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ON PICARD VALUES OF ALGEBROID FUNCTIONS IN A NEIGHBOURHOOD OF A TOTALLY DISCONNECTED COMPACT SET

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1. Let E be a totally disconnected compact set in the z-plane and Ω its complement with respect to the extended z-plane. Then Ω is a region. Let $\{\Omega_n\}_{n=0}^{\infty}$ be an exhaustion of Ω satisfying the following conditions:

1. $\Omega_{n+1} \supset \bar{\Omega}_n$ for every n,

2. for each n, the boundary $\partial\Omega_n$ of Ω_n consists of a finite number of closed analytic curves,

3. each component of the open set $\mathscr{C}\Omega_n$, the complement of Ω_n, contains points of E,

4. the open set $\Omega_n - \bar{\Omega}_{n-1}$ $(n \geqslant 1)$ consists of a finite number of doubly connected regions $R_{n,k}$ $(k = 1, 2, \cdots, N(n))$.

We consider the graph associated with this exhaustion in the sense of Noshiro [6]. Let $u(z) + iv(z)$ be the mapping function of $\Omega - \bar{\Omega}_0$ into it and L its Length (see [3]).

Let β_r be the level line $\{z \mid u(z) = r,\ 0 < r < L\}$ on Ω. Then β_r consists of a finite number of Jordan curves $\beta_{r,m}(m = 1, 2, \cdots, n(r))$, except for a countably many values of r which we shall exclude. Calling each component of the open set $\Omega_n - \bar{\Omega}_k$, $n > k \geqslant 0$, an R-chain, we consider for such a $\beta_{r,m}$ the longest doubly connected R-chain $R(\beta_{r,m})$ such that $\beta_{r,m}$ is contained in $R(\beta_{r,m})$ or is one of two components of $\partial R(\beta_{r,m})$ and denote by $\mu(\beta_{r,m})$ its modulus. We can take a sequence $\{r_n\}_{n=0}^{\infty}(0 < r_n < L)$ such that $r_n \to L$ as $n \to \infty$ and for any two level lines

$$\beta_{r_n} = \{\beta_{r_n,i}\}_{i=1}^{n(r_n)} \quad \text{and} \quad \beta_{r_{n+1}} = \{\beta_{r_{n+1},j}\}_{j=1}^{n(r_{n+1})}$$

each $R(\beta_{r_n,i})$ $(i = 1$ to $n(r_n))$ has one boundary component in common with

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some $R(\beta_{r_{n+1},j})$'s and we set

$$\mu_n = \min_{1 \leqq i \leqq n(r_n)} \mu(\beta_{r_n,i}).$$

Generally $R_{n,k}$ may branches off into a finite number of $R_{n+1,m}$. If every $R_{n,k}$ $(n = 1, 2, \cdots;\ k = 1, 2, \cdots, N(n))$ branches off into at most ρ regions $R_{n+1,m}$, we say that the exhaustion $\{\Omega_n\}$ branches off at most ρ times everywhere.

In his paper [4], Matsumoto proves the following theorem: *If Ω has an exhaustion which branches off at most ρ times everywhere and*

$$\lim_{n\to\infty} \mu_n = \infty,$$

then every single-valued meromorphic function in Ω with at least one essential singularity in E has at most $\rho+1$ Picard values in Ω at each singularity.

Here a Picard value in Ω at a singularity $\zeta \in E$ is a value which is not taken in the intersection of Ω and some neighbourhood of ζ.

In this note, we shall establish an extension of the above theorem to the case of algebroid functions, which we state here in the following

THEOREM. *If Ω has an exhaustion which branches off at most ρ times everywhere and*

$$\lim_{n\to\infty} \frac{\mu_n}{1 + (\log k)\log n} = \infty, \tag{1}$$

then every k-valued algebroid function in Ω with at least one essential singularity in E has at most $k(\rho+1)$ Picard values in Ω at each singularity.

Owing to the result of Dufresnoy [2], we can apply the arguments of [4] to our case. In the sequel, we shall give a proof only for $\rho = 2$. When $\rho = 1$, the set E reduces to one point and our assertion is just the Picard theorem for algebroid functions (Rémoundos [8]). The other cases when $\rho > 2$ can be taken care of in the same way as in $\rho = 2$.

2. Before proving our theorem, we give two lemmas. We shall consider the Riemann sphere Σ with radius 1/2 tangent to the ζ-plane at the origin. For ζ and ζ' in the extended ζ-plane we denote by $[\zeta, \zeta']$ their chordal distance. Further let $C(\zeta, \delta)$ with $\delta > 0$ be the spherical disk with center ζ and chordal radius δ.

