

BANACH SPACES OF BOUNDED SOLUTIONS OF $\Delta u = Pu$ ($P \geq 0$) ON HYPERBOLIC RIEMANN SURFACES

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Consider a nonnegative Hölder continuous 2-form $P(z)dxdy$ on a hyperbolic Riemann surface R ($z = x + iy$). We denote by $PB(R)$ the Banach space of solutions of the equation $\Delta u = Pu$ on R with finite supremum norms. We are interested in the question how the Banach space structure of $PB(R)$ depends on P . Precisely we consider two such 2-forms P and Q on R and compare $PB(R)$ and $QB(R)$. If there exists a bijective linear isometry T of $PB(R)$ to $QB(R)$, then we say that $PB(R)$ and $QB(R)$ are isomorphic. If moreover $|u - Tu|$ is dominated by a potential p_u on R for every u in $PB(R)$, then we say that $PB(R)$ and $QB(R)$ are canonically isomorphic. Intuitively this means that u and Tu have the same ideal boundary values. Using these terminologies our problem is formulated as follows:

- 1° *When are Banach spaces $PB(R)$ and $QB(R)$ isomorphic?*
- 2° *When are Banach spaces $PB(R)$ and $QB(R)$ canonically isomorphic?*

For this purpose we consider the set Δ_P , which we call the nondensity point set associated with $P(z)dxdy$, of points z^* in the Wiener harmonic boundary Δ of R such that there exists a neighborhood U^* of z^* in the Wiener compactification R^* of R such that

$$\int_{U^* \cap R} G(z, \zeta) P(\zeta) d\xi d\eta < \infty \quad (\zeta = \xi + i\eta)$$

for some and hence for every z in R where $G(z, \zeta)$ is the harmonic Green's function on R . We shall see that the set Δ_P is a compact Stonean space with the relative topology of R^* . Our main result in this paper is that the Banach space structure of $PB(R)$ is completely determined by the space Δ_P . More precisely the questions 1° and 2° can be answered in terms of Δ_P as follows:

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THE MAIN THEOREM. *Banach spaces $PB(R)$ and $QB(R)$ are isomorphic (canonically isomorphic, resp.) if and only if spaces Δ_P and Δ_Q are homeomorphic (identical, resp.).*

Sufficient conditions to the question 2° thus far obtained by Royden [18], Nakai [14], Maeda [9], Lahtinen [7], etc. are then direct consequences of the above theorem. These will be discussed in nos. 15–18. In nos. 1–5 we shall discuss behaviors of functions in $PB(R)$ on the Wiener compactification R^* of R . The Heins canonical extension λ_P^W is one of the important tools in our study. Another important tool, the reduction operator T_P , is discussed in nos. 6–7. The structure of Δ_P will be studied in nos. 8–11. The main theorem will then be proven in nos. 12–13. An alternate definition of Δ_P is appended in no. 14.

Wiener compactification

1. Let $P(z)dxdy$ be a 2-form on a Riemann surface R such that $P(z)$ is a Hölder continuous function of each local parameter $z = x + iy$, i.e. $|P(z_1) - P(z_2)| \leq K|z_1 - z_2|^\alpha$ for every pair of points z_1 and z_2 in the parametric disk of z with a $K \in (0, \infty)$ and an $\alpha \in (0, 1]$. Then the elliptic equation $\Delta u(z) = P(z)u(z)$ can be invariantly defined on R where $\Delta u(z)dxdy = d*du(z)$. Denote by $P(U)$ the linear space of solutions of $\Delta u = Pu$ on an open subset U of R . We also use the standard notation $H(U)$ for $P(U)$ with $P \equiv 0$.

LEMMA. *The absolute value $|u|$ of any u in $P(U)$ is subharmonic on U for every open subset U of R if and only if $P(z)dxdy$ is nonnegative, i.e. $P(z) \geq 0$ for every choice of local parameters z .*

Proof. Suppose $P \geq 0$ and $u \in P(U)$. Then $\Delta |u(z)| = P(z)|u(z)| \geq 0$ on the open set $U' = \{z \in U; u(z) \neq 0\}$, i.e. $|u|$ is subharmonic on U' . The submean value property is clearly valid for $|u|$ at each point of $U - U'$. Therefore $|u|$ is subharmonic on U . To show that $P \geq 0$ is necessary, let z be an arbitrary point in R . If we take a sufficiently small regular parametric disk U with z its center, then the Dirichlet problem of $\Delta u = Pu$ is solvable for U with continuous boundary values φ and the solution is nonnegative (positive, resp.) for $\varphi \geq 0$ ($\varphi > 0$, resp.) (cf. Miranda [10]). If u is the solution with $\varphi \equiv 1$, then, since $|u| = u > 0$ is subharmonic, $P(z) = \Delta u(z)/u(z) \geq 0$. Q.E.D.

