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AN OPERATOR VALUED FUNCTION SPACE INTEGRAL APPLIED TO MULTIPLE INTEGRALS OF FUNCTIONS OF CLASS L_1

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§0 Introduction: In a recent paper [2], an operator valued function space integral was defined by the authors as follows.

DEFINITION: Let $C[a,b]$ be the set of real continuous functions defined on $[a,b]$ and $C_0[a,b]$ the subset of $C[a,b]$ whose elements vanish at a. Let F be a real or complex functional defined on $C[a,b]$, let ψ be a measurable function on $(-\infty,\infty)$, let ξ be a real variable and λ a positive parameter. Let $I_\lambda(F)\psi$ be the function whose value at ξ is

$$(0.1)\qquad (I_\lambda(F)\psi)(\xi) \equiv \int_{C_0[a,b]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(b)+\xi)dx$$

and let $I_\lambda(F)$ be the linear operator that takes ψ into $I_\lambda(F)\psi$.

Here the integral is understood to be Wiener's integral over the function space $C_0[a,b]$. In [2] and also in [3] and [4], the function ψ was taken to be of class $\mathscr{L}_2$ and the classes of functionals were such that $I_\lambda(F)$ was a bounded operator taking $\mathscr{L}_2$ into itself. In [5] ψ was taken to be in $\mathscr{L}_1$ and functionals that made $I_\lambda(F)$ a bounded operator from $\mathscr{L}_1$ into $\mathscr{L}_\infty$ were studied.

In the present paper we shall continue to study the operator valued function space integral defined in [5] for new classes of integrals. In part I we shall study integrals of functionals of the form

$$(0.2)\qquad F(x) = f\left[\int_a^b\int_a^b \theta(s,t,x(s),x(t))dsdt\right]$$

for f an entire function. In part II we shall study integrals of functionals of the form

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$$F(x) = f\left(\int_a^b \theta(s, x(s))ds, \int_a^b \varphi(t, x(t))dt\right) \tag{0.3}$$

for f an entire function of two complex variables. In part III we shall apply the results of part II to obtain the solution of a pair of simultaneous integral equations in terms of our integrals in function space and vice versa.

An operator $I_\lambda^{\mathrm{an}}(F)$ was defined in [2] as the analytic continuation of $I_\lambda(F)$ to $\operatorname{Re}\lambda > 0$ when such an extension exists, and it was proved to exist for certain classes of functionals. In particular, it was shown that $I_\lambda^{\mathrm{an}}(F)$ exists as an analytic vector valued function of λ on $\operatorname{Re}\lambda > 0$ for functionals of the form

$$F(x) = \exp\left\{\int_a^b \theta(s, x(s))ds\right\} \tag{0.4}$$

where $\theta(t, u)$ is continuous almost everywhere on $R \equiv [a, b] \times (-\infty, \infty)$ and $\operatorname{Re}\theta(t, u) \leqq B$ on R and $\theta(t, u)$ is bounded on every compact subset of R. This result was obtained by defining an operator $I_\lambda^{\mathrm{seq}}(F)$ as a weak limit of a certain sequence of operators and then showing that $I_\lambda^{\mathrm{seq}}(F) = I_\lambda(F)$ for real positive λ and $I_\lambda^{\mathrm{seq}}(F)$ is analytic on $\operatorname{Re}\lambda > 0$. Thus I_λ^{seq} was shown to satisfy the definition of $I_\lambda^{\mathrm{an}}(F)$ so $I_\lambda^{\mathrm{an}}(F) \equiv I_\lambda^{\mathrm{seq}}(F)$ on $\operatorname{Re}\lambda > 0$. Moreover it was shown that for $\operatorname{Re}\lambda > 0$ and $\psi \in \mathscr{L}_2$, $\|I_\lambda^{\mathrm{an}}(F)\psi\|_2 \leqq \exp[B(b - a)]\cdot\|\psi\|_2$. In [5] and in the present paper, $I_\lambda^{\mathrm{an}}(F)$ is defined as an analytic extension of $I_\lambda(F)$ to $\operatorname{Re}\lambda > 0$ or to $\{\operatorname{Re}\lambda > 0\} \cap \{|\lambda| < \lambda_0\}$, and for each fixed λ, $I_\lambda^{\mathrm{an}}(F)$ takes $\mathscr{L}_1$ into $\mathscr{L}_\infty$. An existence theorem is given for $I_\lambda^{\mathrm{an}}(F)$ for functionals of the form (0.2) where $\theta(s, t, u, v)$ is measurable in all its variables, $\theta(s, t, \cdot, \cdot) \in \mathscr{L}_1$ and $\|\theta(s, t, \cdot, \cdot)\|_1 \leqq B$. For functionals of the form (0.3), θ and φ are both required to be measurable and of class $\mathscr{L}_1$. In both cases $I_\lambda^{\mathrm{an}}(F)$ is shown to be a bounded operator from $\mathscr{L}_1$ to $\mathscr{L}_\infty$. The proof is not given in terms of the analogue of $I_\lambda^{\mathrm{seq}}(F)$ as defined in [2], [3], [4], but follows the technique of [5] where it was shown that $I_\lambda^{\mathrm{seq}}(F)$ might fail to exist.

By a limiting procedure, $I_\lambda^{\mathrm{an}}(F)$ is extended in [2] and in the present paper to the imaginary axis $\operatorname{Re}\lambda = 0$ or to $\{\operatorname{Re}\lambda = 0\} \cap \{|\lambda| < \lambda_0\}$, and the resulting operator is called $J_q^{\mathrm{an}}(F)$, where q corresponds to $-i\lambda$. An existence theorem for this operator was given in [5] for all $q \neq 0$ and in the present paper a similar extension to the imaginary axis is given.

In [2] the authors present motivation and relationship to the early work of Babbitt, Nelson, and others.

I Analytic functions of double integrals

§1 Preliminary Lemmas: In order to deal with functionals of the form (0.1) where $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is an entire function, it is convenient to start with the case $f(z)=z^n$. Let us consider the functional

$$F_n(x)=\left[\int_a^b\int_a^b\theta(s,t,x(s),x(t))dsdt\right]^n \tag{1.1}$$

where $\theta(s,t,u,v)$ is measurable in all its variables and $\theta(s,t,\cdot,\cdot)\in\mathscr{L}_1(\boldsymbol{R}^2)$ and $\|\theta(s,t,\cdot,\cdot)\|_1\leqq B$. We shall later show the existence of $I_\lambda(F_n)$ for all positive λ, but in order to motivate the rather complicated expressions appearing in Lemma 1, we first evaluate $I_\lambda(F_n)$ formally.

NOTATION: throughout the paper, $\lambda^{-1/2}=\rho$.

From the definition of $I_\lambda(F)$ given in the introduction we have

$$\begin{aligned}I_\lambda(F_n)\psi)(\xi)&=\int_{C_0[a,b]}\left\{\int_a^b\int_a^b\theta(s,t,\rho x(s)+\xi,\rho x(t)+\xi)dsdt\right\}^n\\&\quad\cdot\psi(\rho x(b)+\xi)dx=\int_{C_0[a,b]}\int_a^b\overset{(2n)}{\cdots}\int_a^b\prod_{j=1}^{n}\theta(s_j,t_j,\rho x(s_j)\\&\quad+\xi,\rho x(t_j)+\xi)ds_1\cdots ds_n dt_1\cdots dt_n\psi(\rho x(b)+\xi)dx\\&=\int_a^b\overset{(2n)}{\cdots}\int_a^b\int_{C_0[a,b]}\left\{\prod_{j=1}^{n}\theta(s_j,t_j,\rho x(s_j)+\xi,\rho x(t_j)+\xi)\right\}\\&\quad\cdot\psi(\rho x(b)+\xi)dxds_1\cdots ds_n dt_1\cdots dt_n\,,\end{aligned}$$

where $\theta(s,t,\cdot,\cdot)\in\mathscr{L}_1(\boldsymbol{R}^2)$ and $\|\theta(s,t,\cdot,\cdot)\|_1\leqq B$ and $\psi\in\mathscr{L}_1(-\infty,\infty)$. We now let $r_1,\cdots,r_{2n}$ be the set of numbers $s_1,\cdots,s_n,\ t_1,\cdots,t_n$ in some rearrangement, and let P be the set of all permutations of $\{1,\cdots,2n\}$. If we set $s_j=r_{m_j}$ and $t_j=r_{k_j}$ and evaluate the Wiener integrals we obtain

$$\begin{aligned}(I_\lambda(F_n)\psi)(\xi)&=\sum_{(r_{m_1},\cdots,r_{m_n},r_{k_1},\cdots,r_{k_n})\in P}[(2\pi)^{2n+1}(r_1-a)(r_2-r_1)\cdots(r_{2n}-r_{2n-1})\\&\quad\cdot(b-r_{2n})]^{-1/2}\int_a^b\int_a^{r_{2n}}\cdots\int_a^{r_2}\int_{-\infty}^{\infty}\overset{(2n+1)}{\cdots}\int_{-\infty}^{\infty}\prod_{j=1}^{n}\theta(r_{m_j},r_{k_j},\rho u_{m_j}+\xi,\rho u_{k_j}+\xi)\\&\quad\cdot\psi(\rho u_{2n+1}+\xi)\exp\left\{-1/2\sum_{j=1}^{2n+1}\frac{(u_j-u_{j-1})^2}{r_j-r_{j-1}}\right\}du_1\cdots du_{2n}du_{2n+1}dr_1\cdots dr_{2n}\,,\end{aligned}$$

where $r_0=a,\ r_{2n+1}=b$, and $u_0=0$. If we make the transformation

$v_j = \rho u_j + \xi = \lambda^{-1/2} u_j + \xi$ we obtain

$$(1.3)\quad (I_\lambda(F_n)\psi)(\xi) = \sum_P \int_a^b \int_a^{r_{2n}} \cdots \int_a^{r_2} \left(\frac{\lambda}{2\pi}\right)^{(2n+1)/2} [(r_1 - a)\cdots(b - r_{2n})]^{-1/2}$$
$$\cdot \int_{-\infty}^{\infty} \overset{(2n+1)}{\cdots} \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} \theta(r_{m_j}, r_{k_j}, v_{m_j}, v_{k_j})\right] \psi(v_{2n+1})$$
$$\cdot \exp\left\{-1/2 \sum_{j=1}^{2n+1} \frac{\lambda(v_j - v_{j-1})^2}{(r_j - r_{j-1})}\right\} dv_1 \cdots dv_{2n+1} dr_1 \cdots dr_{2n} .$$

In the following lemma we shall establish the existence, continuity and analyticity of the sum appearing in equation (1.3) and thus justify the preceding manipulations.

