.*

Proof. Let x_0 be a δ-regular point. Then there exists $R > 0$ such that

$$\int_{B_R(x_0)} f_n\,dx < \frac{16\sigma_4}{L_0 + \delta}$$

for n sufficiently large.

Split $w_n = w_{1n} + w_{2n}$, where w_{1n} is a solution of

$$\begin{cases} \Delta^2 w_{1n} = f_n \quad \text{in } B_R(x_0), \\ w_{1n}|_{\partial B_R(x_0)} = \Delta w_{1n}|_{\partial B_R(x_0)} = 0, \end{cases}$$

and w_{2n} is a solution of

$$\begin{cases} \Delta^2 w_{2n} = 0 \quad \text{in } B_R(x_0), \\ w_{2n}|_{\partial B_R(x_0)} = w_n|_{\partial B_R(x_0)}, \\ \Delta w_{2n}|_{\partial B_R(x_0)} = \Delta w_n|_{\partial B_R(x_0)}. \end{cases}$$

Note that $w_{1n} > 0$, $w_{2n} > 0$ in $B_R(x_0)$ by the maximum principle.

First, we will show that w_{2n} is uniformly bounded near x_0. Indeed, by the mean value theorem for biharmonic function, we know

$$\|w_{2n}\|_{L^\infty(B_{R/2}(x_0))} \le C\|w_n\|_{L^1(B_R(x_0))} \le C\|w_n\|_{L^1(\Omega)} \le C$$

holds true. Here the last inequality follows by Lemma 4.1.

So we need only to derive the boundedness of w_{2n}.

For this purpose, choose $t > 1$ such as $t' = t/(t-1) = L_0 + (\delta/2)$. Note that the fact $L_0 > 1$ permits the existence of such $t > 1$. Then we have

$$\int_{B_R(x_0)} f_n\,dx < \frac{16\sigma_4}{L_0 + \delta} < \frac{16\sigma_4}{t'}.$$

Brezis-Merle type L^1-estimate (Lemma 4.1) implies

$$\int_{B_R(x_0)} \exp(t' w_{1n}(x))\,dx < C, \tag{4.2}$$

where $C = C(\delta) \to \infty$ as $\delta \to 0$.

By an elementary inequality $\log x \le x/e$ for $x > 0$, we have

$$\begin{aligned}\log f_n &= \log \frac{u_n^{p_n}}{\lambda_n} = p_n \log \frac{u_n}{\lambda_n^{1/p_n}} \le p_n \frac{u_n}{e\lambda_n^{1/p_n}}\\ &\le \frac{L_0 + (\delta/3)}{\lambda_n}\frac{u_n}{\lambda_n^{1/p_n}} = \frac{t' - (\delta/6)}{\lambda_n^{1/p_n}}\frac{u_n}{\lambda_n} \le t' w_n(x),\end{aligned}$$

here, the second inequality follows by definition of L_0:

$$\frac{p_n\lambda_n}{e} < L_0 + \frac{\delta}{3}$$

for n large, and the third inequality follows from the fact

$$\lim_{n\to\infty} \lambda_n^{1/p_n} = 1$$

which in turn follows from $\lambda_n = O(1/p_n)$ (Corollary 3.1).

Thus, we get a pointwise estimate

$$f_n(x) < \exp\bigl(t' w_n(x)\bigr)$$

which implies

$$(f_n e^{-w_{1n}})^t < Ce^{t' w_{1n}} \tag{4.3}$$

on $B_{R/2}(x_0)$, because w_{2n} is bounded uniformly on $B_{R/2}(x_0)$.

Rewrite the equation satisfied by w_{1n} as

$$\begin{cases} \Delta^2 w_{1n} = \underbrace{f_n e^{-w_{1n}(x)}}_{V_n(x)} e^{w_{1n}} & \text{in } B_R(x_0), \\ w_{1n}|_{\partial B_R(x_0)} = \Delta w_{1n}|_{\partial B_R(x_0)} = 0. \end{cases}$$

Now, we check the assumptions of Corollary 4.2.

(1) $V_n = f_n e^{-w_{1n}}$ is uniformly bounded in $L^t(B_{R/2}(x_0))$ by (4.2) and (4.3).