2. Hereafter, we always assume that $P(z) \geq 0$. For simplicity such a 2-form $P(z)dxdy$, i.e. Hölder continuous and nonnegative, will be referred to as a *density* on R . We denote by $PB(R)$ the subspace of $P(R)$ consisting of solutions u with finite supremum norms:

$$\|u\| = \|u\|_R = \sup_{z \in R} |u(z)|.$$

We also use the standard notation $HB(R)$ for $PB(R)$ with $P \equiv 0$. Then $(PB(R), \|\cdot\|)$ is a Banach space. We wish to determine the Banach space structure of $PB(R)$. We say that R is *hyperbolic* if there exists the harmonic Green's function on R . Nonhyperbolic surfaces are called *parabolic*. The Ahlfors-Ohtsuka characterization (cf. [19]) says that R is parabolic if and only if there does not exist any nonconstant positive superharmonic function on R . Therefore if R is parabolic and $u \in PB(R)$, then, since $\|u\| - |u|$ is a nonnegative superharmonic function on R , $\|u\| - |u|$ and hence u is a constant. This proves the following BreLOT [1, 2]-Ozawa [17] theorem:

LEMMA. *If R is parabolic, then $PB(R) = \{0\}$ for densities $P \not\equiv 0$ and $HB(R) = \mathbb{R}$ (the real number field).*

In view of this lemma the Banach space $PB(R)$ is of no interest if R is parabolic, and for this reason, hereafter, we always assume that R is hyperbolic.

3. The *Wiener compactification* R^* of a hyperbolic Riemann surface R is a compact Hausdorff space containing R as its open and dense subset such that $C(R^*) = \{f|_R; f \in C(R^*)\}$ is the totality of bounded continuous Wiener functions on R . If $F \in C(R^*)$, then $f = F|_R$ is of course defined only on R but we always make the convention that $f(z^*) = F(z^*)$ for $z^* \in R^* - R$. Typical examples of Wiener functions on R are subharmonic functions s on R such that $|s|$ is bounded or more generally dominated by superharmonic functions. Denote by $\mathcal{A}$ the set of points z^* in R^* such that

$$\liminf_{z \in R, z \rightarrow z^*} p(z) = 0$$

for every potential p on R , i.e. a superharmonic function p on R whose greatest harmonic minorant is zero. The set $\mathcal{A}$ is contained in the Wiener boundary $R^* - R$ of R and referred to as the *Wiener harmonic boundary*

of R . For a subset A of R^* we denote by $\bar{A}$ the closure of A in R^* and by ∂A the relative boundary $(\bar{A} - \text{Int } A) \cap R$ with respect to R . Let U be an open subset of R and s be a subharmonic function U bounded from above. The *maximum principle* says that if

$$(1) \quad \limsup_{z \in U, z \rightarrow z^*} s(z) \leq M$$

for every z^* in $(\partial U) \cup (\bar{U} \cap \Delta)$, then $s \leq M$ on U .

An open subset W of R will be called *normal* if ∂W is analytic. We do not exclude W with $\partial W = \phi$, i.e. $W = R$, from our family of normal open sets. We denote by Δ^W the open subset $\Delta \cap (\bar{W} - \overline{\partial W})$ of Δ . We can define an operator $\pi_W: C(R^*) \rightarrow C(R^*)$ such that $\pi_W f|_W \in H(W)$, and $\pi_W f|(R^* - \bar{W}) \cup (\partial W) \cup \Delta^W = f$ for every $f \in C(R^*)$. For details of Wiener compactification and the operator π_W , we refer to Constantinescu-Cornea [3] or Sario-Nakai [19].

4. For a normal open subset W of R (including the case $W = R$) we denote by $PB(W; \partial W)$ the family $\{u \in PB(W) \cap C(R); u|_{R-W} = 0\}$. We also use the notation $HB(W; \partial W)$ for $PB(W; \partial W)$ with $P \equiv 0$. If $W = R$, then $PB(W; \partial W) = PB(W; \phi) = PB(W)$. By the same proof as in no. 1 we see that $u \cup 0 = \max(u, 0)$, $-(u \cap 0) = -\min(u, 0)$, and $|u| = u \cup 0 - u \cap 0$ are subharmonic on R for every $u \in PB(W; \partial W)$. Therefore $PB(W; \partial W) \subset C(R^*)$. By the maximum principle (1), $\|u\|_R = \|u\|_\Delta$ for every $u \in PB(W; \partial W)$. For a regular region Ω , i.e. relatively compact and normal region in R , and a continuous function φ in $C(\partial\Omega)$, we denote by P_φ^Ω the function in $P(\Omega) \cap C(\bar{\Omega})$ such that $P_\varphi^\Omega|_{\partial\Omega} = \varphi$. We also use the notation H_φ^Ω for P_φ^Ω with $P \equiv 0$. We define a linear operator $\lambda_P^W: PB(W; \partial W) \rightarrow PB(R)$ by

$$(2) \quad \lambda_P^W u = \lim_{\Omega \rightarrow R} P_u^\Omega$$

for every $u \in PB(W; \partial W)$. Then it satisfies

$$(3) \quad \lambda_P^W u|_\Delta = u|_\Delta.$$

In particular λ_P^W is isometric and hence injective. We use the notation λ_H^W for λ_P^W with $P \equiv 0$. These operators are referred to as *canonical extensions* (Heins [6]).