LEMMA 1: *Let $\theta(s,t,u,v)$ be measurable in the region $R = \{(s,t,u,v) \mid a \le s \le b,\ a \le t \le b,\ -\infty < u < \infty,\ -\infty < v < \infty\}$, let $\theta(s,t,\cdot,\cdot)$ be of class $\mathscr{L}_1(\boldsymbol{R}^2)$ and $\|\theta(s,t,\cdot,\cdot)\|_1 \leqq B$, i.e. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\theta(s,t,u,v)| du dv \le B$, for almost every $s, t \in [a,b]$, and let $\psi \in \mathscr{L}_1(-\infty,\infty)$, $\|\psi\|_1 \le M$. Let* $\operatorname{Re}\lambda \ge 0$ *and* $K_0 = K_0(\xi,\lambda) = \dfrac{1}{\sqrt{b-a}} \displaystyle\int_{-\infty}^{\infty} \psi(v_1) \exp\left\{\frac{-\lambda v_1^2}{2(b-a)}\right\} dv_1$,

$$(1.4)\quad K_n = K_n(\xi,\lambda) = \sum_{(m_1,\cdots,m_n,k_1,\cdots,k_n)\in P} \int_a^b \int_a^{r_{2n}} \cdots \int_a^{r_2} [(r_1 - a)(r_2 - r_1)\cdots$$
$$\cdot (b - r_{2n})]^{-1/2} \int_{-\infty}^{\infty} \overset{(2n+1)}{\cdots} \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n} \theta(r_{m_j}, r_{k_j}, v_{m_j}, v_{k_j})\right] \psi(v_{2n+1})$$
$$\cdot \exp\left\{-1/2 \sum_{j=1}^{2n+1} \frac{\lambda(v_j - v_{j-1})^2}{(r_j - r_{j-1})}\right\} dv_1 \cdots dv_{2n+1} dr_1 \cdots dr_{2n}$$

where P represents the set of all permutations of $1, \cdots, 2n$, and $v_0 \equiv \xi$ and $a = r_0 < r_1 < \cdots < r_{2n+1} = b$; for $n = 1, 2, \cdots$. Then for $n = 0, 1, 2, \cdots$, $K_n(\xi,\lambda)$ exists and is continuous for all real ξ and $\operatorname{Re}\lambda \ge 0$; *and for each real ξ, $K(\xi,\lambda)$ is analytic in λ for* $\operatorname{Re}\lambda > 0$. *For* $\operatorname{Re}\lambda \ge 0$ *and* $n = 0, 1, 2, \cdots$,

$$(1.5)\qquad |K_n| \leqq M_n \equiv 2^{2n}\pi^n (b-a)^{(2n-1)/2}(n!) M B^n .$$

Proof of Lemma 1: The integral for the multiple integral defining K_n is measurable so that we need only establish the integrability of the absolute value of the integrand to prove the existence of K_n. This will be established in the inequalities below. Since $\operatorname{Re}\lambda \ge 0$, the function $|K_n(\xi,\lambda)|$ satisfies

$$|K_n(\xi, \lambda)| \leqq \sum_P \int_a^b \int_a^{r_{2n}} \cdots \int_a^{r_2} [(r_1 - a)(r_2 - r_1) \cdots (b - r_{2n})]^{-1/2}$$

$$\cdot \int_{-\infty}^{\infty} \overset{(2n+1)}{\cdots} \int_{-\infty}^{\infty} \left| \prod_{j=1}^{n} \theta(r_{m_j}, r_{k_j}, v_{m_j}, v_{k_j}) \psi(v_{2n+1}) \right| dv_1 \cdots dv_{2n+1} dr_1 \cdots dr_{2n}$$

$$\leqq \sum_P B^n M \int_a^b \int_a^{r_{2n}} \cdots \int_a^{r_2} [(r_1 - a) \cdots (b - r_{2n})]^{-1/2} dr_1 \cdots dr_{2n}$$

$$= (2n)!\, B^n M \int_a^b \int_a^{r_{2n}} \cdots \int_a^{r_2} [(r_1 - a) \cdots (b - r_{2n})]^{-1/2} dr_1 \cdots dr_{2n} .$$

If we set $u_j = (r_j - r_{j-1})/(b - a)$, so $r_j = a + (u_1 + \cdots + u_j)(b - a)$, $|K_n(\xi, \lambda)| \leqq (2n)!\, B^n M (b - a)^{(2n-1)/2}$.

$$\cdot \int_{\Delta} \overset{(2n)}{\cdots} \int (u_1 \cdots u_{2n})^{-1/2} (1 - u_1 - u_2 - \cdots - u_{2n})^{-1/2} du_1 \cdots du_{2n} ,$$

where $\Delta = \{(u_1, \cdots, u_{2n}) | u_1 + u_2 + \cdots + u_{2n} \leqq 1\}$.
Evaluating this integral as Dirichlet's Integral (see [9; p. 258]) we obtain

$$|K_n(\xi, \lambda)| \leqq (2n)!\, B^n M (b - a)^{(2n-1)/2} \frac{\Gamma(1/2)^{2n}}{\Gamma(n)} \int_0^1 (1 - \tau)^{-1/2} \tau^{n-1} d\tau$$

$$= (2n)!\, B^n M (b - a)^{(2n-1)/2} \frac{\Gamma(1/2)^{2n}}{\Gamma(n)} \frac{\Gamma(1/2)\Gamma(n)}{\Gamma((2n + 1)/2)}$$

$$= (2n)!\, B^n M (b - a)^{(2n-1)/2} \frac{(\pi)^{(2n+1)/2}}{\Gamma((2n + 1)/2)} .$$

Legendre's duplication formula for the Γ function, $2^{2z-1}\Gamma(z)\Gamma(z + 1/2) = \sqrt{\pi}\,\Gamma(2z)$ (see [9; p. 240]) becomes when $z = (2n + 1)/2$, $2^{2n}\Gamma((2n + 1)/2)\Gamma(n + 1) = \sqrt{\pi}\,\Gamma(2n + 1) = \sqrt{\pi}\,(2n)!$ and so

$$\frac{(2n)!}{\Gamma((2n + 1)/2)} = \frac{2^{2n} n!}{\sqrt{\pi}} .$$

Thus $|K_n(\xi, \lambda)| \leqq (b - a)^{(2n-1)/2} B^n M \pi^n 2^{2n} n! \equiv M_n$.
Because $K_n(\xi, \lambda)$ is expressed as a multiple integral with λ appearing in the integral only in an exponential we may conclude that $K_n(\xi, \lambda)$ is continuous for all real ξ and $\operatorname{Re} \lambda \geq 0$ and that for each real ξ, $K_n(\xi, \lambda)$ is analytic in λ for $\operatorname{Re} \lambda > 0$.

Corollary to Lemma 1: *Let* $F_n(x) = \left[\int_a^b \int_a^b \theta(s, t, x(s), x(t)) ds dt \right]^n$; *then* $(I_\lambda(F_n)\psi)(\xi)$ *exists for real* $\lambda > 0$ *and* $(I_\lambda(F_n)\psi)(\xi) = K_n(\xi, \lambda)(\lambda/2\pi)^n$.

This *corollary* suggests the following definitions (see also [8] for a discussion of analytic vector valued functions).

DEFINITION: Let $A(\cdot\,;\lambda)$ be of class $\mathscr{L}_\infty(-\infty,\infty)$ for each λ in a domain Ω of the complex λ-plane. We shall say that $A(\cdot\,;\lambda)$ is a weakly analytic vector valued function of λ throughout Ω if $\int_{-\infty}^{\infty} A(\xi,\lambda)\varphi(\xi)d\xi$ is analytic for $\lambda\in\Omega$ for each $\varphi\in\mathscr{L}_1(-\infty,\infty)$.

DEFINITION: Let Ω be a simply connected domain of the complex λ-plane whose intersection with the positive real axis is a single non-empty open interval (α,β). Let F be a functional such that $I_\lambda(F)$ exists for $\lambda\in(\alpha,\beta)$. For each $\psi\in\mathscr{L}_1(-\infty,\infty)$ let a function $A(\lambda;\psi)$ exist as a weakly analytic vector valued function of λ for $\lambda\in\Omega$, $A(\lambda;\psi)\in\mathscr{L}_\infty(-\infty,\infty)$ and let $A(\lambda;\psi)=I_\lambda(F)\psi$ for $\lambda\in(\alpha,\beta)$ and $\psi\in\mathscr{L}_1(-\infty,\infty)$. Thus we define

$$I_\lambda^{\mathrm{an}}(F)\psi = A(\lambda;\psi)$$

for $\lambda\in\Omega$ and $\psi\in\mathscr{L}_1(-\infty,\infty)$. Let I_λ^{an} be the operator that takes ψ into $I_\lambda^{\mathrm{an}}(F)\psi$ for $\psi\in\mathscr{L}_1(-\infty,\infty)$.

We note that if it exists, $I_\lambda^{\mathrm{an}}(F)$ is uniquely defined and is a linear operator that takes $\mathscr{L}_1(-\infty,\infty)$ into $\mathscr{L}_\infty(-\infty,\infty)$.

LEMMA 2: *Let θ,ψ satisfy the conditions of Lemma 1 and let $K_n(\xi,\lambda)$ be defined as in Lemma 1. Let*

$$f(z)\equiv\sum_{n=0}^{\infty}a_nz^n \tag{1.6}$$

be an entire function of growth $(1,\tau)$ where $\tau<\infty$. Then*

Case I: growth $(1,0)$. In this case

$$\sum_{n=0}^{\infty}a_nK_n(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^n \tag{1.7}$$

converges for all real ξ and all λ, $\operatorname{Re}\lambda\geqq 0$. Moreover for each $\lambda_0>0$, (1.7) converges uniformly for all real ξ and $|\lambda|\leqq\lambda_0$, $\operatorname{Re}\lambda\geq 0$. Thus for each real ξ, (1.7) represents an analytic function in $\operatorname{Re}\lambda>0$ which is continuous for $\operatorname{Re}\lambda\geq 0$.

Case II: order one, type τ, $0<\tau<\infty$. In this case (1.7) converges for all real ξ and all λ, $\operatorname{Re}\lambda\geq 0$, $|\lambda|<\lambda_0=(2(b-a)B\tau)^{-1}$ and converges uniformly for all real ξ and all λ, $\operatorname{Re}\lambda\geq 0$, $|\lambda|\leq\lambda_0'<\lambda_0$. Thus for each

* An entire function is said to be of growth (μ',τ') iff it is of order not exceeding μ' and if its order is μ', its type does not exceed τ', (see [1; p. 8]).

ξ, (1.7) *represents an analytic function in* $|\lambda| < \lambda_0$, $\operatorname{Re}\lambda > 0$ *which is continuous in* $|\lambda| < \lambda_0$, $\operatorname{Re}\lambda \geq 0$.

Proof of Lemma 2: We begin with the proof of Case I. Let μ be the order of $f(z)$; then

$$\mu = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)} \tag{1.8}$$

(where the quotient on the right is taken as 0 if $a_n = 0$). If $\mu < 1$, we have for n sufficiently large that

$$\frac{n \log n}{\log(|1/a_n|)} < \mu_0 < 1\,,$$

and so $|a_n|^{\mu_0} < n^{-n}$.
Thus for n sufficiently large, by Stirling's formula we see that

$$\begin{aligned} n\,!\,|a_n| &\leqq n\,!\, n^{-n/\mu_0} \\ &\leqq e^{-n}\sqrt{2\pi n}\, e^{1/12n} n^{-n((1-\mu_0)/\mu_0)} \end{aligned}$$

Hence by (1.5), for n sufficiently large

$$\left| a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n \right| \leqq (b-a)^{(2n-1)/2} B^n M 2^n |\lambda|^n e^{-n}\sqrt{2\pi n}\, e^{1/12n} n^{-n((1-\mu_0)/\mu_0)}\,.$$

Because of the last factor, we see that (1.7) converges for all real ξ, and all λ, $\operatorname{Re}\lambda \geqq 0$. Clearly (1.7) converges uniformly for all real ξ and $|\lambda| \leq \lambda_0$, $\operatorname{Re}\lambda \geq 0$ and so for each real ξ, (1.7) represents an analytic function in $\operatorname{Re}\lambda > 0$ which is continuous for $\operatorname{Re}\lambda \geq 0$.

In order to establish Case I when $\mu = 1$ and $\tau = 0$, we observe that $\limsup_{n\to\infty} n|a_n|^{1/n} = 0$. (see [1, p. 11]), and hence $\sum_{n=0}^{\infty} n^n a_n z^n$ is an entire function. By Stirling's formula it follows that $\sum_{n=0}^{\infty} n\,!\, a_n z^n$ is an entire function.

Now

$$\begin{aligned} \left| a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n \right| &\leqq M(b-a)^{(2n-1)/2} B^n \pi^n 2^{2n} n\,!\,|a_n| \left|\frac{\lambda}{2\pi}\right|^n \\ &= M(b-a)^{-1/2} n\,!\,|a_n| Z^n \end{aligned} \tag{1.9}$$

where $Z = 2(b-a)B|\lambda|$. Therefore (1.7) converges for all real ξ and all λ, $\operatorname{Re}\lambda \geq 0$ and the convergence is uniform on compact subsets and Case I is established.