(2) $\|w_{1n}\|_{L^1(B_{R/2}(x_0))} \le C$ by Lemma 4.1.

(3) $\int_{B_{R/2}} V_n e^{w_{1n}}\,dx = \int_{B_{R/2}} f_n\,dx \le \exists\varepsilon_0 < 16\sigma_4/t'$ by the definition of t.

Therefore by applying Corollary 4.2 to w_{1n} on $B_{R/2}(x_0)$, we conclude that $\{w_{1n}\}$ is uniformly bounded in $L^\infty(B_{R/4}(x_0))$. □

Lemma 4.4. $S = \Sigma(\delta)$ *for any* $\delta > 0$.

Proof. $S \subset \Sigma(\delta)$ is clear from Lemma 4.3.

On the other hand, suppose $x_0 \in \Sigma(\delta)$ and $\|w_n\|_{L^\infty(B_{R_0}(x_0))} < C$ for some C independent of n. Then $f_n = \lambda_n^{p_n-1} w_n^{p_n} \to 0$ uniformly on $B_{R_0}(x_0)$, which implies x_0 is a δ-regular point; $x_0 \notin \Sigma(\delta)$.

This contradiction shows that for every $R > 0$,

$$\lim_{n\to\infty} \|w_n\|_{L^\infty(B_R(x_0))} = \infty$$

holds at least for a subsequence. So $x_0 \in S$. □

By this Lemma, we obtain

$$1 \ge \mu(\Omega) \ge \frac{16\sigma_4}{L_0 + 2\delta}\,\mathrm{card}\big(\Sigma(\delta)\big) = \frac{16\sigma_4}{L_0 + 2\delta}\,\mathrm{card}(S).$$

Combining with the estimate $L_0 \le 32\sigma_4$, we have

$$\mathrm{card}(S) \le \frac{L_0 + 2\delta}{16\sigma_4} \le \frac{32\sigma_4 + 2\delta}{16\sigma_4},$$

hence

$$1 \le \mathrm{card}(S) \le 2.$$

From now on, we prove the latter half of Theorem 1.2, so assume $\mathrm{card}(S) = 1$ and $S = \{x_0\}$, $x_0 \in \Omega$.

Then $w_n(x) \le C$ on any compact set $K \subset \overline{\Omega}\backslash\{x_0\}$, which implies $f_n \to 0$ compact uniformly on $\overline{\Omega} \setminus \{x_0\}$.

Take $\varphi \in C_0(\Omega)$. For given $\varepsilon > 0$, first we choose $r > 0$ so small and then $n \to \infty$, we have

$$\left|\int_\Omega f_n\varphi\,dx - \varphi(x_0)\right| \le \int_\Omega f_n|\varphi(x)-\varphi(x_0)|\,dx$$
$$\le \int_{B_r(x_0)} f_n|\varphi(x)-\varphi(x_0)|\,dx + \int_{\Omega\setminus B_r(x_0)} f_n|\varphi(x)-\varphi(x_0)|\,dx \le \varepsilon.$$

Therefore

$$f_n \overset{*}{\rightharpoonup} \delta_{x_0} \tag{4.4}$$

in the sense of Radon measures of Ω. Thus Theorem 1.2 (1) is proved.

Set $\tilde{w}_n := -\Delta w_n$. Then

$$\begin{cases} -\Delta\tilde{w}_n = f_n & \text{in } \Omega,\\ \tilde{w}_n|_{\partial\Omega} = 0, \end{cases}$$

and $f_n \to 0$ uniformly on any compact $K \subset \overline{\Omega}\setminus\{x_0\}$.

By elliptic regularity, there is a subsequence $\{w_n\}, \{\tilde{w}_n\}$ (still denoted by the same symbols) and a function G such that

$$w_n \to G \text{ in } C^{4,\alpha}(K), \quad \tilde{w}_n \to -\Delta G \text{ in } C^{2,\alpha}(K),$$
$$w_n \to G \text{ in } L^1(\Omega), \quad \tilde{w}_n \to -\Delta G \text{ in } L^1(\Omega)$$

since $W^{1,q}(\Omega)$ boundedness of w_n, $\tilde{w}_n$ $(1 < q < 4/3)$ and the compactness of the imbedding $W^{1,q}(\Omega) \hookrightarrow L^1(\Omega)$.