To see that (2) is well defined and (3) is valid, set $v_1 = u \cup 0$ and $v_2 = -(u \cap 0)$. Since $\|u\| \geq H_{v_i}^\Omega \geq P_{v_i}^\Omega \geq v_i \geq 0$, $\{P_{v_i}^\Omega\}$ is increasing and

thus $P_{v_i}^\Omega$ converges to a $u_i \in PB(R)$ as Ω exhausts R ($i = 1, 2$) and therefore $P_u^\Omega = P_{v_1}^\Omega - P_{v_2}^\Omega$ to $u_1 - u_2$, i.e. (1) is well defined. Similarly $H_{v_i}^\Omega$ converges increasingly to an $h_i \in HB(R)$ which is the least harmonic majorant of the subharmonic function v_i on R and hence $p_i = h_i - v_i$ is a potential on R ($i = 1, 2$). Since

$$|\lambda_P^W u - u| \leq |u_1 - v_1| + |u_2 - v_2| \leq (h_1 - v_1) + (h_2 - v_2) = p$$

with $p = p_1 + p_2$, a potential on R ,

$$|\lambda_P^W u(z^*) - u(z^*)| = \lim_{z \in R, z \rightarrow z^*} |\lambda_P^W u(z) - u(z)| \leq \liminf_{z \in R, z \rightarrow z^*} p(z) = 0$$

for every point $z^* \in \Delta$, i.e. (3) is valid.

5. Denote by $PB(W; \partial W)^+$ the family $\{u \in PB(W; \partial W); u \geq 0\}$. We maintain that $PB(W; \partial W)^+$ generates $PB(W; \partial W)$; more precisely, for any $u \in PB(W; \partial W)$ there exist $u_i \in PB(W; \partial W)^+$ ($i = 1, 2$) such that

$$(4) \quad u = u_1 - u_2, \quad u_1|_\Delta = u \cup 0|_\Delta, \quad u_2|_\Delta = -(v \cap 0)|_\Delta.$$

The proof of (4) is similar to that of (2) and (3). Let $v_1 = u \cup 0$ and $v_2 = -(u \cap 0)$. As in no. 4 we see that $H_{v_i}^\Omega \geq u_{i\Omega} \geq v_i \geq 0$, where $u_{i\Omega} = P_{v_i}^{W \cap \Omega}$ on $W \cap \Omega$ and $u_{i\Omega} = 0$ on $\Omega - W \cap \Omega$, and that $u_i = \lim_{\Omega \rightarrow R} u_{i\Omega}$ exists in $PB(W; \partial W)^+$ and $h_i = \lim_{\Omega \rightarrow R} H_{v_i}^\Omega$ exists in $HB(R)^+$ ($i = 1, 2$). Since $u = u_{1\Omega} - u_{2\Omega}$, we deduce $u = u_1 - u_2$. Moreover $0 \leq u_i - v_i \leq h_i - v_i = p_i$ and p_i is a potential ($i = 1, 2$). Therefore (4) is true.

Reduction operator

6. Since we have assumed that our base Riemann surface R is hyperbolic, there exists the harmonic Green's function $G(z, \zeta) = G_R(z, \zeta)$ on R . Let W be a normal open subset of R . The harmonic Green's function $G_W(z, \zeta)$ on W is defined as follows. Let $W = \bigcup_n W_n$ be the decomposition of W into connected components W_n such that each W_n is a normal region. If both of z and ζ belong to the same W_n , then $G_W(z, \zeta) = G_{W_n}(z, \zeta)$; otherwise $G_W(z, \zeta) = 0$. Including the case $W = R$, we define a linear operator $T_P^W: PB(W; \partial W) \rightarrow HB(W; \partial W)$ by

$$(5) \quad T_P^W u = u + \frac{1}{2\pi} \int_R G_W(\cdot, \zeta) u(\zeta) P(\zeta) d\bar{\zeta} d\eta.$$

To see that (5) is well defined, first let $u \in PB(W; \partial W)^+$. Then by the Green formula

$$H_u^{w \cap \Omega} = u + \frac{1}{2\pi} \int_{\Omega} G_{w \cap \Omega}(\cdot, \zeta) u(\zeta) P(\zeta) d\xi d\eta.$$

Since u is subharmonic, $\lim_{\Omega \rightarrow R} H_u^{w \cap \Omega}$ exists in $HB(W; \partial W)$. Observe that $\{G_{w \cap \Omega}(\cdot, \zeta) u(\zeta)\}_{\Omega}$ is increasing, and therefore the Lebesgue-Fatou theorem yields (5) for $u \geq 0$. The general case follows from the decomposition (4). Similarly, as above,

$$H_u^{\Omega} = P_u^{\Omega} + \frac{1}{2\pi} \int_{\Omega} G_{\Omega}(\cdot, \zeta) P_u^{\Omega}(\zeta) P(\zeta) d\xi d\eta$$

for $u \in PB(W; \partial W)^+$ and by making Ω tend to R we deduce

$$\lambda_H^w u = \lambda_P^w u + \frac{1}{2\pi} \int_R G_R(\cdot, \zeta) (\lambda_P^w u)(\zeta) P(\zeta) d\xi d\eta.$$

Since $0 \leq u \leq \lambda_P^w u$ and $G_w \leq G_R$, we conclude that

$$0 \leq T_P^w u - u \leq \frac{1}{2\pi} \int_R G_R(\cdot, \zeta) (\lambda_P^w u)(\zeta) P(\zeta) d\xi d\eta = p$$

with p a potential on R . By the decomposition (4) we can also conclude that $|T_P^w u - u|$ is dominated by a potential on R for general $u \in PB(W; \partial W)$. Therefore we see that

$$(6) \quad T_P^w u|_{\Delta} = u|_{\Delta}$$

for every $u \in PB(W; \partial W)$. In particular T_P^w is isometric and hence injective.