We proceed to the proof of Case II, where $\mu = 1$, and $0 < \tau < \infty$, and observe that

$$\limsup_{n\to\infty} n|a_n|^{1/n} = e\tau .$$

By Stirling's formula, with $0 < \delta_n < 1$,

$$\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{(2\pi n)^{1/2n}\exp(\delta_n/12n^2)}{e} = \frac{1}{e} \tag{1.10}$$

Thus

$$\limsup_{n\to\infty}|n!\,a_n|^{1/n} = \frac{1}{e}e\tau = \tau ,$$

and thus $\sum_{n=0}^{\infty} n!\,a_n z^n$ is an analytic function in the disk $|z| < \tau^{-1}$. By (1.9), (1.7) converges for all real ξ and all λ, $\operatorname{Re}\lambda \geq 0$ such that $2(b-a)B|\lambda| < \tau^{-1}$, i.e. for $|\lambda| < \lambda_0 = (2(b-a)B\tau)^{-1}$. Moreover the convergence of (1.7) is uniform for all real ξ and all λ, $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leqq \lambda_0' < \lambda_0$. Hence Lemma 2 is proved.

§2 Existence Theorems for $I_\lambda(F)$ and $I_\lambda^{\text{an}}(F)$: We begin by establishing the existence of $I_\lambda^{\text{an}}(F)$ for real $\lambda > 0$.

LEMMA 3: *Let θ, ψ, and $K_n(\xi,\lambda)$ be as in Lemma 1. Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.0}$$

be an entire function of growth $(1,0)$ *and let*

$$F(x) = f\left[\int_a^b\int_a^b \theta(s,t,x(s),x(t))\,ds\,dt\right] \quad \text{and} \quad \lambda > 0 . \tag{2.1}$$

Then $(I_\lambda(F)\psi)(\xi)$ exists for all real ξ, and

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) &= \int_{C_0[a,b]} F(\lambda^{-1/2}x+\xi)\psi(\lambda^{-1/2}x(b)+\xi)dx \\ &= \left(\frac{\lambda}{2\pi}\right)^{1/2}\sum_{n=0}^{\infty} a_n K_n(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^n .\end{aligned} \tag{2.2}$$

Proof of Lemma 3: Case I: Let θ, ψ, $a_0, a_1, \cdots$ be real and non-negative. Then the calculations leading to (1.3) are justified and we have

$$\int_{C_0[a,b]} \left\{ \int_a^b \int_a^b \theta(s, t, \rho x(s) + \xi, \ \rho x(t) + \xi) ds dt \right\}^n \psi(\rho x(b) + \xi) dx \tag{2.3}$$
$$= \left(\frac{\lambda}{2\pi} \right)^{(2n+1)/2} K_n(\xi, \lambda) ,$$

where as before $\rho = \lambda^{-1/2}$.
Hence,

$$I = \int_{C_0[a,b]} F(\rho x + \xi) \psi(\rho x(b) + \xi) dx \tag{2.4}$$
$$= \int_{C_0[a,b]} \sum_{n=0}^{\infty} a_n \left\{ \int_a^b \int_a^b \theta(s, t, \rho x(s) + \xi, \ \rho x(t) + \xi) ds dt \right\}^n \psi(\rho x(b) + \xi) dx$$
$$= \sum_{n=0}^{\infty} a_n \left(\frac{\lambda}{2\pi} \right)^{(2n+1)/2} K_n(\xi, \lambda) \leqq +\infty .$$

By lemma 2, Case I

$$\sum_{n=0}^{\infty} a_n K_n(\xi, \lambda) \left(\frac{\lambda}{2\pi} \right)^n \tag{2.5}$$

converges for all real ξ and all λ, $\operatorname{Re} \lambda \geq 0$. Thus we may conclude that the last member of (2.4) is finite for all real ξ and $\lambda \geq 0$.

Case II (The general case): Let $\theta, \psi, a_0, a_1, \cdots$ be complex valued. Note that $|\theta|, |\psi|, |a_0|, |a_1|, \cdots$ satisfy the special hypotheses of Case I. Each of the integrals and sums that would occur in the proof of Case I applied to these positive quantities would be finite and hence the proof given in Case I may be applied to complex $\theta, \psi, a_0, a_1, \cdots$ because all the steps would be justified by the domination which would be provided by Case I applied to $|\theta|, |\psi|, |a_0|, |a_1|, \cdots$.

THEOREM 1: *Let $\theta(s, t, u, v)$ be measurable in the region $R \equiv [a, b]^2 \times \boldsymbol{R}^2$, Let $\theta(s, t, \cdot, \cdot)$ be of class $\mathscr{L}_1(\boldsymbol{R}^2)$ and $\|\theta(s, t, \cdot, \cdot)\|_1 \leqq B$, and let $\psi \in \mathscr{L}_1(-\infty, \infty)$, $\|\psi\|_1 \leqq M$. Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.6}$$

be an entire function of growth $(1, 0)$, *and let*

$$F(x) = f\left[\int_a^b \int_a^b \theta(s, t, x(s), x(t)) ds dt \right] . \tag{2.7}$$

Then $I_\lambda^{\mathrm{an}}(F)$ exists and is a bounded operator taking $\mathscr{L}_1$ into $\mathscr{L}_\infty$ and $I_\lambda^{\mathrm{an}}(F)\psi$ is weakly analytic as a vector valued function of λ for $\operatorname{Re} \lambda > 0$.

Moreover for each λ, $\operatorname{Re}\lambda > 0$, the function $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)$ has the representation

$$(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = \sqrt{\frac{\lambda}{2\pi}} \sum_{n=0}^{\infty} a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n \tag{2.8}$$

for all real ξ; and for each fixed real ξ, $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)$ is analytic in λ for $\operatorname{Re}\lambda > 0$.

Proof: By Lemma 2, we see that (2.5) converges for all real ξ and $\operatorname{Re}\lambda \geq 0$ and converges uniformly for all real ξ, $\operatorname{Re}\lambda \geq 0$, $|\lambda| < \lambda_0$ for each finite λ_0. Set

$$k(\xi, \lambda) = \sqrt{\frac{\lambda}{2\pi}} \sum_{n=0}^{\infty} a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n . \tag{2.9}$$

Thus by Lemma 1, $k(\xi, \lambda)$ is analytic for $\operatorname{Re}\lambda > 0$ and

$$|k(\xi, \lambda)| \leqq \sqrt{\frac{|\lambda|}{2\pi}} \sum_{n=0}^{\infty} |a_n||M_n|\left|\frac{\lambda}{2\pi}\right|^n \tag{2.10}$$

where M_n is defined in (1.5). Let $\varphi \in \mathscr{L}_1(-\infty, \infty)$. We consider

$$\int_{-\infty}^{\infty} k(\xi, \lambda)\varphi(\xi)d\xi \tag{2.11}$$

and a sequence $\lambda_1, \lambda_2, \lambda_3, \cdots \to \lambda$. By Lebesgue's convergence Theorem, we see that $\int_{-\infty}^{\infty} k(\xi, \lambda)\varphi(\xi)d\xi$ is continuous in λ for $\operatorname{Re}\lambda > 0$. By the Fubini Theorem and Morera's Theorem we see that (2.11) is analytic in λ for $\operatorname{Re}\lambda > 0$. By Lemma 3, for λ real and positive,

$$k(\xi, \lambda) = (I_\lambda(F)\psi)(\xi) . \tag{2.12}$$

Hence by the definition, $I_\lambda^{\mathrm{an}}(F)\psi$ exists for $\psi \in \mathscr{L}_1$ and $\operatorname{Re}\lambda > 0$ and $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = k(\xi, \lambda)$ and Theorem 1 is proved.

LEMMA 4: *Let θ, ψ, and $K_n(\xi, \lambda)$ be as in Lemma 1. Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.13}$$

be an entire function of order one, type τ, $0 < \tau < \infty$, and let

$$F(x) = f\left[\int_a^b \int_a^b \theta(s, t, x(s), x(t))dsdt\right] . \tag{2.14}$$

Then for all real ξ and real λ satisfying $0 \leq \lambda_0 = [2(b-a)B\tau]^{-1}$, $(I_\lambda(F)\psi)(\xi)$ *exists and*

$$(2.15) \qquad (I_\lambda(F)\psi)(\xi) = \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx$$
$$= \left(\frac{\lambda}{2\pi}\right)^{1/2} \sum_{n=0}^{\infty} a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n$$

Proof of Lemma 4: As in the proof of Lemma 3, if $\theta, \psi, a_0, a_1, \cdots$ are real and non-negative,

$$(2.16) \qquad (I_\lambda(F)\psi)(\xi) = \left(\frac{\lambda}{2\pi}\right)^{1/2} \sum_{n=0}^{\infty} a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n \leqq +\infty\,.$$

Since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of order one, type τ, $0 < \tau < \infty$, we know by Lemma 2, Case II, that the series in (2.15) converges for all real ξ, and all λ, $\operatorname{Re}\lambda \geq 0$ such that $|\lambda| < \lambda_0 = [2(b-a)B\tau]^{-1}$ and Lemma 4 is established in case θ, ψ and the a_n's are all real and non-negative.

The proof for complex θ, ψ and a_n's follows exactly as the proof in Case II, Lemma 3.

THEOREM 2: *Let $\theta(s,t,u,v)$ be measurable in the region $R \equiv [a,b]^2 \times \boldsymbol{R}^2$, let $\theta(s,t,\cdot,\cdot)$ be of class $\mathscr{L}_1(\boldsymbol{R}^2)$ and $\|\theta(s,t,\cdot,\cdot)\|_1 \leqq B$, and let $\psi \in \mathscr{L}_1(-\infty,\infty)$, $\|\psi\|_1 \leqq M$. Let*

$$(2.17) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an entire function of order one, type τ, $0 < \tau < \infty$, and let

$$(2.18) \qquad F(x) = f\left[\int_a^b \int_a^b \theta(s, t, x(s), x(t))dsdt\right].$$

Then $I_\lambda^{\mathrm{an}}(F)$ exists and is a bounded operator taking $\mathscr{L}_1$ into $\mathscr{L}_\infty$ and $I_\lambda^{\mathrm{an}}(F)\psi$ is weakly analytic as a vector valued function of λ for $\lambda \in \Omega = \{\lambda | \operatorname{Re}\lambda > 0,\ |\lambda| < \lambda_0\}$, where $\lambda_0 = [2(b-a)B\tau]^{-1}$. Moreover for each $\lambda \in \Omega$, the function $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)$ has the representation

$$(2.19) \qquad (I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = \left(\frac{\lambda}{2\pi}\right)^{1/2} \sum_{n=0}^{\infty} a_n K_n(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^n$$

for all real ξ, and for each fixed real ξ, $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)$ is analytic in λ for $\lambda \in \Omega$.

Proof of Theorem 2: From Lemma 2, we observe that the series in (2.15) converges for all real ξ and $\lambda \in \Omega$. The argument in the proof of Theorem 1 may now be repeated to establish the weak analyticity and pointwise analyticity of the sum (2.19).

The exponential function provides an interesting special case which we mention in the following corollary.

COROLLARY TO THEOREM 2: *Let θ, ψ satisfy the hypothesis of Theorem* 2, *and let*

$$F(x) = \exp\left[\int_a^b\int_a^b \theta(s, t, x(s), x(t))dsdt\right] .$$

Then $I_\lambda^{\mathrm{an}}(F)\psi$ exists and is weakly analytic as a vector valued function for $\operatorname{Re}\lambda > 0$, $|\lambda| < [2(b-a)B]^{-1}$.

§3 **Existence Theorem for $J_q(F)$:** We begin with the definition of $J_q(F)$.

DEFINITION: Let q be a real number and F be a functional such that $I_\lambda^{\mathrm{an}}(F)$ exists for every $\psi \in \mathscr{L}_1(-\infty, \infty)$. If $Q(\cdot)$ is of class $\mathscr{L}_\infty(-\infty, \infty)$ and if ψ is a given element of $\mathscr{L}_1$ such that

$$\lim_{\substack{\lambda\to -iq\\ \operatorname{Re}\lambda>0}}\int_{-\infty}^{\infty}[(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) - Q(\xi)]\varphi(\xi)d\xi = 0$$

for every $\varphi \in \mathscr{L}_1(-\infty, \infty)$, then we define

$$J_q(F)\psi = Q .$$

If $J_q(F)\psi$ exists for every $\psi \in \mathscr{L}_1$, we denote by $J_q(F)$ the operator that takes ψ into $J_q(F)\psi$ and we note that $J_q(F)$ is a linear operator and is uniquely defined by the equation above.