By integration by parts and (4.4), we see $G = G_4(\cdot, x_0)$. So Theorem 1.2 (2) is proved.

Finally, to prove the characterization of blow up point, we need the famous

Lemma 4.5 (Pohozaev identity). *Let u be a C^4-solution of $\Delta^2 u = f(u)$ in Ω. Then we have for any $y \in \mathbf{R}^4$,*

$$\begin{aligned} 4\int_\Omega F(u)\,dx = {} & \int_{\partial\Omega}(x+y,\nu)F(u)\,d\sigma + \frac{1}{2}\int_{\partial\Omega} v^2(x+y,\nu)\,d\sigma + 2\int_{\partial\Omega}\frac{\partial u}{\partial\nu}v\,d\sigma \\ & + \int_{\partial\Omega}\frac{\partial v}{\partial\nu}(x+y,\nabla u) + \frac{\partial u}{\partial\nu}(x+y,\nabla v)\,d\sigma \\ & - \int_{\partial\Omega}(\nabla u,\nabla v)(x+y,\nu)\,d\sigma. \end{aligned}$$

In particular, we have

$$\int_{\partial\Omega}\nu F(u)\,d\sigma + \frac{1}{2}\int_{\partial\Omega} v^2\nu\,d\sigma + \int_{\partial\Omega}\left\{\frac{\partial v}{\partial\nu}\nabla u + \frac{\partial u}{\partial\nu}\nabla v - (\nabla u,\nabla v)\nu\right\}d\sigma = \vec{0}.$$

Here, $F(u) = \int_0^u f(s)\,ds$, $v = -\Delta u$ *and* $\nu(x)$ *is the unit outer normal vector at* $x \in \partial\Omega$.

More genaral version of this formula can be seen, for example, in [7]. In our case, integrating the identity on Ω

$$\begin{aligned}&\operatorname{div}\big((x+y,\nabla v)\nabla u + (x+y,\nabla u)\nabla v - (\nabla u,\nabla v)(x+y)\big)\\ &\quad = (x+y,\nabla v)\Delta u + (x+y,\nabla u)\Delta v - 2(\nabla u,\nabla v)\end{aligned}$$

for $u, v \in C^2(\overline{\Omega})$, $\nabla = \nabla_x$, and noting that

$$\operatorname{div}\big((x+y)F(u)\big) = f(u)(x+y,\nabla u) + 4F(u)$$

and

$$\operatorname{div}\left(\frac{1}{2}v^2(x+y) + 2v\nabla u\right) = v(\nabla v, x+y) + 2(\nabla u,\nabla v)$$

if $v = -\Delta u$, we get the desired formula.

Take $r > 0$ so small that $B_r(x_0) \subset \Omega$. First, apply the Pohozaev identity to w_n on Ω. Note that w_n solves $\Delta^2 w_n = f_n = \lambda_n^{p_n-1} w_n^{p_n}$ in Ω. So $F(w_n) = (\lambda_n^{p_n-1}/(p_n+1))w_n^{p_n+1}$ in the following identity.

$$\begin{aligned}&\int_{\partial\Omega} \nu F(w_n)\,d\sigma + \frac{1}{2}\int_{\partial\Omega}(\Delta w_n)^2\nu\,d\sigma + \int_{\partial\Omega}\left\{\frac{\partial(-\Delta w_n)}{\partial\nu}\nabla w_n + \frac{\partial w_n}{\partial\nu}\nabla(-\Delta w_n)\right\}d\sigma\\ &- \int_{\partial\Omega}\big\{(\nabla w_n, \nabla(-\Delta w_n))\nu\big\}d\sigma\\ &= \vec{0}.\end{aligned}$$

Since $w_n = \Delta w_n = 0$ on $\partial\Omega$, we have

$$\int_{\partial\Omega}\left\{\frac{\partial(-\Delta w_n)}{\partial\nu}\nabla w_n + \frac{\partial w_n}{\partial\nu}\nabla(-\Delta w_n) - \big(\nabla w_n, \nabla(-\Delta w_n)\big)\nu\right\}d\sigma = \vec{0}.$$