Let f be an n-valued algebroid function in an annulus R: $1 < |z| < e^{\sigma}$ $(\sigma > 0)$ omitting $3n+1$ values ζ_i $(i = 1, \cdots, 3n+1)$ and let $\mathfrak{X}_R$ be the n-sheeted covering surface over R generated by f. We denote by $C_1, C_2, \cdots,$ $C_p (p \leqslant n)$ the closed curves on $\mathfrak{X}_R$ over the circle $|z| = e^{\sigma/2}$ in R and by $P_1(z), \cdots, P_q(z)$ $(q \leqslant n)$ the points on $\mathfrak{X}_R$ over a point z in R. We take $\delta > 0$ so small that the spherical disks $C(\zeta_i, \delta)$ are disjoint by pairs.

In order to obtain the first lemma, we use the Schottky theorem for algebroid functions established by Dufresnoy [2]:

Let $g(z)$ be an n-valued algebroid function in $|z| < R$ omitting $2n+1$ values $w_i (i = 1$ to $2n+1)$, one of them being the point at infinity. If $|g_\nu(0)| < M_0$ for all branches g_ν of g and some $M_0 > 0$, then

$$\max_{|z|=r} |g(z)| \leqslant \exp\left[\frac{An \log(1+M_0) + BK^2}{\frac{R-r}{R}} + \frac{1}{2}\log n\right]$$

for all $r < R$, where A and B are absolute constants and K depends only on $\min\limits_{1 \leqslant i < j \leqslant 2n+1} [w_i, w_j]$ and not on g.

Lemma 1. *There exists a positive constant δ' such that, if the values $f(P_\lambda(z))$ $(\lambda = 1$ to $q)$ lie outside of $C(\zeta_i, \delta)$ for some $i (1 \leqslant i \leqslant 3n+1)$ at a point z on $|z| = e^{\sigma/2}$, then the images of $C_\mu (\mu = 1$ to $p)$ lie completely outside of the concentric disk $C(\zeta_i, \delta')$. Here δ' depends only on n, δ and $\min\limits_{1 \leqslant i < j \leqslant 3n+1} [\zeta_i, \zeta_j]$ and not on f.*

Proof. By the Schottky theorem, we can see easily that, if an n-valued algebroid function $g(z)$ in $1 < |z| < e^{\sigma}$ omits $2n+1$ values $w_i (i = 1$ to $2n+1)$ one of them being the point at infinity, and $\min\limits_{|z|=e^{\sigma/2}} g^*(z) < M$ with $g^*(z) = \max\limits_{1 \leqslant \nu \leqslant n} |g_\nu(z)|$ for some $M > 0$, then there exists a positive constant M' depending only on M and σ such that

$$\max_{|z|=e^{\sigma/2}} |g(z)| \leqslant M'.$$

We may assume that $\zeta_i = \infty$, for otherwise, it is sufficient for us to consider the transformation $R_i(\zeta) = \dfrac{1+\bar{\zeta}_i\zeta}{\zeta - \zeta_i}$ under which the chordal distance remains invariant. Let $|\zeta| > M$ be the region in the extended ζ-plane corresponding to $C(\zeta_i, \delta)$. For this M and fixed $2n+1$ exceptional values $\zeta_{j_1}, \cdots, \zeta_{j_{2n+1}}$ of $\zeta_l (l = 1$ to $3k+1)$, $f(z)$ has the same properties as g stated above and hence

$$|f(z)| < M' \quad \text{on} \quad |z| = e^{\sigma/2} \tag{2}$$

with $M' > 0$ depending on M, σ and $\min_{1 \leqslant p < q \leqslant 2n+1}[\zeta_{j_p}, \zeta_{j_q}]$. Since $\binom{2n+1}{3n+1}$ is finite, we can take $M^* > 0$ independently of the choice of $2n+1$ exceptional values $\zeta_{j_1}, \cdots, \zeta_{j_{2n+1}}$ such that $f(z)$ satisfies (2) with M^*. If we denote by δ' the radius of the spherical disk corresponding to the region $|\zeta| > M^*$ in the extended ζ-plane, then this δ' satisfies all the conditions of the lemma.

3. In order to establish the second lemma, we shall be concerned with the distance between two systems, which was introduced by Dufresnoy [2].