THEOREM 3: *Let $\theta(s, t, u, v)$ be measurable in the region $R \equiv [a, b]^2 \times \boldsymbol{R}^2$, let $\theta(s, t, \cdot, \cdot)$ be of class $\mathscr{L}_1(\boldsymbol{R}^2)$ and $\|\theta(s, t, \cdot, \cdot)\|_1 \leqq B$, and let $\psi \in \mathscr{L}_1(\boldsymbol{R})$, $\|\psi\|_1 \leqq M$. Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3.0}$$

be an entire function of growth $(1, \tau)$ where $\tau < \infty$, and let

$$F(x) = f\left[\int_a^b\int_a^b \theta(s, t, x(s), x(t))dsdt\right] \tag{3.1}$$

Then:

Case I: *growth* (1,0). *In this case* $J_q(F)\psi$ *exists and is of class* $\mathscr{L}_\infty$ *for each* $\psi \in \mathscr{L}_1$ *and each real* q, $q \neq 0$. *Moreover* $J_q(F)\psi$ *has the representation*

$$(3.2) \qquad (J_q(F)\psi)(\xi) = \left(\frac{q}{2\pi i}\right)^{+1/2} \sum_{n=0}^{\infty} a_n K_n(\xi, -iq)\left(\frac{q}{2\pi i}\right)^n$$

for each real ξ *and each real* q, $q \neq 0$.

Case II: *order one, type* τ, $0 < \tau < \infty$. *In this case* $J_q(F)\psi$ *exists and is of class* $\mathscr{L}_\infty$ *for each* $\psi \in \mathscr{L}_1$ *and each real* q, $0 < |q| < \lambda_0 = [2(b-a)B\tau]^{-1}$, *and* $(J_q(F)\psi)(\xi)$ *has the representation* (3.2).

Proof of Theorem 3: For each real q (appropriate to either Case I or Case II), by Lemma 2, the series in (1.7) converges uniformly to a continuous function for $\lambda \in N_\lambda \equiv \{\lambda \,|\, |\lambda + iq| < \delta,\ \mathrm{Re}\,\lambda \geq 0\}$ for some sufficiently small δ and each real ξ. Let $\varphi \in \mathscr{L}_1(-\infty, \infty)$ and consider

$$\int_{-\infty}^{\infty}\left[\left(\frac{\lambda}{2\pi}\right)^{+1/2}\sum_{n=0}^{\infty} K_n(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^n - \left(\frac{-iq}{2\pi}\right)^{+1/2}\sum_{n=0}^{\infty} K_n(\xi,-iq)\left(\frac{-iq}{2\pi}\right)^n\right]\varphi(\xi)d\xi\,.$$

Since the sums in the integrand converge to a continuous function, the integrand approaches zero as $\lambda \to -iq$, $\mathrm{Re}\,\lambda \geq 0$ for each ξ for which $\varphi(\xi)$ is finite. Since the first factor is uniformly bounded for real ξ and $\lambda \in N_q$, by Lebesgue's convergence Theorem the integral has the limit zero as $\lambda \to -iq$, $\mathrm{Re}\,\lambda > 0$. Thus by (3.2), $J_q(F)$ exists for each $\psi \in \mathscr{L}_1$ and has the representation (3.2), and Theorem 3 is proved.

We remark that if $f(z) = \exp z$, from Case II of Theorem 3, $J_q(F)\psi$ exists and is of class $\mathscr{L}_\infty$ for each $\psi \in \mathscr{L}_1$ and each real q, $0 < |q| < [2(b-a)B]^{-1}$.

Remark: Under the hypotheses of Theorem 3, $\lim_{\lambda \to -iq}(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = (J_q(F)\psi)(\xi)$ uniformly in ξ and q for all real ξ and $|q| \leq q_0 < \lambda_0$, where we take $\lambda_0 = \infty$ in Case I.

II Analytic functions of two single integrals

§4 Preliminary Lemmas: We shall now consider functionals of the form

$$(4.0) \qquad F(x) = f\left(\int_a^b \theta(s, x(s))ds,\ \int_a^b \varphi(t, x(t))dt\right)$$

where f is an entire function of two complex variables, $\theta(s,u)$ and $\varphi(s,u)$ are measurable in the strip $[a,b]\times(-\infty,\infty)$, $\theta(s,\cdot)\in\mathscr{L}_1(-\infty,\infty)$, $\varphi(s,\cdot)\in\mathscr{L}_1(-\infty,\infty)$ for $s\in[a,b]$, $\|\theta(s,\cdot)\|_1\le B_1$, $\|\varphi(s,\cdot)\|_1\le B_2$, $\psi\in\mathscr{L}_1(-\infty,\infty)$ and

$$f(z,w)=\sum_{m,n=0}^{\infty}a_{m,n}z^m w^n\,. \tag{4.1}$$

As in the previous sections we begin by considering the special case given by

$$f_{m,n}(z,w)=z^m w^n \tag{4.2}$$

and obtain formally that

$$\begin{aligned} F_{m,n}(x) &\equiv f_{m,n}\left(\int_a^b\theta(s,x(s))ds,\ \int_a^b\varphi(t,x(t))dt\right)\\ &=\left[\int_a^b\theta(s,x(s))ds\right]^m\left[\int_a^b\varphi(t,x(t))dt\right]^n\\ &=\int_a^b\overset{(m)}{\cdots}\int_a^b\theta(s_1,x(s_1))\cdots\theta(s_m,x(s_m))ds_1\cdots ds_m\\ &\quad\cdot\int_a^b\overset{(n)}{\cdots}\int_a^b\varphi(t_1,x(t_1))\cdots\varphi(t_n,x(t_n))dt_1\cdots dt_n\\ &=\int_a^b\overset{(m+n)}{\cdots}\int_a^b\theta(s_1,x(s_1))\cdots\theta(s_m,x(s_m))\\ &\quad\cdot\varphi(t_1,x(t_1))\cdots\varphi(t_n,x(t_n))ds_1\cdots ds_m dt_1\cdots dt_n\,. \end{aligned} \tag{4.3}$$

We now apply the definition to obtain

$$\begin{aligned} (I_\lambda(F_{m,n})\psi)(\xi) &\equiv\int_{C_0[a,b]}F_{m,n}(\rho x+\xi)\psi(\rho x(b)+\xi)dx\\ &=\int_{C_0[a,b]}\int_a^b\overset{(m+n)}{\cdots}\int_a^b\theta(s_1,\rho x(s_1)+\xi)\cdots\theta(s_m,\rho x(s_m)+\xi)\\ &\quad\cdot\varphi(t_1,\rho x(t_1)+\xi)\cdots\varphi(t_n,\rho x(t_n)+\xi)\psi(\rho x(b)+\xi)\\ &\quad\cdot ds_1\cdots ds_m dt_1\cdots dt_n dx\\ &\quad\sum_{(j_1,\cdots,j_m,k_1\cdots k_n)\in P}\int_a^b\int_a^{\tau_{m+n}}\cdots\int_a^{\tau_2}\int_{C_0[a,b]}\theta(\tau_{j_1},\rho x(\tau_{j_1})+\xi)\cdots\\ &\quad\cdot\theta(\tau_{j_m},\rho x(\tau_{j_m})+\xi)\varphi(\tau_{k_1},x(\tau_{k_1})+\xi)\cdots\varphi(\tau_{k_n},\rho x(\tau_{k_n})+\xi)\\ &\quad\cdot\psi(\rho x(b)+\xi)dxd\tau_1\cdots d\tau_{m+n}\,, \end{aligned} \tag{4.4}$$

where P is the set of all permutations of $\{1,\cdots,m+n\}$, $\{\tau_1,\cdots,\tau_{m+n}\}$ is the set $\{s_1,\cdots,s_m,t_1,\cdots,t_n\}$ in some order, $s_1=\tau_{j_1},\cdots,s_m=\tau_{j_m}$, $t_1=\tau_{k_1},\cdots,t_n=\tau_{k_n}$.

We now evaluate the Wiener integral to obtain

$$(I_\lambda(F_{m,n})\psi)(\xi) = \sum_P \int_a^b \int_a^{\tau_{m+n}} \cdots \int_a^{\tau_2} [(2\pi)^{m+n+1}(\tau_1 - a)\cdots(b - \tau_{m+n})]^{-1/2}$$
$$\cdot \int_{-\infty}^{\infty} \overset{(m+n+1)}{\cdots} \int_{-\infty}^{\infty} \theta(\tau_{j_1}, \rho u_{j_1} + \xi)\cdots\theta(\tau_{j_m}, \rho u_{j_m} + \xi)\varphi(\tau_{k_1}, \rho u_{k_1} + \xi)\cdots$$
$$\cdot \varphi(\tau_{k_n}, \rho u_{k_n} + \xi)\psi(\rho u_{m+n+1} + \xi) \exp\left\{-\sum_{p=1}^{m+n+1} \frac{(u_p - u_{p-1})^2}{2(\tau_p - \tau_{p-1})}\right\}$$
$$\cdot du_1 \cdots du_{m+n+1} d\tau_1 \cdots d\tau_{m+n} ,$$

where $u_0 \equiv 0$, $\tau_0 \equiv a$, $\tau_{m+n+1} \equiv b$. We now set $v_0 \equiv \xi$, $v_p = \rho u_p + \xi$, for $p = 0, \cdots, m + n + 1$ to obtain

$$(I_\lambda(F_{m,n})\psi)(\xi) = \sum_P \int_a^b \int_a^{\tau_{m+n}} \cdots \int_a^{\tau_2} \left[\frac{\lambda^{m+n+1}}{(2\pi)^{m+n+1}(\tau_1 - \tau_0)\cdots(\tau_{m+n+1} - \tau_{m+n})}\right]^{1/2}$$
$$(4.5) \quad \cdot \int_{-\infty}^{\infty} \overset{(m+n+1)}{\cdots} \int_{-\infty}^{\infty} \theta(\tau_{j_1}, v_{j_1})\cdots\theta(\tau_{j_m}, v_{j_m})\varphi(\tau_{k_1}, v_{k_1})\cdots\varphi(\tau_{k_n}, v_{k_n})$$
$$\cdot \psi(v_{m+n+1}) \exp\left\{-\sum_{p=1}^{m+n+1} \frac{\lambda(v_p - v_{p-1})^2}{2(\tau_p - \tau_{p-1})}\right\} dv_1 \cdots dv_{m+n+1} d\tau_1 \cdots d\tau_{m+n} .$$

We now estimate

$$|(I_\lambda(F_{m,n})\psi)(\xi)| \leqq \sum_P \int_a^b \int_a^{\tau_{m+n}} \cdots \int_a^{\tau_2} \left[\frac{\lambda^{m+n+1}}{(2\pi)^{m+n+1}(\tau_1 - \tau_0)\cdots(\tau_{m+n+1} - \tau_{m+n})}\right]^{1/2}$$
$$\cdot B_1^m \cdot B_2^n \cdot \|\psi\|_1 d\tau_1 \cdots d\tau_{m+n}$$
$$(4.6) \quad = (m + n)! \left|\frac{\lambda}{2\pi}\right|^{(m+n+1)/2} B_1^m B_2^n \|\psi\|_1 \int_a^b \int_a^{\tau_{m+n}} \cdots$$
$$\cdot \int_a^{\tau_2} \frac{d\tau_1 \cdots d\tau_{m+n}}{\sqrt{(\tau_1 - \tau_0)\cdots(\tau_{m+n+1} - \tau_{m+n})}}$$
$$= \left(\frac{|\lambda|}{2\pi}\right)^{(m+n+1)/2} B_1^m B_2^n \|\psi\|_1 \pi^{(m+n)/2} \Gamma\left(\frac{m + n + 2}{2}\right) 2^{m+n} (b - a)^{(m+n-1)/2}$$

where we have evaluated the $m + n$ fold integral as Dirichlet's integral (see [9, p. 258]) and simplified the resulting expression. We now state a lemma analogous to our Lemma 1 for this case.