7. The operator $T_P = T_P^R$ is referred to as a *reduction operator* since T_P reduces the study of $PB(R)$ to that of more tractable class $HB(R)$ (cf. Singer [20]). In this sense it is important to determine when T_P is surjective. As a preparatory observation we state the following: If

$$(7) \quad \int_W G_R(\cdot, \zeta) P(\zeta) d\xi d\eta < \infty$$

for some $z \in R$ and hence by the Harnack inequality for every $z \in R$, then T_P^w is surjective. To prove this take an $h \in HB(W; \partial W)^+$. Since $0 \leq P_h^{w \cap \Omega} \leq h$, $\{P_h^{w \cap \Omega}\}$ is decreasing and converges to a $u \in PB(W; \partial W)^+$. Then

$$h = P_h^{w \cap \Omega} + \frac{1}{2\pi} \int_{\Omega} G_{w \cap \Omega}(\cdot, \zeta) P_h^{w \cap \Omega}(\zeta) P(\zeta) d\xi d\eta.$$

In view of (7) the Lebesgue convergence theorem is applicable to deduce

$$h = u + \frac{1}{2\pi} \int_R G_W(\cdot, \zeta) u(\zeta) P(\zeta) d\xi d\eta,$$

i.e. $T_P^W u = h$. Since $HB(W; \partial W)^+$ generates $HB(W; \partial W)$, we obtain the required conclusion.

Nondensity points

8. We introduce the set Δ_P of points z^* in Δ such that there exists a neighborhood U^* of z^* in R^* such that

$$(8) \quad \int_{U^* \cap R} G_R(z, \zeta) P(\zeta) d\xi d\eta < \infty$$

for some and hence for every point z in R . Such a point z^* will be referred to as a *nondensity point* of P with the weight G . If we denote by Δ_H for Δ_P with $P \equiv 0$, then $\Delta_H = \Delta$. Clearly the nondensity point set Δ_P is *open*. Since Δ is a Stonean space i.e. every point in Δ has a base of compact and open neighborhoods in Δ , the same is true of Δ_P . Another kind of nondensity point was first introduced in Glasner-Katz [5] (cf. also [15]) for the Royden harmonic boundary. In the definition (8) we can moreover assume that $U^* \cap R$ is a normal open subset of R since we can replace U^* by a smaller neighborhood (cf. [3], [19]). First we remark that

$$(9) \quad u|_{\Delta} - \Delta_P = 0$$

for every $u \in PB(R)$. In fact, let $z^* \in \Delta$ with $u(z^*) \neq 0$. We can choose a neighborhood U^* of z^* such that $|u(z)| > |u(z^*)|/2$ on $U^* \cap R$. By (5), u is $G_R(\cdot, \zeta)P(\zeta)d\xi d\eta$ -integrable on R . Therefore

$$\int_{U^* \cap R} G(z, \zeta) P(\zeta) d\xi d\eta \leq \frac{2}{|u(z^*)|} \int_{U^* \cap R} G(z, \zeta) |u(\zeta)| P(\zeta) d\xi d\eta < \infty,$$

i.e. $z^* \in \Delta_P$. This proves (9).

9. Let K be a compact and open set in Δ_P . We can find a neighborhood W^* of K in R^* such that $W = W^* \cap R$ is normal in R and (7) is valid since K is compact. Choose a $\varphi \in C(R^*)$ such that $\varphi|_K = 1$ and $\varphi|(R^* - W^*) \cup (\Delta - K) = 0$. Such a φ exists since K is open and compact in Δ and $K \cap (R^* - W^*) = \emptyset$. Observe that $\pi_W \varphi \in HB(W; \partial W)$ with $\pi_W \varphi|_K = 1$ and $\pi_W \varphi|_{K^c} = 0$ (cf. no. 3). By (7), $T_P^W; PB(W; \partial W) \rightarrow HB(W; \partial W)$ is surjective and hence $(T_P^W)^{-1} \circ \pi_W \varphi \in PB(W; \partial W)$ with

$(T_P^W)^{-1} \circ \pi_W \varphi|K = 1$ and $(T_P^W)^{-1} \circ \pi_W \varphi|\mathcal{A} - K = 0$ (cf. (6)). Finally set $e_K = \lambda_P^W \circ (T_P^W)^{-1} \circ \pi_W \varphi \in PB(R)$. By (3), $e_K|K = 1$ and $e_K|\mathcal{A} - K = 0$. Put

$$e_P = \sup_{K \subset \mathcal{A}_P} e_K ,$$

where K runs over all compact open subsets of $\mathcal{A}_P$. The conditionally monotone compactness of $PB(R)$ assures that $e_P \in PB(R)$. Clearly $0 \leq e_P \leq 1$ on $\mathcal{A}$. Since every point $z^* \in \mathcal{A}_P$ has such a K with $z^* \in K$, we see that $e_P|\mathcal{A}_P = 1$. With (9) we now conclude that

$$(10) \quad e_P \in PB(R) , \quad 0 \leq e_P \leq 1 , \quad e_P|\mathcal{A}_P = 1 , \quad e_P|\mathcal{A} - \mathcal{A}_P = 0 .$$

The function e_P will be referred to as the *P-unit* (cf. Singer [21]). It is easy to see that e_P is the greatest function in $PB(R)$ dominated by 1 and actually

$$e_P = \lim_{\alpha \rightarrow R} P_1^\alpha .$$

Therefore e_P is the *P-elliptic measure* in the terminology of Royden [18]. Since $e_P|\mathcal{A} \in C(\mathcal{A})$ and $e_P|\mathcal{A}$ is the characteristic function of $\mathcal{A}_P$, we obtain the following