We consider only the systems consisting of $n+1$ complex numbers, all of which are not zero simultaneously. Let

$$w^{(1)} = (w_0^{(1)}, w_1^{(1)}, \cdots, w_n^{(1)}) \text{ and } w^{(2)} = (w_0^{(2)}, w_1^{(2)}, \cdots, w_n^{(2)})$$

be two systems and if $w^{(1)}$ is proportional to $w^{(2)}$, i.e. for all i and some constant c $(c \neq 0)$, $w_i^{(1)} = cw_i^{(2)}$, we identify $w^{(1)}$ with $w^{(2)}$. We set

$$[[w^{(1)}, w^{(2)}]] = \sqrt{\frac{\sum_{i<j} |w_i^{(1)} w_j^{(2)} - w_i^{(2)} w_j^{(1)}|^2}{\sum_i |w_i^{(1)}|^2 \sum_i |w_i|^2}}\,. \tag{3}$$

Then this satisfies three axioms for the distance and we call $[[w^{(1)}, w^{(2)}]]$ the distance between two systems $w^{(1)}$ and $w^{(2)}$. From (3), we can easily deduce that an inequality

$$[[w^{(1)}, w^{(2)}]]^2 \leqslant \frac{\sum |w_i^{(1)} - w_i^{(2)}|^2}{\sqrt{\sum |w_i^{(1)} \sum |w_i^{(2)}|^2}} \tag{4}$$

holds.

Let

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0 \tag{5}$$

$$a_0^* z^n + a_1^* z^{n-1} + \cdots + a_n^* = 0 \tag{6}$$

be two algebraic equations whose coefficients consist of the systems

$$a = (a_0, a_1, \cdots, a_n) \text{ and } a^* = (a_0^*, a_1^*, \cdots, a_n^*),$$

respectively. Then using the distance (3), the theorem [2] on continuity of roots of algebraic equations is described as follows:

Let $z_1, z_2, \cdots, z_n$ and $z_1^*, z_2^*, \cdots, z_n^*$ be the roots of the equations (5) and (6), respectively. Then if $[[a, a^*]]$ is sufficiently small, we can associate each

$z_i (i = 1 \text{ to } n)$ with some $z_i^* (1 \leqslant i \leqslant n)$, say z_i with $z_{\alpha_i}^*$, such that

$$[z_i, z_{\alpha_i}^*] < 8e[[a, a^*]]^{\frac{1}{n}} \qquad (i = 1 \text{ to } n)$$

where [,] is the chordal distance.

4. Let f be an n-valued algebroid function in $\bar{R}: 1 \leqslant |z| \leqslant e^\mu$, $\mu > 0$, and $\mathfrak{X}_{\bar{R}}$ be the same as in §2 corresponding to $\bar{R}$. We denote again by $C_1, C_2, \cdots, C_p$ $(p \leqslant n)$ the closed curves on $\mathfrak{X}_{\bar{R}}$ over $|z| = e^{\mu/2}$. Then we have:

LEMMA 2. *If f takes no value in a spherical disk $C(\zeta_0, \delta)$, $\delta > 0$, then there exists a positive constant A depending only on δ and n such that for every i $(1 \leqslant i \leqslant p)$ the diameter of the image of C_i under f in terms of the chordal distance is dominated by $Ae^{-\mu/2n}$ for sufficiently large μ.*

In particulier, if δ is sufficiently close to 1, i.e. the spherical disk $C(-1/\bar{\zeta}_0, d)$ complementary to $C(\zeta_0, \delta)$ has a sufficiently small radius d, then

$$A < Bd^{\frac{1}{n}}$$

where B is a positive constant depending only on n.

Proof. We may assume without loss of generality that the center of $C(\zeta_0, \delta)$ is the point at infinity, for otherwise we can map ζ_0 to the point at infinity by the linear transformation $\dfrac{1 + \bar{\zeta}_0 \zeta}{\zeta - \zeta_0}$ under which the chordal distance remains invariant. Let $|\zeta| > M$ be the region in the extended ζ-plane corresponding to $C(\zeta_0, \delta)$. Then $|f(z)| \leqslant M$ on $1 \leqslant |z| \leqslant e^\mu$.

We consider the defining equation of f:

$$f^n + S_1(z) f^{n-1} + \cdots + S_n(z) = 0$$

where each $S_i(z)$ is single-valued and meromorphic in $1 \leqslant |z| \leqslant e^\mu$. Since each $S_i(z)$ is a fundamental symmetric function of n branches $f_\nu (\nu = 1 \text{ to } n)$ of f and $|f_\nu| \leqslant M$ for all $\nu (\nu = 1 \text{ to } n)$, $|S_i(z)| \leqslant \binom{i}{n} M^i$ and hence for all $i (i = 1 \text{ to } n)$ and $M_1 = \max\limits_{1 \leqslant i \leqslant n} \binom{i}{n} M^i$,