LEMMA 5: *Let $\theta(s,u)$ and $\varphi(s,u)$ be measurable in the strip $[a,b] \times (-\infty,\infty)$, let $\theta(s,\cdot)$ and $\varphi(s,\cdot)$ be of class $\mathscr{L}_1(-\infty,\infty)$ for $s \in [a,b]$ and $\|\theta(s,\cdot)\|_1 \leqq B_1$ and $\|\varphi(s,\cdot)\|_1 \leqq B_2$ for almost every $s \in [a,b]$, and let $\psi \in \mathscr{L}_1(-\infty,\infty)$, $\|\psi\|_1 \leqq M$. Let λ be a complex number,* $\operatorname{Re}\lambda \geq 0$ *and let*

$$(4.7)\qquad K_{0,0} \equiv K_{0,0}(\xi, \lambda) \equiv (b-a)^{-1/2}\int_{-\infty}^{\infty} \psi(v) \exp\left\{\frac{-\lambda v^2}{2(b-a)}\right\} dv\,,$$

$$K_{m,n} \equiv K_{m,n}(\xi,\lambda) = \sum_{(j_1,\cdots,j_m,k_1,\cdots,k_n)\in P} \int_a^b \int_a^{\tau_{m+n}} \cdots$$
$$\cdot \int_a^{\tau_2} \Big[(\tau_1-\tau_0)\cdots(\tau_{m+n+1}-\tau_{m+n})\Big]^{-1/2}$$
$$\cdot \int_{-\infty}^{\infty} \overset{(m+n+1)}{\cdots} \int_{-\infty}^{\infty} \theta(\tau_{j_1}, v_{j_1})\cdots\theta(\tau_{j_m}, v_{j_m})\varphi(\tau_{k_1}, v_{k_1})\cdots\varphi(\tau_{k_n}, v_{k_n})$$
$$\cdot \psi(v_{m+n+1}) \exp\left\{-\sum_{p=1}^{m+n+1} \frac{\lambda(v_p - v_{p-1})^2}{2(\tau_p - \tau_{p-1})}\right\} dv_1\cdots dv_{m+n+1} d\tau_1 \cdots d\tau_{m+n}\,,$$

for $m, n = 0, 1, 2, \cdots$, *where* P *represents the set of all permutations of* $1, \cdots, m+n$, $v_0 \equiv \xi$ *and* $a_0 = \tau_0 < \tau_1 < \cdots < \tau_{m+n+1} = b$. *Then for all* m, n, $K_{m,n}(\xi, \lambda)$ *exists and is continuous for all real* ξ *and* $\mathrm{Re}\,\lambda \geq 0$; *and for each real* ξ, $K_{m,n}(\xi,\lambda)$ *is analytic in* λ *for* $\mathrm{Re}\,\lambda > 0$. *For* $\mathrm{Re}\,\lambda > 0$ *and all* m, n

$$(4.8)\qquad |K_{m,n}(\xi,\lambda)| \leqq 2^{m+n}\pi^{(m+n)/2}(b-a)^{(m+n-1)/2}\Gamma\left(\frac{m+n+2}{2}\right)MB_1^m B_2^n\,.$$

The proof of Lemma 5 parallels that of Lemma 1 and we shall omit it.

COROLLARY TO LEMMA 5: *Let* $F_{m,n}(x)$ *be defined as in* (4.3) *then* $(I_\lambda(F_{m,n})\psi)(\xi)$ *exists for real* $\lambda > 0$ *and*

$$(I_\lambda(F_{m,n})\psi)(\xi) = K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}$$

Our next lemma will depend on the order and type of the entire function of several complex variables. For the convenience of the reader we apply the definitions as given by A. A. Gol'dberg [7; p. 338] to the case of an entire function of two complex variables where the domain D as used in his definition is taken to be the bicylinder:

$$(4.9)\qquad D = D(R_1, R_2) = \{(z,w) \,|\, |z| < R_1 < \infty,\ |w| < R_2 < \infty\}\,.$$

Thus if

$$f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n\,,$$

and

$$D_R \equiv D_R(R_1, R_2) = \{(z, w) | (z/R, w/R) \in D\}$$

and

$$M_f(R) = \sup_{z \in D_R} |f(z, w)|,$$

the order μ and the type τ are defined thus

$$\mu \equiv \mu_D = \limsup_{R \to \infty} \frac{\log \log M_f(R)}{\log R} \tag{4.10}$$

$$\tau \equiv \tau_D = \limsup_{R \to \infty} \frac{\log M_f(R)}{R^\mu}. \tag{4.11}$$

A theorem of Gol'dberg enables us to express the order and type in terms of the coefficients, indeed

$$\mu = \limsup_{m+n \to \infty} \frac{(m+n) \log (m+n)}{-\log |a_{m,n}|} \tag{4.12}$$

$$(e\mu\tau_D)^{1/\mu} = \limsup_{m+n \to \infty} \{(m+n)^{1/\mu}[|a_{m,n}| R_1^m R_2^n]^{1/(m+n)}\}. \tag{4.13}$$

LEMMA 6: *Let θ, φ, ψ satisfy the conditions of Lemma 5 and let $K_{m,n}(\xi, \lambda)$ be defined as in Lemma 5. Let*

$$f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \tag{4.14}$$

be an entire function of growth $(2, \tau)$ where $\tau \equiv \tau_D < \infty$, and τ and D are as defined in (4.11) and (4.9). Then*

Case I: *growth* $(2, 0)$ *In this case*

$$\sum_{m,n=0}^{\infty} a_{m,n} K_{m,n}(\xi, \lambda) \left(\frac{\lambda}{2\pi} \right)^{(m+n+1)/2} \tag{4.15}$$

converges for all real ξ and all λ, $\operatorname{Re} \lambda \geq 0$. *Moreover for each $\lambda_0 > 0$,* (4.15) *converges uniformly for all real ξ and $|\lambda| \leq \lambda_0$,* $\operatorname{Re} \lambda \geq 0$. *Thus for each real ξ,* (4.15) *represents an analytic function in* $\operatorname{Re} \lambda > 0$ *which is continuous for* $\operatorname{Re} \lambda \geq 0$.

Case II: *order two, type $\tau \equiv \tau_{D(R_1,R_2)}$, $0 < \tau < \infty$; R_1, R_2 any two*

* An entire function is said to be of growth (μ', τ') iff it is of order not exceeding μ', and if its order is μ', its type does not exceed τ'. The type $\tau \equiv \tau_{D(R_1,R_2)}$ is not specified in Case I since if $\tau_{D(R_1,R_2)} = 0$ for a particular pair of positive numbers (R_1, R_2), $\tau_D = 0$ for all pairs (R_1, R_2).

positive numbers. In this case (4.15) *converges for all real* ξ *and all* λ, $\operatorname{Re}\lambda \geqq 0$, $|\lambda| < \lambda_0$

$$\lambda_0 = \frac{1}{2(b-a)\tau}\left[\min\left(\frac{R_1}{B_1}, \frac{R_2}{B_2}\right)\right]^2 . \tag{4.16}$$

The convergence of (4.15) *is uniform for all real* ξ *and all* λ, $\operatorname{Re}\lambda \geqq 0$, $|\lambda| \leqq \lambda_0' < \lambda_0$. *Thus for each* ξ, (4.15) *represents an analytic function in* $|\lambda| < \lambda_0$, $\operatorname{Re}\lambda > 0$ *which is continuous in* $|\lambda| < \lambda_0$, $\operatorname{Re}\lambda \geq 0$.

Proof of Lemma 6: We begin with the proof of Case I. Let μ be the order of $f(z,w)$; then by Gol'dberg's Theorem, [7; p. 339],

$$\mu = \limsup_{(m+n)\to\infty} \frac{(m+n)\log(m+n)}{-\log|a_{m,n}|} \tag{4.17}$$

(where the quotient on the right is taken as zero if $a_{m,n} = 0$).

If $\mu < 2$, we have for $m+n$ sufficiently large that

$$\frac{(m+n)\log(m+n)}{-\log|a_{m,n}|} < \mu_0 < 2$$

and so

$$|a_{m,n}|^{\mu_0} < (m+n)^{-(m+n)} .$$

Thus for $(m+n)$ sufficiently large by Stirling's formula we see that

$$\begin{aligned}\Gamma\left(\frac{m+n+2}{2}\right)|a_{m,n}| &\leqq \Gamma\left(\frac{m+n+2}{2}\right)(m+n)^{-(m+n)/\mu_0}\\ &\leqq e^{-((m+n)/2)}\left(\frac{m+n}{2}\right)^{((m+n)/2)}\sqrt{\pi(m+n)}\,e^{1/6(m+n)}(m+n)^{-((m+n)/\mu_0)}\\ &= e^{-((m+n)/2)}2^{-((m+n)/2)}(\pi(m+n))^{1/2}e^{1/6(m+n)}(m+n)^{(m+n)(1/2-1/\mu_0)} .\end{aligned}$$

Hence, by (4.8), for $(m+n)$ sufficiently large

$$\begin{aligned}&\left|a_{m,n}K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\right|\\ &\quad\leqq 2^{m+n}\pi^{(m+n)/2}(b-a)^{(m+n-1)/2}MB_1^mB_2^ne^{-((m+n)/2)}2^{-((m+n)/2)}\\ &\qquad\cdot[\pi(m+n)]^{1/2}e^{1/(6(m+n)}(m+n)^{(m+n)(1/2-1/\mu_0)}\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\end{aligned}$$

Because of the factor $(m+n)^{(1/2-1/\mu_0)(m+n)}$, since $(1/2-1/\mu_0) < 0$, we see that (4.15) converges for all real ξ and all λ, $\operatorname{Re}\lambda \geqq 0$. Clearly (4.15) converges uniformly for all real ξ and $|\lambda| \leqq \lambda_0 < \infty$, $\operatorname{Re}\lambda \geqq 0$ and so for each real ξ, (4.15) represents an analytic function in $\operatorname{Re}\lambda > 0$ which is continuous for $\operatorname{Re}\lambda \geqq 0$.

In order to establish Case I when $\mu = 2$ and $\tau = 0$, we observe that by Gol'dberg's Theorem

$$\limsup_{m+n\to\infty} (m+n)^{1/2}[|a_{m,n}|R_1^m R_2^n]^{1/(m+n)} = 0\,. \tag{4.18}$$

Now by Stirling's formula,

$$\begin{aligned} &\left|a_{m,n}K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\right| \\ &\leqq 2^{(m+n)}\pi^{(m+n)/2}(b-a)^{(m+n-1)/2}\Gamma\left(\frac{m+n+2}{2}\right)MB_1^mB_2^n|a_{m,n}| \\ &\cdot\left|\frac{\lambda}{2\pi}\right|^{(m+n+1)/2} \leqq 2^{(m+n)}\pi^{(m+n)/2}(b-a)^{(m+n-1)/2}MB_1^mB_2^n|a_{m,n}| \\ &\cdot\left|\frac{\lambda}{2\pi}\right|^{(m+n+1)/2} e^{-((m+n)/2)}\left(\frac{m+n}{2}\right)^{(m+n)/2}\sqrt{\pi(m+n)}\,e^{1/6(m+n)} \end{aligned} \tag{4.19}$$

Thus

$$\begin{aligned} &\left|a_{m,n}K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\right|^{1/(m+n)} \\ &\leqq 2^{1/2}\pi^{1/2}(b-a)^{1/2}(b-a)^{-1/(2(m+n))}M^{1/(m+n)}\left(\frac{B_1}{R_1}\right)^{m/(m+n)} \\ &\cdot\left(\frac{B_2}{R_2}\right)^{n/(m+n)} e^{-1/2}[\pi(m+n)]^{1/(2(m+n))}e^{1/(6(m+n)^2)} \\ &\cdot\left|\frac{\lambda}{2\pi}\right|^{1/2+1/(2(m+n))}(m+n)^{1/2}[|a_{m,n}|R_1^mR_2^n]^{1/(m+n)}\,, \end{aligned}$$

and by (4.18), (4.15) converges for all real ξ and all λ, $\operatorname{Re}\lambda \geq 0$. The convergence is uniform for all real ξ, and $|\lambda| \leqq \lambda_0 < \infty$, $\operatorname{Re}\lambda \geq 0$ and so for each real ξ, (4.15) represents an analytic function in $\operatorname{Re}\lambda > 0$ which is continuous for $\operatorname{Re}\lambda \geqq 0$.