Now we know that

$$w_n \to G_4(\cdot, x_0) \quad \text{in } C^4_{loc}(\overline{\Omega}\setminus\{x_0\}), \quad -\Delta w_n \to G_2(\cdot,x_0) \quad \text{in } C^2_{loc}(\overline{\Omega}\setminus\{x_0\})$$

as $n \to \infty$. Therefore letting $n \to \infty$, we have:

$$\int_{\partial\Omega}\left\{\frac{\partial G_2}{\partial\nu}\nabla G_4 + \frac{\partial G_4}{\partial\nu}\nabla G_2 - (\nabla G_4, \nabla G_2)\nu\right\}d\sigma = \vec{0}. \tag{4.5}$$

On the other hand, apply the Pohozaev identity to $G_4(\cdot, x_0)$ on $\Omega\setminus B_r(x_0)$. Note that $G_4(\cdot,x_0)$ solves $\Delta^2 G_4 = 0$ in $\Omega\setminus B_r(x_0)$ and $G_4 = G_2 = 0$ on $\partial\Omega$. Therefore we have

$$\frac{1}{2}\int_{\partial(\Omega\setminus B_r(x_0))} G_2^2\nu\, d\sigma + \int_{\partial(\Omega\setminus B_r(x_0))} \left\{\frac{\partial G_2}{\partial\nu}\nabla G_4 + \frac{\partial G_4}{\partial\nu}\nabla G_2 - (\nabla G_4, \nabla G_2)\nu\right\} d\sigma = \vec{0},$$

that is,

$$\begin{aligned}&\frac{1}{2}\int_{\partial B_r(x_0)} G_2^2\nu\, d\sigma + \int_{\partial B_r(x_0)} \left\{\frac{\partial G_2}{\partial\nu}\nabla G_4 + \frac{\partial G_4}{\partial\nu}\nabla G_2 - (\nabla G_4, \nabla G_2)\nu\right\} d\sigma \\ &\quad = \int_{\partial\Omega} \left\{\frac{\partial G_2}{\partial\nu}\nabla G_4 + \frac{\partial G_4}{\partial\nu}\nabla G_2 - (\nabla G_4, \nabla G_2)\nu\right\} d\sigma. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have:

$$\begin{aligned}\vec{0} = &\underbrace{\frac{1}{2}\int_{\partial B_r(x_0)} G_2^2\nu\, d\sigma}_{[A]} + \underbrace{\int_{\partial B_r(x_0)} \frac{\partial G_2}{\partial\nu}\nabla G_4\, d\sigma}_{[B]} \\ &+ \underbrace{\int_{\partial B_r(x_0)} \frac{\partial G_4}{\partial\nu}\nabla G_2\, d\sigma}_{[C]} - \underbrace{\int_{\partial B_r(x_0)} (\nabla G_4, \nabla G_2)\nu\, d\sigma}_{[D]}. \end{aligned} \tag{4.7}$$

Substituting

$$\begin{aligned} G_4(x, x_0) &= H_4(x, x_0) - \frac{1}{4\sigma_4}\log|x - x_0|, \\ G_2(x, x_0) &= -\Delta G_4(x, x_0) = H_2(x, x_0) + \frac{1}{2\sigma_4}\frac{1}{|x - x_0|^2} \end{aligned}$$

in the above identity (4.7), we compute:

- **Estimate of** $[A]$

$$\int_{\partial B_r(x_0)} G_2^2\nu\, d\sigma = \int_{\partial B_r(x_0)} \left(\frac{1}{2\sigma_4|x - x_0|^2} + O(1)\right)^2 \nu\, d\sigma = O(r).$$

Note that $\nu = (x - x_0)/|x - x_0|$ on $\partial B_r(x_0)$ and

$$\int_{\partial B_r(x_0)} h(r)\nu\, d\sigma = \vec{0}$$

for any function $h(r)$, $r = |x - x_0|$.