$$|S_i(z)| \leqslant M_1.$$

By Cauchy's integral formula,

$$S_i'(z) = \frac{1}{2\pi i} \left\{ \int\limits_{|t| = e^\mu} \frac{S_i(t)}{(t - z)^2} dt - \int\limits_{|t| = 1} \frac{S_i(t)}{(t - z)^2} dt \right\}$$

for every z on $|z| = e^{\mu/2}$ and hence, if $\mu \geqslant 2$, then

$$|S_i'(z)| \leqslant \frac{M_1}{2\pi}\left\{\frac{2\pi e^{\mu}}{(e^{\mu}-e^{\mu/2})^2}+\frac{2\pi}{(e^{\mu/2}-1)^2}\right\} \leqslant \frac{2e^2}{(e-1)^2} M_1 e^{-\mu},$$

and
$$\int_{|z|=e^{\mu/2}} |S_i'(z)|\cdot|dz| \leqslant \frac{2e^2}{(e-1)^2} M_1 e^{-\mu},\ 2\pi e^{\mu/2} = \frac{4\pi e^2}{(e-1)^2} M_1 e^{-\mu/2}.$$

Therefore we have for any two points z_0 and z_1 on $|z| = e^{\mu/2}$,

$$(7) \qquad |S_i(z_0) - S_i(z_1)| \leqslant \frac{4\pi e^2}{(e-1)^2} M_1 e^{-\mu/2} \quad (i = 1 \text{ to } n).$$

For these two points z_0 and z_1, we consider two algebraic equations:

$$f^n + S_1(z_0)f^{n-1} + \cdots + S_n(z_0) = 0 \quad \text{and} \quad f^n + S_1(z_1)f^{n-1} + \cdots + S_n(z_1) = 0.$$

Then the roots of these equations are values taken by n branches of f at $z = z_0$ and z_1:

$$f_1(z_0),\ f_2(z_0), \cdots, f_n(z_0);\ f_{\alpha_1}(z_1),\ f_{\alpha_2}(z_2), \cdots, f_{\alpha_n}(z_1).$$

By the theorem in §3, we can associate each $f_i(z_0)$ $(i = 1$ to $n)$ with some $f_{\alpha_i}(z_1)$ $(1 \leqslant i \leqslant n)$, say $f_i(z_0)$ with $f_{\alpha_i}(z_1)$, such that for all i,

$$(8) \qquad [f_i(z_0),\ f_{\alpha_i}(z_1)] \leqslant 8e[[S(z_0),\ S(z_1)]]^{\frac{1}{n}}$$

where $S(z_0)$ and $S(z_1)$ are the systems

$$(1,\ S_1(z_0),\ S_2(z_0), \cdots, S_n(z_0)) \quad \text{and} \quad (1,\ S_1(z_1),\ S_2(z_1), \cdots, S_n(z_1))$$

respectively.

Applying the inequality (4) to (8) and using (7), we have for all i,

$$(9) \qquad [f_i(z_0),\ f_{\alpha_i}(z_1) \leqslant 8en^{\frac{1}{2n}}\left[\frac{4\pi e^2}{(e-1)^2} M_1\right]^{\frac{1}{n}} e^{-\frac{\mu}{2n}} = M_2 e^{-\frac{\mu}{2n}}.$$

Next we denote by K_i the spherical disk with center $f_i(z_0)$ and radius $M_2 e^{-\frac{\mu}{2n}}$. We continue a branch f_i analytically along $|z| = e^{\mu/2}$ from z_0 to z_1. If we obtain a branch f_{α_j} by this continuation, then we have

$$[f_i(z_0),\ f_{\alpha_j}(z_1)] < nM_2 e^{-\mu/2n}.$$

In fact, if we denote by γ the curve on the sphere corresponding to this analytic continuation, γ is covered by at most p spherical disks $K_i(i = 1$ to $p \leqslant n)$, because each point on γ must be contained in at least one spherical disk of K_i from (9). Now noting that z_1 is an arbitrary point on $|z| = e^{\mu/2}$, we have the desired result with

$$A = A(n, \delta) = nM_2.$$

If $d < \frac{1}{2}$, then $M < 2d < 1$, $M_1 < \binom{\left[\frac{n}{2}\right]}{n} M$ and hence

$$B = B(n) = 8en^{\frac{1}{2n}} \left[\frac{4\pi e^2}{(e-1)^2} \cdot \binom{\left[\frac{n}{2}\right]}{n} \cdot 2 \right]^{\frac{1}{n}}$$

satisfies our condition.