THEOREM. *The nondensity point set $\mathcal{A}_P$ is compact and open in $\mathcal{A}$.*

10. We denote by $C(\mathcal{A}; \mathcal{A}_P)$ the family $\{\varphi \in C(\mathcal{A}); \varphi|\mathcal{A} - \mathcal{A}_P = 0\}$. Clearly $C(\mathcal{A}; \mathcal{A}_P)$ is isomorphic to $C(\mathcal{A}_P)$ as Banach spaces by the natural correspondence $\tau_P: C(\mathcal{A}; \mathcal{A}_P) \rightarrow C(\mathcal{A}_P)$ given by $\tau_P u = u|\mathcal{A}_P$. We define an operator, the *boundary restriction*, $\rho_P: PB(R) \rightarrow C(\mathcal{A}; \mathcal{A}_P)$ by $\rho_P u = u|\mathcal{A}$ for every $u \in PB(R)$. We write ρ_H for ρ_P with $P \equiv 0$. By (1), ρ_P is isometric. We prove

THEOREM. *The boundary restriction ρ_P is surjective.*

Proof. Let $\varphi \in C(\mathcal{A}; \mathcal{A}_P)$ and W^* be an open neighborhood of $\mathcal{A}_P$ in R^* such that $W = W^* \cap R$ is normal and (7) is valid for W . We can extend φ to R^* so that $\varphi \in C(R^*)$ with $\varphi(R^* - W^*) \cup (\mathcal{A} - \mathcal{A}_P) = 0$. Then as in no. 9 $u = \lambda_P^W \circ (T_P^W)^{-1} \circ \pi_W \varphi$ belongs to $PB(R)$ and $u|\mathcal{A} = \varphi$, i.e. $\rho_P u = \varphi$.
Q.E.D.

11. The surjectiveness question of the canonical extension λ_P^W can now be settled in terms of $\mathcal{A}_P$:

THEOREM. *The canonical extension λ_P^W is surjective if and only if $\overline{W} - \overline{\partial W}$ is a neighborhood of Δ_P in R^* .*

Proof. Suppose λ_P^W is surjective. Then there exists a $u \in PB(W; \partial W)$ such that $\lambda_P^W u = e_P$. Since $u|_{\Delta} = \lambda_P^W u|_{\Delta} = e_P|_{\Delta}$, we see that $u|_{\Delta_P} = 1$. In view of $u|_{R^* - \overline{W}} = 0$, $\overline{W} - \overline{\partial W}$ is a neighborhood of Δ_P . Conversely if $\overline{W} - \overline{\partial W}$ is a neighborhood of Δ_P , then we can choose an open neighborhood W_0^* of Δ_P such that $\overline{W} - \overline{\partial W} \supset W_0^*$, $W_0 = W_0^* \cap R$ is normal in R , and (7) is valid for W_0 . The canonical extension $\lambda_{P^0, W}^W: PB(W_0; \partial W_0) \rightarrow PB(W; \partial W)$ relative to W can be defined by

$$\lambda_{P^0, W}^W u = \lim_{\Omega \rightarrow R} P_u^{\Omega \cap W}$$

on W and 0 on $R - W$. As in no. 4, $\lambda_{P^0, W}^W u|_{\Delta} = u|_{\Delta}$. Let $v \in PB(R)$ and $\varphi \in C(R^*)$ with $\varphi|(R^* - \overline{W}_0) \cup (\Delta - \Delta_P) = 0$ and $\varphi|_{\Delta_P} = 1$. Then $u = (T_P^{W_0})^{-1} \circ \pi_{W_0}(\varphi v) \in PB(W_0, \partial W_0)$ and $u|_{\Delta} = v|_{\Delta}$. Thus $\lambda_P^{W_0}: PB(W_0; \partial W_0) \rightarrow PB(R)$ is surjective. Since $\lambda_P^{W_0} = \lambda_P^W \circ \lambda_{P^0, W}^W$ and $\lambda_P^{W_0}$

$$\begin{array}{ccc} PB(W_0; \partial W_0) & \xrightarrow{\lambda_P^{W_0}} & PB(R) \\ \lambda_{P^0, W}^W \downarrow & & \nearrow \lambda_P^W \\ PB(W; \partial W) & & \end{array}$$

is surjective, λ_P^W must be surjective.

Q.E.D.

Canonical isomorphisms

12. Let P and Q be two densities on a hyperbolic Riemann surface R . A linear isomorphism $T_{Q,P}$ of $PB(R)$ onto $QB(R)$ will be referred to as a *canonical isomorphism* if $|T_{Q,P}u - u|$ is dominated by a potential $p = p_u$ on R for every $u \in PB(R)$. This is equivalent to that

$$(11) \quad T_{Q,P}u|_{\Delta} = u|_{\Delta}$$

for every $u \in PB(R)$ (cf. Constantinescu-Cornea [3]). By (1) we see that $T_{Q,P}$ is an isometry and thus $T_{Q,P}$ is a special Banach space isomorphism of $PB(R)$ onto $QB(R)$. In such a case we say that $PB(R)$ and $QB(R)$ are *canonically isomorphic*. We are ready to prove one of our main result in this paper:

THEOREM. *Banach spaces $PB(R)$ and $QB(R)$ are canonically isomorphic if and only if nondensity point sets Δ_P and Δ_Q are identical.*

Proof. Suppose $PB(R)$ and $QB(R)$ are canonically isomorphic. Let $z^* \in \Delta_P$. There exists a $\varphi \in C(\Delta; \Delta_P)$ with $\varphi(z^*) = 1$. By Theorem in no. 10, there exists a $u \in PB(R)$ with $\rho_P u = \varphi$. Then $T_{Q,P}u|_{\Delta} = u|_{\Delta}$ shows that $(T_{Q,P}u)(z^*) = u(z^*) = \varphi(z^*) = 1$ and (9) yields that $z^* \in \Delta_Q$, i.e. $\Delta_P \subset \Delta_Q$. Since the reverse inclusion can be shown similarly, we conclude that $\Delta_P = \Delta_Q$. Conversely assume that $\Delta_P = \Delta_Q$. Then the operator $T = \rho_Q^{-1}\rho_P: PB(R) \rightarrow QB(R)$ can be defined as a bijective mapping and $Tu|_{\Delta} = u|_{\Delta}$ for every $u \in PB(R)$. Therefore T fulfills the condition of canonical isomorphism and $T = T_{Q,P}$, i.e. $PB(R)$ and $QB(R)$ are canonically isomorphic.

Q.E.D.

$$\begin{array}{ccc} PB(R) & \xrightarrow{T} & QB(R) \\ \downarrow \rho_P & & \uparrow \rho_Q^{-1} \\ C(\Delta; \Delta_P) & \xrightarrow{\text{id.}} & C(\Delta; \Delta_Q) \end{array}$$

13. We simply say that $PB(R)$ and $QB(R)$ are *isomorphic* if there exists a Banach space isomorphism (i.e. bijective linear isometry) of $PB(R)$ onto $QB(R)$. Then we obtain another of our main result:

THEOREM. *Banach spaces $PB(R)$ and $QB(R)$ are isomorphic if and only if nondensity point sets Δ_P and Δ_Q are homeomorphic.*

Actually we can prove a bit more general assertion without adding any elaboration. Let $P(Q, \text{resp.})$ be a density on a hyperbolic Riemann surface $R(S, \text{resp.})$. We can speak of isomorphisms of $PB(R)$ onto $QB(S)$ as Banach spaces and also nondensity point sets Δ_P and Δ_Q relative to Wiener compactifications R^* and S^* of R and S , respectively. The above theorem is, then, a special case, i.e. $R = S$, of the following:

THEOREM. *Banach spaces $PB(R)$ and $QB(S)$ are isomorphic if and only if nondensity point sets Δ_P and Δ_Q are homeomorphic.*

Proof. Suppose there exists a homeomorphism α of Δ_P onto Δ_Q . Then $\varphi \rightarrow A\varphi = \varphi \circ \alpha^{-1}$ is a Banach space isomorphism of $C(\Delta_P)$ onto $C(\Delta_Q)$.

$$\begin{array}{ccc} PB(R) & \xrightarrow{T} & QB(S) \\ \downarrow \rho_P & & \uparrow \rho_Q^{-1} \\ C(\Delta; \Delta_P) & & C(\Delta; \Delta_Q) \\ \downarrow \tau_P & & \uparrow \tau_Q^{-1} \\ C(\Delta_P) & \xrightarrow{A} & C(\Delta_Q) \end{array}$$

Then $T = \rho_P^{-1} \circ \tau_Q^{-1} \circ A \circ \tau_P \circ \rho_P$ is a Banach space isomorphisms of $PB(R)$ onto $QB(S)$. Conversely assume that there exists a Banach space isomorphism T of $PB(R)$ onto $QB(R)$. Then $A = \tau_Q \circ \rho_Q \circ T \circ \rho_P^{-1} \circ \tau_P^{-1}$ is a Banach space isomorphism of $C(\Delta_P)$ onto $C(\Delta_Q)$. In such a case there exists a

$$\begin{array}{ccc} PB(R) & \xrightarrow{T} & QB(S) \\ \uparrow \rho_P^{-1} & & \downarrow \rho_Q \\ C(\Delta; \Delta_P) & & C(\Delta; \Delta_Q) \\ \uparrow \tau_P^{-1} & & \downarrow \tau_Q \\ C(\Delta_P) & \xrightarrow{A} & C(\Delta_Q) \end{array}$$

homeomorphism α of Δ_P and Δ_Q and an $a \in C(\Delta_Q)$ with $|a| = 1$ such that $A\varphi = a \cdot \varphi \circ \alpha^{-1}$ (cf. e.g. Dunford-Schwartz [4]); in particular Δ_P and Δ_Q are homeomorphic. Q.E.D.