We proceed to the proof of Case II, where $\mu = 2$ and $0 < \tau < \infty$, and observe that

$$\limsup_{m+n\to\infty} (m+n)[|a_{m,n}|R_1^mR_2^n]^{2/(m+n)} = 2e\tau \tag{4.20}$$

where R_1 and R_2 are the radii in the bicylinder $D(R_1, R_2)$ used to define the type τ.

In order to establish the convergence of the series (4.15), we shall take the $((m+n)/2)$ root of the general term and apply (4.8) and Stirling's Theorem to obtain for $\operatorname{Re}\lambda > 0$

$$\begin{aligned}(4.21)\quad &\left|a_{m,n}K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\right|^{2/(m+n)} \\ &\leqq 2\pi(b-a)e^{-1}\frac{|\lambda|}{2\pi}\left[(b-a)^{-1}\pi(m+n)M^2e^{1/(3(m+n))}\frac{|\lambda|}{2\pi}\right]^{1/(m+n)} \\ &\cdot\left(\frac{B_1}{R_1}\right)^{(2m)/(m+n)}\left(\frac{B_2}{R_2}\right)^{(2n)/(m+n)}(m+n)[|a_{m,n}|R_1^mR_2^n]^{2/(m+n)} \\ &\leqq (b-a)|\lambda|Q^2\left[(b-a)^{-1}\pi(m+n)M^2e^{1/(3(m+n))}\frac{|\lambda|}{2\pi}\right]^{1/(m+n)} \\ &\cdot\frac{m+n}{e}[|a_{m,n}|R_1^mR_2^n]^{2/(m+n)}, \quad \text{where} \quad Q = \max\left(\frac{B_1}{R_1},\frac{B_2}{R_2}\right).\end{aligned}$$

Let the limit superior,

$$\limsup_{m+n\to\infty}\left|a_{m,n}K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\right|^{1/(m+n)} = L.$$

It follows from (4.21) and (4.20) that if $\operatorname{Re}\lambda \geq 0$ and $|\lambda| < \lambda_0$, then $L < 1$ and hence the series (4.15) converges absolutely. The remaining statements in the conclusion of Case II follow in the usual way.

By choosing the shape of the region $D(R_1, R_2)$ so as to maximize λ_0, the formula for λ_0 will be simplified.

LEMMA 7: *Let θ, φ, B_1, B_2, ψ and $f(z,w)$ satisfy the conditions of Lemmas* 5 *and* 6. *Let τ and D be defined as* (4.9) *and* (4.11), *and let $\lambda_0 \equiv \lambda_0(R_1, R_2)$ be defined as in* (4.16). *Then for fixed B_1, B_2,*

$$(4.22)\qquad \max_{R_1,R_2>0}\lambda_0(R_1,R_2) = \lambda_0(B_1,B_2).$$

Thus

$$(4.23)\qquad \max_{R_1,R_2>0}\lambda_0(R_1,R_2) = [2(b-a)\tau_{D(B_1,B_2)}]^{-1}.$$

Proof of Lemma 7: We observe from (4.20) that τ is a monotonically increasing function of R_1 for fixed R_2 and vice versa. We further observe that τ is positive homogeneous of degree 2 in (R_1, R_2). Hence by the definition (4.16), λ_0 is homogeneous of degree zero in (R_1, R_2). Thus if $R_1/B_1 = R_2/B_2$, $\lambda_0(R_1, R_1) = \lambda_0(B_1, B_2)$. If $R_1/B_1 < R_2/B_2$, take $R_2' =$

$(B_2/B_1)R_1$, then $R_2' < R_2$ and $R_1/B_1 = R_2'/B_2$. Since τ is monotone increasing in R_2, it follows that $\lambda_0(R_1, R_2) \leq \lambda_0(R_1, R_2') = \lambda_0(B_1, B_2)$. Also if $R_1/B_1 > R_2/B_2$, we have $\lambda_0(R_1, R_2) \leqq \lambda_0(B_1, B_2)$ by the same argument and we have proved (4.22); and (4.23) follows from (4.16) and (4.22).

§ 5 Existence Theorems for $I_\lambda(F)$ and $I_\lambda^{\mathrm{an}}(F)$:

LEMMA 8: *Let θ, φ, ψ and $K_{m,n}(\xi, \lambda)$ be as in Lemma 5. Let*

$$f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \tag{5.0}$$

be an entire function of growth (2, 0) *and let*

$$F(x) = f\left[\int_a^b \theta(s, x(s))ds,\ \int_a^b \phi(t, x(t))dt\right] \tag{5.1}$$

and let λ be real and positive. Then $(I_\lambda(F)\psi)(\xi)$ exists for all real ξ, and

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) &\equiv \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b)) \\ &= \sum_{m,n=0}^{\infty} a_{m,n} K_{m,n}(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)},\end{aligned} \tag{5.2}$$

Proof of Lemma 8: Case I: Let $\theta, \varphi, \psi, a_{m,n}$ be real and non-negative and let $f_{m,n}(z, w)$ and $F_{m,n}(x)$ be defined as in (4.2) and (4.3). Then the calculations at the beginning of section 4 are justified, and by (4.5) and (4.7) we have

$$\int_{C_0[a,b]} F_{m,n}(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx = K_{m,n}(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}$$

Hence, setting $\rho = \lambda^{-1/2}$, we have by Lemma 6 and the Fubini theorems

$$\begin{aligned} I \equiv (I_\lambda(F)\psi)(\xi) &= \int_{C_0[a,b]} F_{m,n}(\rho x + \xi)\psi(\rho x(b) + \xi)dx \\ &= \int_{C_0[a,b]} \sum_{m,n\geq 0} a_{m,n} F_{m,n}(\rho x + \xi)\psi(\rho x(b) + \xi)dx \\ &= \sum_{m,n\geq 0} a_{m,n} \int_{C_0[a,b]} F_{m,n}(\rho x + \xi)\psi(\rho x(b) + \xi)dx \\ &= \sum_{m,n\geq 0} a_{m,n} K_{m,n}(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2} < +\infty\,. \end{aligned}$$

Thus Case I is established.

Case II (The general case): The proof is exactly the same as the proof of the general case of Lemma 3.

THEOREM 4: *Let* $\theta(s,u)$ *and* $\varphi(s,u)$ *be measurable in the strip* $[a,b] \times (-\infty,\infty)$, *let* $\theta(s,\cdot)$ *and* $\varphi(s,\cdot)$ *be of class* $\mathscr{L}_1(-\infty,\infty)$ *for* $s \in [a,b]$ *and* $\|\theta(s,\cdot)\|_1 \leqq B_1$ *and* $\|\varphi(s,\cdot)\|_1 \leqq B_2$ *for almost every* $s \in [a,b]$ *and let* $\psi \in \mathscr{L}_1(-\infty,\infty)$, $\|\psi\|_1 \leqq M$. *Let*

$$f(z,w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \tag{5.3}$$

be an entire function of growth $(2,0)$ *and let*

$$F(x) = f\left[\int_a^b \theta(s, x(s))ds, \int_a^b \varphi(t, x(t))dt\right] . \tag{5.4}$$

Then $I_\lambda^{\mathrm{an}}(F)$ *exists and is a bounded operator taking* $\mathscr{L}_1$ *into* $\mathscr{L}_\infty$; *and* $I_\lambda^{\mathrm{an}}(F)\psi$ *is weakly analytic as a vector valued function of* λ *for* $\mathrm{Re}\,\lambda > 0$. *Moreover for each* λ, $\mathrm{Re}\,\lambda > 0$, *the function* $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)$ *has the representation*

$$(I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = \sum_{m,n=0}^{\infty} a_{m,n} K_{m,n}(\xi,\lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2} \tag{5.5}$$

for all real ξ, *and for each fixed real* ξ, $(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)$ *is analytic in* λ *for* $\mathrm{Re}\,\lambda > 0$.

COROLLARY 1 TO THEOREM 4: *Let* $\theta, \varphi, B_1, B_2, \psi, M$ *and* $f(z,w)$ *and* $F(x)$ *satisfy the hypothesis of Theorem* 4. *Let*

$$f^*(z,w) = \sum_{m,n=0}^{\infty} |a_{m,n}| z^m w^n .$$

Let τ' *be any positive number and let* $N(\tau')$ *be an integer greater than one such that when* $m + n > N(\tau')$,

$$(m+n)[|a_{m,n}| B_1^m B_2^n]^{2/(m+n)} < 2e\tau' .$$

Then whenever $|\lambda| < [2\tau'(b-a)]^{-1}$, *it follows that for all real* ξ,

$$\begin{aligned} &|(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)| \\ &\quad \leqq M\left(\frac{|\lambda|}{2\pi(b-a)}\right)^{1/2} \Gamma(\tfrac{1}{2}N(\tau') + 1) f^*(B_1\sqrt{2(b-a)|\lambda|}, B_2\sqrt{2(b-a)|\lambda|}) \\ &\qquad + Me\left(\frac{2|\lambda|}{b-a}\right)^{1/2}[1 - \sqrt{(b-a)2\tau'|\lambda|}]^{-3} \end{aligned}$$

We note that the existence of $N(\tau')$ in the hypotheses follows from (4.18).

Proof of Corollary 1: By (5.2),

$$|I| \equiv |(I_\lambda^{\mathrm{an}}(F)\psi)(\xi)| \leqq \sum_{m,n=0}^{\infty}\left|\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2} a_{m,n}K_{m,n}\right|.$$

Now from (4.8) we have

$$\begin{aligned}
&\sum_{m+n\leqq N(\tau')}\left|\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2} a_{m,n}K_{m,n}\right| \\
&\leqq \sum_{m+n\leqq N(\tau')}\left|\frac{\lambda}{2\pi}\right|^{(m+n+1)/2}|a_{m,n}|2^{m+n}\pi^{(m+n)/2} \\
&\quad\cdot(b-a)^{(m+n-1)/2}\Gamma\left(\frac{m+n+2}{2}\right)MB_1^mB_2^n \\
&\leqq \Gamma\left(\frac{N(\tau')}{2}+1\right)M\left(\frac{|\lambda|}{2\pi(b-a)}\right)^{1/2} \\
&\quad\cdot f^*(B_1\sqrt{2(b-a)|\lambda|},\ B_2\sqrt{2(b-a)|\lambda|}).
\end{aligned}$$

Also by (4.19) and the hypotheses of the corollary,

$$\begin{aligned}
&\left|\sum_{m+n>N(\tau')}\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2} a_{m,n}K_{m,n}\right| \\
&\leqq \sum_{m+n>N(\tau')} 2^{m+n}\pi^{(m+n)/2}(b-a)^{(m+n-1)/2}MB_1^mB_2^n|a_{m,n}|\left(\frac{|\lambda|}{2\pi}\right)^{(m+n+1)/2} \\
&\quad\cdot e^{-(m+n)/2}\left(\frac{m+n}{2}\right)^{(m+n)/2}\sqrt{\pi(m+n)}\,e^{1/(6(m+n))} \\
&\leqq \sum_{m+n>N(\tau')}\frac{Me}{\sqrt{2}}(b-a)^{(m+n-1)/2}|\lambda|^{(m+n+1)/2}e^{-(m+n)/2}(m+n)^{1/2} \\
&\quad\cdot[(m+n)^{(m+n)/2}B_1^mB_2^n|a_{m,n}|] \leqq \sum_{m+n=0}^{\infty}\frac{Me}{\sqrt{2}}(b-a)^{(m+n-1)/2}|\lambda|^{(m+n+1)/2} \\
&\quad\cdot e^{-(m+n)/2}(m+n)^{1/2}(2e\tau')^{(m+n)/2} \\
&< Me\left(\frac{|\lambda|}{2(b-a)}\right)^{1/2}\sum_{k=0}^{\infty}k(k+1)(b-a)^{k/2}|\lambda|^{k/2}(2\tau')^{k/2} \\
&< 2Me\left(\frac{|\lambda|}{2(b-a)}\right)^{1/2}[1-\sqrt{2\tau'(b-a)|\lambda|}\,]^{-3}
\end{aligned}$$

when $2(b-a)\tau'|\lambda|<1$, and the Corollary is proved.