5. Our theorem will be proved by contradiction. Suppose that there exists a k-valued algebroid function f on Ω with at least one essential singularity in E and with more than $3k$ Picard values at an essential singularity $z_0 \in E$. Let

$$f^k + S_1(z) f^{k-1} + \cdots + S_k(z) = 0 \tag{10}$$

be the defining equation of f with each $S_i(z)$ meromorphic in Ω. Then there is a neighbourhood $U(z_0)$ of z_0 such that f omits $3k+1$ values ζ_i ($i = 1$ to $3k+1$) in $U(z_0) \cap \Omega$. We take a positive δ so small that the spherical disks $C(\zeta_i, \delta)$ ($i = 1$ to $3k+1$) are disjoint by pairs. For this δ and a $\sigma > 0$, Lemma 1 determines $\delta' > 0$. We take this δ' as δ of Lemma 2 and choose μ_0 so large that

$$Ae^{-\mu_0/2k} < K = \min\left[\frac{1}{(6k^2)^2}, \frac{\delta'}{3}\right], \quad Be^{-\mu_0/2k} < K$$

$$\frac{\log k}{2k} \mu_0 = s > 1 \quad (\text{if } k > 1)$$

where A and B are the constants of Lemma 2. By our assumption (1) there is an $n_0 > 0$ such that

$$\frac{\mu_n}{1 + (\log k)\log n} > \mu_0 + 2\sigma \quad (n \geqq n_0), \quad \sum_{n=n_0}^{\infty} \frac{1}{n^s} < 1 \quad (\text{if } k > 1)$$

and so

$$\frac{\mu_n}{2k} > \frac{\mu_0}{2k} + \frac{\log k}{2k} \mu_0 \log n = \frac{\mu_0}{2k} + \log n^s$$

$$Ae^{-\mu_n/2k} < Ae^{-\mu_0/2k} \cdot \frac{1}{n^s} < \frac{K}{n^s}, \quad Be^{-\mu_n/2k} < \frac{K}{n^s}.$$

The level line $\beta_r = \{z \mid u(z) = r\}$ consists of a finite number of Jordan curves $\beta_{r,m}$ with $m = 1$ to $n(r)$, and one of them, say $\beta_{r,1}$ encloses z_0. For r sufficiently near L the longest doubly connected R-chain $R(\beta_{r,1}) = D_{1,1}$ for $\beta_{r,1}$ defined in **§1** coincides with one of $R(\beta_{r_{n_0'},i})$ $(i = 1$ to $n(r_{n_0'}))$ for some $r_{n_0'}$ $(n_0' \geqslant n_0)$ and is contained in $U(z_0)$. Thus the modulus of $D_{1,1}$ is not less than $\mu_{n_0'}$ and hence greater than $\mu_0 + 2\sigma$ but is not infinite for otherwise z_0 would have to be isolated and f could not have $3k+1$ Picard values at z_0. Therefore $D_{1,1}$ must branch off. Now suppose that $D_{1,1}$ is a component of the open set $\Omega_n - \bar{\Omega}_{n'}$, with $n > n'$, and branches off into two regions $R_{n+1,\alpha}$ and $R_{n+1,\alpha'}$. Consider the longest doubly connected R-chain $D_{2,1}$ and $D_{2,2}$ containing $R_{n+1,\alpha}$ and $R_{n+1,\alpha'}$, respectively. They both have moduli not less than $\mu_{n_0'+1}$ and hence greater than $\mu_0 + 2\sigma$ and one of them, say $D_{2,1}$, separates z_0 from $D_{1,1}$. Its modulus is finite for the same reason as above. Hence $D_{2,1}$ is a component of the open set $\Omega_{\tilde{n}} - \bar{\Omega}_n$ for some $\tilde{n} > n$ and branches off into two regions $R_{\tilde{n}+1,\alpha}$ and $R_{\tilde{n}+1,\alpha'}$. We denote by $D_{3,1}$ and $D_{3,2}$ the longest doubly connected R-chains containing them. If the modulus of $D_{2,2}$ is infinite, one of the boundary components of $D_{2,2}$ is a point $z_1 \in E$ and f is algebroid at z_1. If the modulus is finite we obtain two R-chains $D_{3,3}$ and $D_{3,4}$ in the same manners as above. Thus we have at most 2^2 R-chains $D_{3,q}$ such that their harmonic moduli are not less than $\mu_{n_0'+2}$ and so greater than $\mu_0 + 2\sigma$, and one of them encloses z_0. Moreover each of them branches off into two regions if the modulus is finite, or has a point $z_1 \in E$ as one of its boundary components at which f is algebroid if the modulus is infinite.