14. Suppose there exists a neighborhood U^* of $z^* \in \Delta$ such that

$$(12) \quad \int_U G_U(z, \zeta) P(\zeta) d\xi d\eta < \infty \quad (U = U^* \cap R)$$

for every z in U . We can replace U^* by a $W^* \subset U^*$ which is also a neighborhood of z^* such that $W = W^* \cap R$ is normal and (12) with U replaced by W is valid. Let $h \in HB(W; \partial W)$ such that $0 \leq h \leq 1$ and $h(z^*) = 1$. Then as in no. 7 $T_P^W: PB(W; \partial W) \rightarrow HB(W; \partial W)$ is surjective and $T_P^W v|_{\tilde{A}} = v|_{\tilde{A}}$ for every $v \in PB(W; \partial W)$, where $\tilde{A}$ is the Wiener harmonic boundary of W . There exists a projection p of the Wiener compactification $\tilde{W}$ of W onto the closure $\bar{W}$ of W in R^* such that $p|_W$ is an identity, $p^{-1}(\Delta^W) \subset \tilde{A}$, and $p: W \cup p^{-1}(\Delta^W) \rightarrow W \cup \Delta^W$ is homeomorphic (cf. e.g. Constantinescu-Cornea [1] and [19]), where $\Delta^W = \Delta \cap (\bar{W} - \partial W)$. Therefore $T_P^W v|_{\Delta^W} = v|_{\Delta^W}$. Since $T_P^W v|_{\partial W} = v|_{\partial W} = 0$, $T_P^W v|_{\partial W} = v|_{\partial W} = 0$ and hence $T_P^W v|_{\partial W} = v|_{\partial W} = 0$ and hence $T_P^W v|_{\Delta} = v|_{\Delta}$. In particular, if $v = (T_P^W)^{-1}h$, then $v(z^*) = (T_P^W v)(z^*) = h(z^*) = 1$ and $z^* \in \Delta_P$. Thus we have shown the following

THEOREM. *A point z^* in Δ belongs to Δ_P if and only if (12) is valid.*

Applications

15. A subset $K \subset R$ is said to be *B-negligible* if there exists a po-

tential p such that $p \geq 1$ on K . In this case, since $\liminf_{z \in R, z \rightarrow z^*} p(z) = 0$ for $z^* \in \Delta$, we see that $\bar{K} \cap \Delta = \phi$. Conversely, if $\bar{K} \cap \Delta = \phi$, then there exists a $\varphi \in C(R^*)$ such that $0 \leq \varphi \leq 1$, $\varphi|_{\bar{K}} = 1$, and $\varphi|_{\Delta} = 0$. Then there exists a potential p with $\varphi \leq p$ on R (cf. [3]). Hence $p \geq 1$ on K and therefore K is B -negligible. Thus we have the following characterization: *A subset $K \subset R$ is B -negligible if and only if $R^* - \bar{K}$ is a neighborhood of Δ .* Compact sets in R are trivial examples of B -negligible sets. From Theorem in no. 12 the following criterion of Royden follows at once:

COROLLARY (ORDER COMPARISON THEOREM). *If there exists a constant $c \in [1, \infty)$ such that $c^{-1}P \leq Q \leq cP$ on R except possibly for a B -negligible set K , then $PB(R)$ and $QB(R)$ are canonically isomorphic.*

In general, $Q \leq cP$ on $R - K$ implies

$$\int_{R-\bar{K}} G(z, \zeta) Q(\zeta) d\xi d\eta \leq c \int_{R-\bar{K}} G(z, \zeta) P(\zeta) d\xi d\eta.$$

Since $\overline{R - \bar{K}} - (\partial(\overline{R - \bar{K}}))$ is a neighborhood of Δ , the above inequality implies that $\Delta_P \subset \Delta_Q$. In passing we insert here a consequence of $\Delta_P \subset \Delta_Q$, i.e. a consequence of $Q \leq cP$ on $R - K$ with B -negligible K . Since Δ_P is also compact and open in Δ_Q , the function $\tau\varphi$ given by $\tau\varphi = \varphi$ on Δ_P and $\tau\varphi = 0$ on $\Delta_Q - \Delta_P$ belongs to $C(\Delta_Q)$ for every $\varphi \in C(\Delta_P)$, i.e. $\tau: C(\Delta_P) \rightarrow C(\Delta_Q)$ is a linear isometry. Then $T = \rho_Q^{-1} \circ \tau_Q^{-1} \circ \tau \circ \tau_P \circ \rho_P$ is a linear isometry of $PB(R)$ into $QB(R)$ with $Tu|_{\Delta} = u|_{\Delta}$ for every $u \in PB(R)$. Returning to the above corollary, we also see that $\Delta_P \supset \Delta_Q$ from $c^{-1}P \leq Q$. Thus $\Delta_P = \Delta_Q$; $PB(R)$ and $QB(R)$ are canonically isomorphic. This criterion was obtained by Royden [18] for compact exceptional set K and by Loeb [8] in an abstract setting. The present formulation is stated in [16].

16. Let $G^P(z, \zeta)$ be the Green's function on R for the equation $\Delta u = Pu$ whose existence is always assured for any R (even for compact R) if $P \not\equiv 0$ (Myrberg [11, 12, 13]). In the present case, since we have assumed that R is hyperbolic, $G^P(z, \zeta)$ exists for every density P including $P \equiv 0$; as before we write $G(z, \zeta)$ for $G^P(z, \zeta)$ with $P \equiv 0$. Consider conditions

$$(13) \quad \int_{R-K} |P(\zeta) - Q(\zeta)| d\xi d\eta < \infty ;$$

$$(14) \quad \int_{R-K} G(z, \zeta) |P(\zeta) - Q(\zeta)| d\xi d\eta < \infty ;$$

$$(15) \quad \int_{R-K} (G^P(z, \zeta) + G^Q(z, \zeta)) |P(\zeta) - Q(\zeta)| d\xi d\eta < \infty ;$$

$$(16) \quad \int_{R-K} (G^P(z, \zeta)Q(\zeta) + G^Q(z, \zeta)P(\zeta)) d\xi d\eta < \infty .$$