The proof of Theorem 4 is omitted because it is parallel to the proof of Theorem 1 with the Lemmas 5 and 8 taking the place of Lemmas 1 and 3.

We next consider the case of order two, type τ, $0 < \tau < \infty$.

LEMMA 9: *Let $\theta, \varphi, \psi, B_1, B_2$ and $K_{m,n}(\xi, \lambda)$ be as in Lemma 5. Let*

$$f(z) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \tag{5.6}$$

be an entire function of order two, type τ, $0 < \tau < \infty$, where $\tau \equiv \tau_{D(B_1,B_2)}$, and let

$$F(x) = f\left[\int_a^b \theta(s, x(s))ds, \int_a^b \varphi(t, x(t))dt\right]. \tag{5.7}$$

Then for all real ξ and real λ satisfying $0 \leq \lambda < \lambda_0 = [2(b-a)\tau]^{-1}$,

$$\begin{aligned}(I_\lambda(F)\psi)(\xi) &= \int_{C_0[a,b]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(b) + \xi)dx \\ &= \sum_{m,n=0}^{\infty} a_{m,n} K_{m,n}(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2}\end{aligned} \tag{5.8}$$

The proof is similar to that of Lemma 8. The region of convergence of the series is as above where the value of λ_0 is given by equation (4.16).

THEOREM 5: *Let $\theta, \varphi, \psi, B_1, B_2$ be as in Theorem 4. Let*

$$f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \tag{5.9}$$

be an entire function of order two, type τ, $0 < \tau < \infty$, where $\tau \equiv \tau_{D(B_1,B_2)}$, and let

$$F(x) = f\left[\int_a^b \theta(s, x(s))ds, \int_a^b \varphi(t, x(t))dt\right]. \tag{5.10}$$

Then $I_\lambda^{\text{an}}(F)$ exists and is a bounded operator taking $\mathscr{L}_1$ into $\mathscr{L}_\infty$ and $I_\lambda^{\text{an}}(F)\psi$ is weakly analytic as a vector-valued function of λ for $\lambda \in \Omega = \{\lambda | \operatorname{Re} \lambda > 0, |\lambda| < \lambda_0\}$ where $\lambda_0 = [2(b-a)\tau]^{-1}$. Moreover for each $\lambda \in \Omega$, the function $(I_\lambda^{\text{an}}(F)\psi)(\xi)$ has the representation

$$(I_\lambda^{\text{an}}(F)\psi)(\xi) = \sum_{m,n=0}^{\infty} a_{m,n} K_{m,n}(\xi, \lambda)\left(\frac{\lambda}{2\pi}\right)^{(m+n+1)/2} \tag{5.11}$$

for all real ξ, and for each fixed real ξ, $(I_\lambda^{\text{an}}(F)\psi)(\xi)$ is analytic λ for $\lambda \in \Omega$.

The proof of Theorem 5 which parallels the proof of Theorem 2 is omitted.

The exponential function $f(z, w) = \exp(zw)$ provides an interesting special case. Direct computation shows that the order of f is two and its type $\tau \equiv \tau_{D(R_1,R_2)} = R_1R_2$.

COROLLARY 1 TO THEOREM 5: *Let $\theta, \varphi, \psi, B_1, B_2$ be as in Theorem 5 and let* $F(x) = \exp\left[\int_a^b \theta(s, x(s))ds \int_a^b \varphi(t, x(t))dt\right]$. *Then $I_\lambda^{\mathrm{an}}(F)\psi$ exists and is weakly analytic as a vector valued function for* $\mathrm{Re}\,\lambda > 0$, $|\lambda| < [2(b-a)B_1B_2]^{-1}$.

COROLLARY 2 TO THEOREM 5: *Under the hypotheses of Theorem* 5, *the estimate obtained in the Corollary to Theorem* 4 *still holds provided that $\tau' > \tau$.*

§6 Existence Theorem for $J_q(F)$: We now proceed to the limiting case where λ is purely imaginary.

THEOREM 6: *Let $\theta(s, u)$ and $\varphi(s, u)$ be measurable in the strip* $[a, b] \times (-\infty, \infty)$, *let $\theta(s, \cdot)$ and $\varphi(s, \cdot)$ be of class $\mathscr{L}_1(-\infty, \infty)$ for $s \in [a, b]$ and $\|\theta(s, \cdot)\|_1 \leqq B_1$ and $\|\varphi(s, \cdot)\|_1 \leqq B_2$ for almost every $s \in [a, b]$ and let $\psi \in \mathscr{L}_1(-\infty, \infty)$, $\|\psi\|_1 \leqq M$. Let*

$$f(z, w) = \sum_{m,n=0}^{\infty} a_{m,n} z^m w^n \tag{6.0}$$

be an entire function of growth $(2, \tau)$ where $\tau \equiv \tau_{D(B_1,B_2)} < \infty$, and let

$$F(x) = f\left[\int_a^b \theta(s, x(s))ds,\ \int_a^b \varphi(t, x(t))dt\right]. \tag{6.1}$$

Then:

Case I: *growth* (2,0). *In this case $J_q(F)\psi$ exists and is of class $\mathscr{L}_\infty$ for each $\psi \in \mathscr{L}_1$ and each real q, $q \neq 0$. Moreover $J_q(F)\psi$ has the representation*

$$(J_q(F)\psi)(\xi) = \sum_{m,n=0}^{\infty} a_{m,n} K_{m,n}(\xi, -iq)\left(\frac{q}{2\pi i}\right)^{(m+n+1)/2} \tag{6.2}$$

for each real ξ and each real q, $q \neq 0$.

Case II: *order two, type τ, $0 < \tau < \infty$. In this case $J_q(F)\psi$ exists and is of class $\mathscr{L}_\infty$ for each $\psi \in \mathscr{L}_1$ and each q, $0 < |q| < \lambda_0 = [2(b-a)\tau]^{-1}$, and $(J_q(F)\psi)(\xi)$ has the representation* (6.2).

The proof of Theorem 6 parallels the proof of Theorem 3 and will be omitted.

Remark: Under the hypotheses of Theorem 6

$$\lim_{\lambda\to -iq} (I_\lambda^{\mathrm{an}}(F)\psi)(\xi) = (J_q(F)\psi)(\xi)$$

uniformly in ξ and q for all real ξ and $|q| \leq q_0 < \lambda_0$, where we take $\lambda_0 = \infty$ in Case I.

III An application to integral equations

§ 7 Integral equations $\mathrm{Re}\,\lambda > 0$**:** We shall now apply our results on operator valued function space integrals to obtain the solution of a pair of simultaneous integral equations. In doing this we shall need to vary the interval over which the functions in our function spaces are defined and continuous. Therefore when it is necessary to specify the interval we shall do so as follows:

$$(I_{\lambda,[a,b]}(F)\psi)(\xi) \equiv (I_\lambda(F)\psi)(\xi)\ .$$

At times it will be convenient to have an element $x(\cdot) \in C_0[a, b]$ defined for all real values of the independent variable. We shall extend the definition of $x(t)$ by requiring the function to be constant on $(-\infty, a]$ and $[b, +\infty)$. Thus $x \in C_0[a, b]$ implies that $x(t)$ is continuous for all real t and $x(t) = 0$ for $t \in (-\infty, a]$ and $x(t) = x(b)$ for $t \in [b, +\infty)$. The following property of Wiener integrals (see E. Cuthill [6]) will be used in the proof of Theorem 7:

$$\int_{C_0[a,b]} F(x)dx = \int_{C_0[a,c]\times C_0[c,b]} F(y + z)d(y \times z)$$

where the existence of either side implies the existence of the other and their equality.

THEOREM 7: *Let θ, φ, and ψ be as in Theorem* 6 *and let*

$$E \equiv E(t, x) = \exp\left\{\int_t^b \theta(s, x(s))ds\right\} \tag{7.0}$$

and let

$$F \equiv F(t, x) = E(t, x)\int_t^b \varphi(s, x(s))ds\ . \tag{7.1}$$

Let $\mathrm{Re}\,\lambda > 0$ *and let*

$$G(t, \xi, \lambda) \equiv (I^{\mathrm{an}}_{\lambda,[t,b]}(E)\psi)(\xi) \tag{7.2}$$

and

$$H(t,\xi,\lambda) \equiv (I^{\mathrm{an}}_{\lambda,[t,b]}(F)\psi)(\xi)\ . \tag{7.3}$$

Then G and H satisfy the following pair of simultaneous integral equations on $[a,b)\times(-\infty,\infty)$:

$$\begin{cases} G(t,\xi,\lambda) = \left[\dfrac{\lambda}{2\pi(b-t)}\right]^{1/2}\displaystyle\int_{-\infty}^{\infty}\psi(v)\exp\left(\dfrac{-\lambda(v-\xi)^2}{2(b-t)}\right)dv \\ \qquad + \left[\dfrac{\lambda}{2\pi}\right]^{1/2}\displaystyle\int_t^b (s-t)^{-1/2}\int_{-\infty}^{\infty}\theta(s,v)G(s,v,\lambda) \\ \qquad \cdot\exp\left[\dfrac{-\lambda(v-\xi)^2}{2(s-t)}\right]dvds \\ H(t,\xi,\lambda) = \left(\dfrac{\lambda}{2\pi}\right)^{1/2}\displaystyle\int_t^b (s-t)^{-1/2}\int_{-\infty}^{\infty}G(s,v,\lambda)\varphi(s,v) \\ \qquad \cdot\exp\left[\dfrac{-\lambda(v-\xi)^2}{2(s-t)}\right]dvds \\ \qquad + \left(\dfrac{\lambda}{2\pi}\right)^{1/2}\displaystyle\int_t^b (s-t)^{-1/2}\int_{-\infty}^{\infty}\theta(s,v)H(s,v,\lambda) \\ \qquad \cdot\exp\left[\dfrac{-\lambda(v-\xi)^2}{2(s-t)}\right]dvds\ . \end{cases} \tag{7.4}$$

Proof of Theorem 7: We shall begin by observing that for λ real and positive,

$$G(t,\xi,\lambda) = (I_\lambda(E)\psi)(\xi) = \int_{C_0[t,b]} E(t,\rho x+\xi)\psi(\rho x(b)+\xi)dx \tag{7.5}$$

and

$$H(t,\xi,\lambda) = (I_\lambda(F)\psi)(\xi) = \int_{C_0[t,b]} F(t,\rho x+\xi)\psi(\rho x(b)+\xi)dx \tag{7.6}$$

where $\lambda^{-1/2}=\rho$.
The Wiener integrals above exist by Lemma 8 where we have taken $f(z,w)=e^z$ or we^z which are of appropriate growth. We next establish the second of the integral equations (7.4) for λ real and positive. Differentiating (7.0) and (7.1), we have for almost all t

$$F_t(t,x) = -\varphi(t,x(t))E(t,x) - \theta(t,x(t))E(t,x)\int_t^b \varphi(s,x(s))ds\ .$$

By the fundamental theorem of the integral calculus for Lebesgue integrals, we have for $a\le\tau\le b$,

$$F(\tau, x) = \int_\tau^b \varphi(t, x(t))E(t, x)dt + \int_\tau^b\int_t^b \varphi(s, x(s))ds\theta(t, x(t))E(t, x)dt .$$

If we replace x by $\rho x + \xi$ and multiply by $\psi(\rho x(b) + \xi)$ where $\rho > 0$ and ξ is real, after taking the Wiener integral over $C_0[\tau, b]$, we obtain

$$(7.7)\quad \begin{cases} \displaystyle\int_{C_0[\tau,b]} F(\tau, \rho x + \xi)\psi(\rho x(b) + \xi)dx \\ \displaystyle\quad = \int_{C_0[\tau,b]}\left[\int_\tau^b \varphi(t, \rho x(t) + \xi)E(t, \rho x + \xi)dt\psi(\rho x(b) + \xi)dx\right. \\ \displaystyle\quad + \int_{C_0[\tau,b]}\int_\tau^b\int_t^b \varphi(s, \rho x(s) + \xi)ds\theta(t, \rho x(t) + \xi)E(t, \rho x + \xi)dt\Big] \\ \quad \cdot\psi(\rho x(b) + \xi)dx . \end{cases}$$

The Wiener integrals on the right exist because (7.7) would hold if φ, θ, and ψ were replaced by their absolute values since the left hand side would still exist by Lemma 8. The new right hand side would then dominate the old. The domination just mentioned permits us to use the Fubini theorem on the right hand side.