Continuing inductively we obtain a set of R-chains $D_{p,q}$ with $p = 1, 2, \cdots$ and $q = 1, 2, \cdots, Q(p) \leqslant 2^{p-1}$, which has the following properties;

(a) $\bigcup_{p=1}^{\infty} \bigcup_{q=1}^{Q(p)} \bar{D}_{p,q} \supset \Delta$, where Δ is the intersection of Ω with the set bounded by the Jordan curve $\beta_{r,1}$,

(b) the modulus of each $D_{p,q}$ is not less than $\mu_{n_0'+p-1}$ and so greater than $\mu_0 + 2\sigma$,

(c) each $D_{p,q}$ branches off into two regions $D_{p+1,q}$ if the modulus of $D_{p,q}$ is finite, or

(c′) each $D_{p,q}$ has a point $z_1 \in E$ as one of its boundary components and f is algebroid at z_1 if the modulus of $D_{p,q}$ is infinite. In this case we shall denote the point z_1 by $z_{p,q}$ and the value $f(z_{p,q})$ by $\zeta_{p,q}$.

Let $\mathfrak{X}_\Omega$ be k-sheeted covering surface over Ω generated by f. For any connected subset C of Ω, the part of $\mathfrak{X}_\Omega$ over the set C consists of at most k, say $p(C) \leqslant k$, connected components and we denote them by $\mathfrak{X}_\Omega^i(C)$ $(i=1$ to $p(C))$.

Each $D_{p,q}$ is conformal equivalent to the annulus $1<|t|<e^\mu$, where μ is the modulus of $D_{p,q}$. If $\mu<\infty$, we denote by $D_{p,q}^1$ $D_{p,q}^2$ and $D_{p,q}^3$ the subregions of $D_{p,q}$ corresponding to the annuli $1<|t|<e^\sigma$, $\sigma^\sigma<|t|<e^{\mu-\sigma}$ and $e^{\mu-\sigma}<|t|<e^\mu$, respectively and by $\beta_{p,q}^1$, $\beta_{p,q}^2$ and $\beta_{p,q}^3$ the closed curves corresponding to $|t|=e^{\sigma/2}$, $|t|=e^{\mu/2}$ and $|t|=e^{\mu-\sigma/2}$, respectively.

We shall see that for every i $(1 \leqslant i \leqslant p(\beta_{p,q}^2))$ the diameter of the image of $\mathfrak{X}_\Omega^i(\beta_{p,q}^2)$ under f with respect to the chordal distance is dominated by $K/(n_0'+p-1)^s$. In fact, for $z' \in \beta_{p,q}^1$ and $z'' \in \beta_{p,q}^3$, the image of $f(P_i(z'))$ and $f(P_j(z''))$ lie outside of at least one $C(\zeta_i, \delta)$, say $C(\zeta_1, \delta)$, where $P_i(z')$ and $P_j(z'')$ are the points of $\mathfrak{X}_\Omega$ over z' and z'', respectively. Applying Lemma 1 to $D_{p,q}^1$ and $D_{p,q}^3$, we see that for all i_1 $(1 \leqslant i_1 \leqslant p(\beta_{p,q}^1))$ and $i_3 (1 \leqslant i_3 \leqslant p(\beta_{p,q}^3))$, the images of $\mathfrak{X}_\Omega^{i_1}(\beta_{p,q}^1)$ and $\mathfrak{X}_\Omega^{i_3}(\beta_{p,q}^3)$ lie completely outside of $C(\zeta_1, \delta')$. Consequently $\mathfrak{X}_\Omega^j(D_{p,q}^2)$ $(j=1$ to $p(D_{p,q}^2))$ lie completely outside of $C(\zeta_1, \delta')$ by the maximum principle. Since for any i_2, $\mathfrak{X}_\Omega^{i_2}(\beta_{p,q}^2)$ is contained in one of the components $\mathfrak{X}_\Omega^j(D_{p,q}^2)$, say $\mathfrak{X}_\Omega^1(D_{p,q}^2)$, and the modulus of $D_{p,q}^2$ is not less than $\mu_{n_0'+p-1}$ and hence greater than $\mu_0+2\sigma$ Lemma 2 applied to $\mathfrak{X}_\Omega^1(D_{p,q}^2)$ leads us to our assertions.

