ON MONTEL'S THEOREM

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1. In this note we shall prove a theorem which is related to Montel's theorem [1] on bounded regular functions. Let E be a measurable set on the positive y-axis in the $z(=x+iy)$-plane, $E(a, b)$ be its part contained in $0 \leqq a \leqq y \leqq b$, and $|E(a, b)|$ be its measure. We define the lower density of E at $y=0$ by

$$\lambda = \varliminf_{r\to 0} \frac{|E(0, r)|}{r}.$$

LEMMA. *Let E be a set of positive lower density λ at $y=0$. Then E contains a subset E_1 of the same lower density at $y=0$ such that $E_1 \cup \{0\}$ is a closed set.*

Proof. Let $r_n = 1/n$ $(n=1, 2, \ldots)$. There exists a closed subset $E_1(r_{n+1}, r_n)$ of $E(r_{n+1}, r_n)$, such that

$$|E_1(r_{n+1}, r_n)| \geqq \delta_n |E(r_{n+1}, r_n)| \qquad (n=1, 2, \ldots),$$

with $\delta_n = 1 - \frac{1}{n}$. We put

$$E_1 = \sum_{n=1}^{\infty} E_1(r_{n+1}, r_n).$$

Then if $r_n < r \leqq r_{n-1}$,

$$|E_1(0, r)| \geqq \sum_{i=n}^{\infty} |E_1(r_{i+1}, r_i)| \geqq \delta_n |E(0, r_n)|,$$

so that

$$\frac{|E_1(0, r)|}{r} \geqq \frac{|E_1(0, r_n)|}{r}\delta_n \geqq \frac{|E(0, r_n)|}{r_n} \cdot \frac{r_n}{r_{n-1}}\delta_n,$$

whence

$$\lambda = \varliminf_{r\to 0} \frac{|E(0, r)|}{r} \geqq \varliminf_{r\to 0} \frac{|E_1(0, r)|}{r} \geqq \varliminf_{n\to\infty} \frac{|E(0, r_n)|}{r_n} \geqq \lambda.$$

Hence

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$$\lim_{r\to 0}\frac{|E_1(0, r)|}{r} = \lambda.$$

2. We shall prove the following theorem.

THEOREM. *Let $f(z)=f(x+iy)$ be regular and bounded in $x>0$, and continuous at a measurable set E of positive lower density λ at $y=0$ on the positive y-axis. If $f(z)\to A$ when $z\to 0$ along E, then $f(z)\to A$ uniformly when $z\to 0$ in the domain $|y|\leqq kx$, where k is any positive constant.*

Proof. By the lemma we assume that $E\cup\{0\}$ is a closed set. Without loss of generality we may assume that $|f(z)|\leqq 1$ and $A=0$. Let $D_\rho : |z|<\rho$, $x>0$ be the half-disc. Let us denote by $u_\rho(z)$ the harmonic measure of $E(0, \rho)\cup\{0\}$ with respect to D_ρ.

If we take $0<\rho<1$ sufficiently small such that $|f(z)|\leqq\varepsilon$ on $E(0, \rho)$, then, by the maximum principle, we have

$$\log|f(z)|\leqq u_\rho(z)\cdot\log\varepsilon \quad \text{for} \quad z\in D_\rho;$$

hence

$$|f(z)|\leqq\varepsilon^{u_\rho(z)} \quad \text{for} \quad z\in D_\rho. \tag{1}$$

As is well known,

$$u_\rho(z)=\frac{1}{2\pi}\int_{E(0,\rho)}\frac{\partial}{\partial n}G(i\eta, z)\,d\eta,$$

where $G_\rho(w, z)$ $(w=\xi+i\eta)$ is the Green's function of D_ρ with pole at $z=x+iy$. By a simple calculation we have

$$\left(\frac{\partial G}{\partial n}\right)_{\xi=0}=\frac{2x(\rho^2-x^2-y^2)(\rho^2-\eta^2)}{\{x^2+(y-\eta)^2\}\{(\rho-y\eta)^2+x^2\eta^2\}}.$$

Hence

$$u_\rho(z)=\frac{1}{2\pi}\int_0^\rho\frac{2x(\rho^2-x^2-y^2)(\rho^2-\eta^2)}{\{x^2+(y-\eta)^2\}\{(\rho^2-y\eta)^2+x^2\eta^2\}}\,d\mu(\eta),$$

where

$$\mu(\eta)=\int_{E(0,\eta)}d\eta=|E(0, \eta)|.$$

If $|z|\leqq\delta\rho$, $\eta\leqq\delta\rho$, $(0<\delta<1)$, then

$$(\rho^2-x^2-y^2)(\rho^2-\eta^2)\geqq\rho^4C_1, \qquad (\rho^2-y\eta)^2+x^2\eta^2\leqq\rho^4C_2,$$

whence

$$u_\rho(z) \geqq C_2 \int_0^{\delta\rho} \frac{x}{x^2+(y-\eta)^2} d\mu(\eta),$$

where C_1, C_2, C_3 are constants, depending on δ only. Hence if $|y| \leqq kx$, we have

$$u_\rho(z) \geqq C_3 \int_0^{\delta\rho} \frac{x}{x^2+(\eta+kx)^2} d\mu(\eta).$$

By the substitution $\eta = xt$, we have

$$\begin{aligned} U_\rho(z) &\geqq \frac{C_3}{x} \int_0^{\delta\rho/x} \frac{1}{1+(t+k)^2} d\mu(xt) \\ &= \frac{C_3}{x} \left[\frac{\mu(xt)}{1+(t+k)^2} \right]_0^{\delta\rho/x} + \frac{2C_3}{x} \int_0^{\delta\rho/x} \frac{\mu(xt)(t+k)}{\{1+(t+k)^2\}^2} dt \\ &\geqq \frac{2C_3}{x} \int_0^1 \frac{\mu(xt)(t+k)}{\{1+(t+k)^2\}^2} dt. \end{aligned}$$

Since $\mu(xt) \geqq \lambda' xt$ for some λ' such that $0 < \lambda' < \lambda$, we have

$$u_\rho(z) \geqq 2C_3 \int_0^1 \frac{\lambda' t(t+k)}{\{1+(t+k)^2\}^2} dt = C,$$

where C is a constant depending on k, δ, and λ' only. Hence by (1)

$$|f(z)| \leqq \varepsilon^C, \quad \text{if} \quad |z| \leqq \delta\rho \quad \text{and} \quad |y| \leqq kx,$$

so that $\lim_{z\to 0} f(z) = 0$ uniformly, when $z \to 0$ in the domain $|y| \leqq kx$.

Remark. The writer has proved that our theorem holds when E satisfies the condition that λ_α is positive, where

$$\lambda_\alpha = \varliminf_{r\to 0} r^{\alpha-1} \int_r^1 \frac{d\mu(t)}{t^\alpha} \qquad (\alpha \geqq 2).$$

However, Professor Ohtsuka kindly informed him that this condition for any $\alpha > 1$ is equivalent to the condition that the lower density of E at $y = 0$ is positive.[1)]

Reference

[1] P. Montel, Sur les familles de fonctions analytiques qui admettent des valeurs exceptionelles dans un domaine, Ann. Sci. Ecole Norm. Sup. (3), 23 (1912), pp. 487-535.

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1) See the paper after the next.