- **Estimate of** $[B]$

$$\int_{\partial B_r(x_0)} \frac{\partial G_2}{\partial\nu}\nabla G_4\, d\sigma$$

$$
\begin{aligned}
&= \int_{\partial B_r(x_0)} \left(\nabla H_2 \cdot \nu - \frac{1}{\sigma_4 |x - x_0|^3} \right) \left(\nabla H_4 - \frac{1}{4\sigma_4} \frac{x - x_0}{|x - x_0|^2} \right) d\sigma \\
&= -\frac{1}{\sigma_4 r^3} \int_{\partial B_r(x_0)} \nabla H_4(x, x_0)\, d\sigma + O(r^2) \\
&= -\nabla H_4(x^*, x_0) + O(r^2)
\end{aligned}
$$

where $\exists x^* \in \partial B_r(x_0)$. The last equality comes from the mean value theorem of integral.

- **Estimate of** $[C]$

$$
\begin{aligned}
&\int_{\partial B_r(x_0)} \frac{\partial G_4}{\partial \nu} \nabla G_2\, d\sigma \\
&= \int_{\partial B_r(x_0)} \left(\nabla H_4 \cdot \nu - \frac{1}{4\sigma_4 |x - x_0|} \right) \left(\nabla H_2 - \frac{1}{\sigma_4} \frac{x - x_0}{|x - x_0|^4} \right) d\sigma \\
&= -\frac{1}{\sigma_4 r^3} \int_{\partial B_r(x_0)} \nu(\nabla H_4 \cdot \nu)\, d\sigma + O(r^2).
\end{aligned}
$$

- **Estimate of** $[D]$

$$
\begin{aligned}
&\int_{\partial B_r(x_0)} (\nabla G_4, \nabla G_2) \nu\, d\sigma \\
&= \int_{\partial B_r(x_0)} \left(\nabla H_4 - \frac{1}{4\sigma_4} \frac{x - x_0}{r^2}, \nabla H_2 - \frac{1}{\sigma_4} \frac{x - x_0}{r^4} \right) \nu\, d\sigma \\
&= -\frac{1}{\sigma_4 r^3} \int_{\partial B_r(x_0)} \nu(\nabla H_4 \cdot \nu)\, d\sigma + O(r^2).
\end{aligned}
$$

Letting $r \to 0$ and noting that $x^* \to x_0$, we have

$$
\vec{0} = -\nabla H_4(x_0, x_0).
$$

This proves Theorem 1.2 (3) □

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NOTE ADDED IN THE PROOF. Recently, we have succeeded to confirm the conjecture in Remark 1.3 (2) affirmatively. Please refer to "Single-point condensation phenomena for a four-dimensional biharmonic Ren-Wei problem" (preprint).

References

[1] D.R. Adams: *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. **128** (1988), 385–398.

[2] H. Brezis and F. Merle: *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. P.D.E. **16** (1991), 1223–1253.

[3] S.Y.A. Chang and P. Yang: *The inequality of Moser and Trudinger and applications to conformal geometry*, Comm. Pure. Apll. Math. **56** (2003), 1135–1150.

[4] Ph. Crement and D. de Figueiredo and E. Mitidieri: *Positive solutions of semilinear elliptic systems*, Comm. P.D.E. **17** (1992), 923–940.

[5] M. Flucher and J. Wei: *Semilinear Dirichlet problem with nearly critical exponent, asymptotic location of hot spots*, Manuscripta Math. **94** (1997), 337–346.

[6] D. Gilbarg and N.S. Trudinger: Elliptic Partial Differential Equations of Second Order, second edition. Springer-Verlag, Berlin-New York, 1983.

[7] E. Mitidieri: *A Rellich type identity and applications*, Comm. P.D.E. **18** (1993), 125–151.

[8] X. Ren and J. Wei: *On a two-dimensional elliptic problem with large exponent in nonlinearlity*, Trans. A.M.S. **343** (1994), 749–763.

[9] X. Ren and J. Wei: *Single-point condensation and least-energy solutions*, Proc. A.M.S. **124** (1996), 111–120.

[10] J. Wei: *Asymptotic behavior of a nonlinear fourth order eigenvalue problem*, Comm. P.D.E. **21** (1996), 1451–1467.

Mathematical Institute
Tohoku university
Sendai, 980-8578, Japan.
e-mail: tfutoshi@math.tohoku.ac.jp