Each $D_{p+1,q'}$ with $p \geqslant 1$ has in common with another $D_{p+1,q''}$ a $D_{p,q}$ branching off into them, and we shall denote by $\Delta_{p,q}$ the triply connected region bounded by $\beta_{p,q}^2$, $\beta_{p+1,q'}^2$ and $\beta_{p+1,q''}^2$ where $\beta_{p+1,q'}^2 = z_{p+1,q'}$ or $\beta_{p+1,q''}^2 = z_{p+1,q''}$, if $D_{p+1,q'}$ or $D_{p+1,q''}$ has infinite modulus. For $\vartheta_\lambda \in f(\mathfrak{X}_\Omega^\lambda(\beta_{p,q}^2))$ $(\lambda=1$ to $p(\beta_{p,q}^2))$, $\vartheta_\mu \in f(\mathfrak{X}_\Omega^\mu(\beta_{p+1,q'}^2))$ $(\mu=1$ to $p(\beta_{p+1,q'}^2))$ and $\vartheta_\nu \in f(\mathfrak{X}_\Omega^\nu(\beta_{p+1,q''}^2))$ $(\nu=1$ to $p(\beta_{p+1,q''}^2))$, we consider at most $3k$ spherical disks $C(\vartheta_\lambda, K/n_0'^s)$ $(\lambda=1$ to $p(\beta_{p,q}^2))$, $C(\vartheta_\mu, K/(n_0'+1)^s)$ $(\mu=1$ to $p(\beta_{p+1,q'}^2))$ and $C(\vartheta_\nu, K/(n_0'+1)^s)$ $(\nu=1$ to $p(\beta_{p+1,q''}^2))$, respectively. Since $K<\delta'/3$ there exists at least one ζ_i, say ζ_1, not contained in the disks. Let $\mathfrak{X}_\Omega^i(\Delta_{p,q})$ be a component above the region $\Delta_{p,q}$. We assume that the boundary curves of $\mathfrak{X}_\Omega^i(\Delta_{p,q})$ consist of $\mathfrak{X}_\Omega^\lambda(\beta_{p,q}^2)$ $(\lambda=1$ to $p^i(\beta_{p,q}^2) \leqslant p(\beta_{p,q}^2))$, $\mathfrak{X}_\Omega^\mu(\beta_{p+1,q'}^2)$ $(\mu=1$ to $p^i(\beta_{p+1,q'}^2) \leqslant p(\beta_{p+1,q'}^2))$ and $\mathfrak{X}_\Omega^\nu(\beta_{p+1,q''}^2)$ $(\nu=1$ to $p^i(\beta_{p+1,q''}^2) \leqslant p(\beta_{p+1,q''}^2))$. Then the union

$$\{\bigcup_{\lambda}^{p^i(\beta_{p,q}^2)} C(\vartheta_\lambda, K/n_0'^s)\} \cup \{\bigcup_{\mu}^{p^i(\beta_{p+1,q'}^2)} C(\vartheta_\mu, K/(n_0'+1)^s)\} \cup \{\bigcup_{\nu}^{p^i(\beta_{p+1,q''}^2)} C(\vartheta_\nu, K/(n_0'+1)^s)\}$$

must be connected. In fact, if this were not the case, there would exist a point $P \in \mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$ such that $f(P)$ can be joined to ζ_1 by a curve Λ in the exterior of the union. We would be led to the contradiction that the element of the inverse function f^{-1} corresponding to P can be continued analytically along Λ up to a point arbitrarily near ζ_1 so that f takes the value ζ_1 in $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$. We conclude:

(α) For every $\Delta_{p,q}$ and every component $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$, there is a spherical disk with the chordal radius $k\left(\frac{K}{n_0'^s} + 2\frac{K}{(n_0'+1)^s}\right) < 3k\cdot\frac{K}{n_0'^s}$ containing its image $f(\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q}))$.

Next consider $\beta_{p,q}^2$ for $p \geqslant 2$. The region $\Delta_{p,q}$ and some $\Delta_{p-1,q'}$ have $\beta_{p,q}^2$ as the common boundary and any component $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q} \cup \Delta_{p-1,q} \cup \beta_{p,q}^2)$ consists of some components $\mathfrak{X}_{\Omega}^{j_1}(\Delta_{p,q})$, and some components $\mathfrak{X}_{\Omega}^{j_2}(\Delta_{p-1,q'})$ and some closed curves $\mathfrak{X}_{\Omega}^{j_3}(\beta_{p,q}^2)$. Therefore, in view of (α), the image of $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q} \cup \beta_{p,q}^2 \cup \Delta_{p-1,q'})$, consequently that of every component $\mathfrak{X}_{\Omega}^{j}(D_{p,q}^2)$ contained in $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q} \cup \beta_{p,q}^2 \cup \Delta_{p-1,q'})$ is contained in a spherical disk with chordal radius $6k^2K/n_0'^s < 1/2$. On applying Lemma 2 to every $\mathfrak{X}_{\Omega}^{i}(D_{p,q}^2)$ for $d = 6k^2K/n_0'^s$, we see that the diameter of every $\mathfrak{X}_{\Omega}^{i}(\beta_{p,q}^2)$ is less than $(6k^2)^{1/k}\cdot(K/n_0'^s)^{1/k}\cdot Be^{-\mu n_0'+1/2k}$. For $p \geqslant 2$ and every component $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$, each boundary component of $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$ has an image with diameter less than $(6k^2)^{1/k}\cdot(K/n_0'^s)^{1/k}\cdot (K/(n_0'+1)^s)$. By the same reasoning as above we infer:

(β) For $p \geqslant 2$, the image of every component $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$ is contained in a spherical disk with chordal redius $3k\cdot(6k^2)^{1/k}\cdot(K/n_0'^s)^{1/k}(K/(n_0'+1)^s)$.