We obtain from (7.6) and (7.7) that

$$(7.8)\quad H(\tau, \xi, \lambda) = \int_\tau^b\int_{C_0[\tau,b]} \varphi(t, \rho x(t) + \xi)E(t, \rho x + \xi)\psi(\rho x(b) + \xi)dxdt$$
$$+ \int_\tau^b\int_{C_0[\tau,b]}\int_t^b \varphi(s, \rho x(s) + \xi)ds\theta(t, \rho x(t) + \xi)E(t, \rho x + \xi)$$
$$\cdot\psi(\rho x(b) + \xi)dxdt ,$$

where both integrals on the right hand side exist as finite numbers. Hence, we have by the Cuthill Theorem,

$$H(\tau, \xi, \lambda) = \int_t^b\int_{C_0[\tau,t]\times C_0[t,b]} \varphi(t, \rho y(t) + \rho z(t) + \xi)\psi(\rho y(b) + \rho z(b) + \xi)$$
$$\cdot\exp\left\{\int_t^b \theta(s, \rho y(s) + \rho z(s) + \xi)ds\right\} d(y \times z)dt$$
$$+ \int_\tau^b\int_{C_0[\tau,t]\times C_0[t,b]}\int_t^b \varphi(s', \rho y(s') + \rho z(s') + \xi)ds'\theta(t, \rho y(t) + \rho z(t) + \xi)$$
$$\cdot\psi(\rho y(b) + \rho z(b) + \xi)\exp\left\{\int_t^b \theta(s, \rho y(s) + \rho z(s) + \xi)ds\right\} d(y \times z)dt$$

where

$$(7.9)\qquad \text{for } s \in [\tau, t],\ z(s) = 0, \text{ and for } s \in [t, b],\ y(s) = y(t).$$

By the Fubini Theorem and definitions (7.9) we have

$$
(7.10)\quad \begin{cases} H(\tau, \xi, \lambda) = \int_\tau^b \int_{C_0[\tau,t]} \int_{C_0[t,b]} \varphi(t, \rho y(t) + \xi) \psi(\rho y(t) + \rho z(b) + \xi) \\ \quad \cdot \exp\left\{\int_t^b \theta(s, \rho y(t) + \rho z(s) + \xi) ds\right\} dz dy dt \\ \quad + \int_\tau^b \int_{C_0[\tau,t]} \int_{C_0[t,b]} \int_t^b \varphi(s', \rho y(t) + \rho z(s') + \xi) ds' \theta(t, \rho y(t) + \xi) \\ \quad \cdot \psi(\rho y(t) + \rho z(b) + \xi) \exp\left\{\int_t^b \theta(s, \rho y(t) + \rho z(s) + \xi) ds\right\} dz dy dt . \end{cases}
$$

In equation (7.5), we replace ξ by $\rho y(t) + \xi$ and thus obtain using (7.0)

$$
\begin{aligned} G(t, \rho y(t) + \xi, \lambda) = \int_{C_0[t,b]} &\exp\left\{\int_t^b \theta(s, \rho y(t) + \rho z(s) + \xi) ds\right\} \\ &\cdot \psi(\rho y(t) + \rho z(b) + \xi) dz . \end{aligned}
$$

Similarly from (7.6) and (7.1) we have

$$
\begin{aligned} H(t, \rho y(t) + \xi, \lambda) = \int_{C_0[t,b]} &\exp\left\{\int_t^b \theta(s, \rho y(t) + \rho z(s) + \xi) ds\right\} \\ &\cdot \int_t^b \varphi(s', \rho y(t) + \rho z(s') + \xi) ds' \psi(\rho y(t) + \rho z(b) + \xi) dz . \end{aligned}
$$

Substituting in (7.10) we obtain

$$
\begin{aligned} H(\tau, \xi, \lambda) &= \int_\tau^b \int_{C_0[\tau,t]} \varphi(t, \rho y(t) + \xi) G(t, \rho y(t) + \xi, \lambda) dy dt \\ &\quad + \int_\tau^b \int_{C_0[\tau,t]} \theta(t, \rho y(t) + \xi) H(t, \rho y(t) + \xi, \lambda) dy dt \\ &= \int_\tau^b \frac{1}{\sqrt{2\pi(t-\tau)}} \int_{-\infty}^{\infty} \varphi(t, \rho u + \xi) G(t, \rho u + \xi, \lambda) \exp\left\{\frac{-u^2}{2(t-\tau)}\right\} du dt \\ &\quad + \int_\tau^b \frac{1}{\sqrt{2\pi(t-\tau)}} \int_{-\infty}^{\infty} \theta(t, \rho u + \xi) H(t, \rho u + \xi, \lambda) \exp\left\{\frac{-u^2}{2(t-\tau)}\right\} du dt \\ &= \left(\frac{\lambda}{2\pi}\right)^{1/2} \int_\tau^b (t-\tau)^{-1/2} \int_{-\infty}^{\infty} \varphi(t, v) G(t, v, \lambda) \exp\left\{\frac{-\lambda(v-\xi)^2}{2(t-\tau)}\right\} dv dt \\ &\quad + \left(\frac{\lambda}{2\pi}\right)^{1/2} \int_\tau^b (t-\tau)^{-1/2} \int_{-\infty}^{\infty} \theta(t, v) H(t, v, \lambda) \exp\left\{\frac{-\lambda(v-\xi)^2}{2(t-\tau)}\right\} dv dt . \end{aligned}
$$

Thus we have obtained the second of the integral equations (7.4) for λ real and positive. The first of the integral equations (7.4) is obtained in a similar manner. We now use analytic extension to show that (7.4) holds for $\operatorname{Re}\lambda > 0$. By Theorem 4 for each fixed (t, ξ) in $[a, b] \times (-\infty, \infty)$ $G(t, \xi, \lambda)$ and $H(t, \xi, \lambda)$ are analytic in λ for $\operatorname{Re}\lambda > 0$ so that the left hand members are analytic. To show that the right hand members are analytic we note that for $\{\operatorname{Re}\lambda > 0\} \cap \{|\lambda| < \lambda_0\}$, the inner integrands are dominated by $|\psi|$, $|\theta G|$, $|\varphi G|$, $|\theta H|$ which are of class $\mathscr{L}_1$ in v for every s on $[a, b)$. Thus the inner integrals are analytic functions of λ for $\operatorname{Re}\lambda > 0$ and each $s \in [a, b)$. Moreover the integrand of the outer integrals are of class $\mathscr{L}_1$ on $[t, b]$ since the inner integrals are dominated by the corresponding $\mathscr{L}_1$ norms which are bounded in s because of the estimates given in the Corollary to Theorem 4. Thus the double integrals and in fact the right hand sides are analytic. Thus the analytic extension argument is complete and the integral equations (7.4) hold for $\operatorname{Re}\lambda > 0$.

§ 8 **Integral equations** $\operatorname{Re}\lambda = 0$: Finally we take limits as $\lambda \to -iq$ in (7.4) to obtain the following theorem.

THEOREM 8: *Let θ, φ, and ψ be as in Theorem* 6 *and let E and F be given as in* (7.0) *and* (7.1). *Let q be any real number, $q \neq 0$ and let*

$$\mathscr{S}(t, \xi, q) \equiv (J_{q,[t,b]}(E)\psi)(\xi) \tag{8.1}$$

$$\mathscr{H}(t, \xi, q) \equiv (J_{q,[t,b]}(F)\psi)(\xi) \tag{8.2}$$

Then $\mathscr{S}$ and $\mathscr{H}$ satisfy the following pair of simultaneous integral equations on $[a, b) \times (-\infty, \infty)$:

$$\begin{cases} \mathscr{S}(t, \xi, q) = \left[\dfrac{q}{2\pi i(b-t)}\right]^{1/2}\displaystyle\int_{-\infty}^{\infty} \psi(v) \exp\left[\dfrac{iq(v-\xi)^2}{2(b-t)}\right]dv \\ \qquad + \left(\dfrac{q}{2\pi i}\right)^{1/2}\displaystyle\int_t^b (s-t)^{-1/2}\int_{-\infty}^{\infty} \theta(s, v)\mathscr{S}(s, v, q) \exp\left[\dfrac{iq(v-\xi)^2}{2(s-t)}\right]dvds \\ \mathscr{H}(t, \xi, q) = \left[\dfrac{q}{2\pi i}\right]^{1/2}\displaystyle\int_t^b (s-t)^{-1/2}\int_{-\infty}^{\infty} \varphi(s, v)\mathscr{S}(s, v, q) \\ \qquad \cdot \exp\left[\dfrac{iq(v-\xi)^2}{2(s-t)}\right]dvds \\ \qquad + \left[\dfrac{q}{2\pi i}\right]^{+1/2}\displaystyle\int_t^b (s-t)^{-1/2}\int_{-\infty}^{\infty} \theta(s, v)\mathscr{H}(s, v, q) \\ \qquad \cdot \exp\left[\dfrac{iq(v-\xi)^2}{2(s-t)}\right]dvds\,. \end{cases} \tag{8.3}$$

Proof of Theorem 8: Let q be any real number, $q \neq 0$. By hypothesis, (7.4) holds for $\operatorname{Re}\lambda > 0$. By the remark after Theoreom 6,

$G \to \mathscr{S}$ and $H \to \mathscr{H}$ as $\lambda \to -iq$, $\operatorname{Re}\lambda > 0$. Again by the Remark after Theorem 6, and the fact that θ and φ are of Class $\mathscr{L}_1$ the inner integrals of (7.4) approach the corresponding inner integrals of (8.3). Moreover the inner integrals in (7.4) are bounded in s by the estimates given in the Corollary to Theorem 4. Thus by Lebesgue's convergence Theorems the result follows.

Remark: It can be shown that the solutions of the integral equations (7.4) and (8.3) are unique* by the standard technique of successive substitution. Theorems 7 and 8 have been obtained and expressed with the purpose of obtaining solutions of integral equations in terms of integrals in function space. We point out in Theorem 9 that the opposite point of view can be taken and integrals in function space can be evaluated in terms of solutions of integral equations.

THEOREM 9: *Let θ, φ, and ψ be given as in Theorem 6 and let q be any real number, $q \neq 0$ and let*

$$A(x) = \exp\left\{\int_a^b \theta(s, x(s))ds\right\}$$

$$B(x) = A(x)\int_a^b \varphi(s, x(s))ds .$$

Then $J_{q,[a,b]}(A)\psi$ and $J_{q,[a,b]}(B)\psi$ exist and are elements of $\mathscr{L}_\infty(-\infty, \infty)$. Moreover they are given by $(J_{q,[a,b]}(A)\psi)(\xi) = \mathscr{S}(a, \xi, q)$ and $(J_{q,[a,b]}(B)\psi)(\xi) = \mathscr{H}(a, \xi, q)$, where $\mathscr{S}(t, \xi, q)$ and $\mathscr{H}(t, \xi, q)$ are the (unique) solutions of (8.3) for $a \leq t < b$, $-\infty < \xi < \infty$.*

Clearly this is a restatement of Theorem 8.

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* Unique in the class of functions whose product with $\sqrt{b-t}$ is essentially bounded.

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