Here K is a B -negligible set in R and (14)–(16) are assumed to be valid for one and hence by the Harnack inequality for every z in S . Since

$$G^P(z, \zeta) \leq G(z, \zeta) \quad \text{and} \quad \int_R G^P(z, \zeta) P(\zeta) d\xi d\eta < \infty$$

because of

$$e_P = 1 - \frac{1}{2\pi} \int_R G^P(\cdot, \zeta) P(\zeta) d\xi d\eta$$

(see the proof of the corollary below) it is clear that the following implications are valid: (13) $\rightarrow$ (14) $\rightarrow$ (15) $\Leftrightarrow$ (16).

COROLLARY (INTEGRAL COMPARISON THEOREM). *If one of conditions (13)–(16) is valid, then $PB(R)$ and $QB(R)$ are canonically isomorphic.*

This was obtained in [14] for $K = \phi$ and in the present form in [16] (cf. also Maeda [9]). The fact that (14) and hence (13) implies $\Delta_P = \Delta_Q$ is entirely clear. To show (15) or (16) implies $\Delta_P = \Delta_Q$, we may assume that $K = \phi$ in (15) since we can replace R by its normal open subset W whose $\tilde{A}$ contains A (cf. no. 14). The Green formula yields

$$Q_1^a = P_1^a + \frac{1}{2\pi} \int_a G_b^Q(\cdot, \zeta) P_1^a(\zeta) (P(\zeta) - Q(\zeta)) d\xi d\eta .$$

Since $e_P = \lim_{a \rightarrow R} P_1^a$ and $e_Q = \lim_{a \rightarrow R} Q_1^a$, (15) and the Lebesgue convergence theorem imply

$$e_Q = e_P + \frac{1}{2\pi} \int_R G^Q(\cdot, \zeta) e_P(\zeta) (P(\zeta) - Q(\zeta)) d\xi d\eta .$$

Similarly

$$Q_1^a = 1 - \frac{1}{2\pi} \int_a G_b^Q(\cdot, \zeta) Q(\zeta) d\xi d\eta$$

and the Lebesgue-Fatou theorem yield

$$e_Q = 1 - \frac{1}{2\pi} \int_R G^Q(\cdot, \zeta) Q(\zeta) d\xi d\eta .$$

Set

$$h = T_P e_P = e_P + \frac{1}{2\pi} \int_R G(\cdot, \zeta) e_P(\zeta) P(\zeta) d\xi d\eta .$$

Observe that

$$\frac{1}{2\pi} \int_R G^Q(\cdot, \zeta) e_P(\zeta) Q(\zeta) d\xi d\eta \leq \frac{1}{2\pi} \int_R G^Q(\cdot, \zeta) Q(\zeta) d\xi d\eta = 1 - e_Q$$

and

$$\frac{1}{2\pi} \int_R G^Q(\cdot, \zeta) e_P(\zeta) P(\zeta) d\xi d\eta \leq \frac{1}{2\pi} \int_R G(\cdot, \zeta) e_P(\zeta) P(\zeta) d\xi d\eta = h - e_P .$$

Therefore $|e_Q - e_P| \leq (h - e_P) + (1 - e_Q)$. Since $h|A = T_P e_P|A = e_P|A$,

$$|e_Q - e_P| \leq 1 - e_Q$$

on A . If $z^* \in A_Q$, then $e_Q(z^*) = 1$ and hence $e_P(z^*) = 1$, i.e. $z^* \in A_P$. Thus we conclude that $A_Q \subset A_P$, and similarly $A_P \subset A_Q$, i.e. $A_P = A_Q$.

17. Each of the conditions (14)–(16) takes the following form for $Q \equiv 0$:

$$(17) \quad \int_{R-K} G(z, \zeta) P(\zeta) d\xi d\eta < \infty ,$$

where again K is a B -negligible set. Clearly $A_P = A$ is equivalent to (17) for some B -negligible K . Thus we have

COROLLARY. *Banach spaces $PB(R)$ and $HB(R)$ are canonically isomorphic, i.e. the reduction operator $T_P: PB(R) \rightarrow HB(R)$ is surjective, if and only if (17) is valid for some B -negligible set K .*

The sufficiency of (17) for $K = \phi$ was obtained in [14]. The condition (17) for $K = \phi$ may not be necessary is remarked by Lahtinen [7]. The assertion in the present form is stated in [16].

18. Let h_P be the least harmonic majorant of the P -unit e_P (the P -elliptic measure). Clearly $h_P = T_P e_P$. Then $e_P|A = h_P|A$. Therefore $A_P = A_Q$ if and only if $h_P = h_Q$ and we have the following

COROLLARY. *Banach spaces $PB(R)$ and $QB(R)$ are canonically iso-*

morphic if and only if $h_P = h_Q$. In particular $PB(R)$ and $HB(R)$ are canonically isomorphic if and only if $h_P = 1$.

That $h_P = h_Q$ is sufficient and that $h_P = 1$ is necessary and sufficient are recent results of Lahtinen [7], in which he also studies the class $PB(R)$ for not necessarily $P \geq 0$ (cf. also Myrberg [13]).

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