By induction we deduce for every ν:

(γ) For $p \geqslant \nu$, the image of every component $\mathfrak{X}_{\Omega}^{i}(\Delta_{p,q})$ is contained in a spherical disk with chordal radius

$$3k\cdot(6k^2)^{\frac{1}{k}+\frac{1}{k^2}+\cdots+\frac{1}{k^{\nu-1}}}\cdot\left(\frac{K}{n_0'^s}\right)^{\frac{1}{k^{\nu-1}}}\cdot\left\{\frac{K}{(n_0'+1)^s}\right\}^{\frac{1}{k^{\nu-2}}}\cdot\cdots\cdot\left\{\frac{K}{(n_0'+\nu-1)^s}\right\}$$

$$\begin{cases} < \dfrac{1}{2}(6K)^{\nu} & (k=1) \\ < \dfrac{(6k^2)^2}{2k}\cdot K\cdot\dfrac{1}{(n_0'+\nu-1)^s} & (k>1). \end{cases}$$

Let Δ' be the intersection of Ω and the region bounded by a Jordan curve $\beta_{1,1}^{2}$ and let z^* be a point of $\beta_{1,1}^{2}$. Then it follows from the property (α) of $\{D_{p,q}\}$ that

$$\Delta' \subset \bigcup_{p=1}^{\infty} \bigcup_{q=1}^{Q(p)} \bar{\Delta}_{p,q}$$

and consequently for any $z' \in \Delta'$ there is a $\Delta_{p',q'}$ whose closure contains z'. Let $P(z^*)$ and $P(z)$ be any two points on $\mathfrak{X}_\Omega$ over z^* and z' and let $\mathfrak{X}_\Omega^1(\Delta_{1,1})$ and $\mathfrak{X}_\Omega^1(\Delta_{p',q'})$ be the components containing $P(z^*)$ and $P(z')$ in their closures, respectively. By (γ) we have for a chain of $\mathfrak{X}_\Omega^i(\Delta_{p,q})$ joining $\mathfrak{X}_\Omega^1(\Delta_{1,1})$ and $\mathfrak{X}_\Omega^1(\Delta_{p',q'})$,

$$[f(P(z^*)),\ f(P(z'))] \leq \sum_{p=1}^{p'} \text{diam.}\ f(\mathfrak{X}_\Omega^i(\Delta_{p,q}))$$

$$\leqslant \frac{(6k^2)^2}{2k} \cdot K \cdot \sum_{\nu=1}^{\infty} \frac{1}{(n_0' + \nu - 1)^s} \leqslant \frac{(6k^2)^2}{2k} \cdot \frac{1}{(6k^2)^2} = \frac{1}{2k} \quad (k > 1)$$

$$[f(P(z^*)),\ f(P(z'))] \leqslant \sum_{\nu=1}^{\infty} \frac{1}{2} (6K)^\nu < \frac{1}{2} \sum_{\nu=1}^{\infty} \left(\frac{1}{6}\right)^\nu < \frac{1}{2} \quad (k = 1).$$

We may assume that $\mathfrak{X}_\Omega(\Delta')$ consists of a single component with k sheets. Therefore the image of $\mathfrak{X}_\Omega(\Delta')$ under f can be covered by some spherical disk with chordal radius 1/2. By means of a linear transformation we conclude that f is bounded on $\mathfrak{X}_\Omega(\Delta')$ and hence all coefficients $S_i(z)$ of defining equation (10) of f are bounded and single-valued in Δ'. On the other hand, on applying the criterion of Pfluger [7]-Mori [5] (cf. App. I of [9]) to the annular regions $\{D_{p,q}\}$ we easily see that the part E' of E contained in the region bounded by $\beta_{1,1}^{2}$ is an $N_{\mathfrak{B}}$-set in the sense of Ahlfors-Beurling [1]. Hence each point of E' must be a removable singularity of $S_i(z)$ ($i = 1$ to k). This contradicts our assumption that $z_0 \in E'$ is an essential singularity of f and we conclude that f cannot omit $3k+1$ values in Ω at z_0. Thus our theorem is proved completely.

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