

ON THE DIMENSION OF THE VECTOR SPACE $C[\theta_m]_4$

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Introduction

Let τ denote a point of the Siegel upper-half space \mathfrak{S}_g , z be a (column) vector in C^g and m a vector with m' resp. m'' in Z^g as its first resp. second entry vectors, then the series

$$\theta_m(\tau, z) = \sum_{p \in Z^g} e\left(\frac{1}{2} {}^t\left(p + \frac{m'}{2}\right)\tau\left(p + \frac{m'}{2}\right) + {}^t\left(p + \frac{m'}{2}\right)\left(z + \frac{m''}{2}\right)\right),$$

in which $e(t) = \exp(2\pi(-1)^{1/2}t)$, represents an holomorphic function on the product $\mathfrak{S}_g \times C^g$ and it is called the theta function of characteristic m . It follows from the definition

$$\theta_{m+2n}(\tau, z) = (-1)^{t m' n'} \theta_m(\tau, z), \quad \theta_m(\tau, -z) = (-1)^{t m' m''} \theta_m(\tau, z).$$

Therefore up to sign there are 2^{2g} theta functions and the function $\theta_m(\tau, z)$ is even or odd according as the characteristic m is even or odd. If we put $z = 0$ we get the so called Thetanullwerte $\theta_m = \theta_m(\tau, 0)$. It is well known that $\theta_m \neq 0$ if and only if m is even, thus we can consider the graded ring $C[\theta_m]$ generated over C by the products of Thetanullwerte. Let us consider the complex vector space $C[\theta_m]_k$, i.e. the vector space spanned over C by the monomials of degree k in the Thetanullwerte, this has finite dimension over C .

Our results are:

- i) $\dim_C C[\theta_m]_4 = \binom{2^{g-1}(2^g + 1) + 3}{4} - \sum_{i=0}^2 \mu_i(\nu_i - \pi_i)$
- ii) All the quartic relations between the Thetanullwerte are consequence of the Riemann's theta-formula.

The action of the symplectic group

For any associative ring R with 1 we shall denote by $Sp(2g, R)$ the

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group of symplectic matrices of degree $2g$ with coefficients in R . Its elements can be written as

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in which a, b, c, d are square matrices of degree g with coefficients in R satisfying $a^t d - b^t c = 1_g$, $a^t b = b^t a$ and $c^t d = d^t c$.

We have denoted by 1_g the identity matrix of degree g . For any square matrix s with (i, j) -th entry s_{ij} we shall denote by s_i the (column) vector $s_{i\cdot}$ as its i -th entry.

Let m be a vector in Z^{2g} , we recall the formula

$$\sigma \cdot m = \begin{pmatrix} d & -c \\ -b & -a \end{pmatrix} \begin{pmatrix} m' \\ m'' \end{pmatrix} + \begin{pmatrix} (c^t d)_0 \\ (a^t b)_0 \end{pmatrix}$$

defines an action of $Sp(2g, F_2)$ (i.e. $Sp(2g, Z) \bmod 2$) on the set $(F_2)^{2g}$.

For any m, m_1, m_2, m_3 in Z^{2g} we put

$$\begin{aligned} e(m) &= (-1)^{t m' m''}, & e(m_1, m_2) &= (-1)^{t m_1 m_2'' - t m_1' m_2} \\ e(m_1, m_2, m_3) &= e(m_1) e(m_2) e(m_3) e(m_1 + m_2 + m_3) = e(m_1 + m_2, m_1 + m_3) \end{aligned}$$

We observe $e(m), e(m_1, m_2), e(m_1, m_2, m_3)$ depend only on $m, m_1, m_2, m_3 \bmod 2$. We say m is even or odd according as $e(m) = 1$ or -1 .

For any m, n in $(F_2)^{2g}$ we shall denote by mn the unique element of $(F_2)^{2g}$ satisfying

$$mn \equiv m + n \pmod{2}.$$

With this law of composition $e(m, n)$ gives rise to a non-degenerate skew-symmetric bicharacter of $(F_2)^{2g} \times (F_2)^{2g}$.

A subspace N of $(F_2)^{2g}$ is totally isotropic if $e(m, n) = 1$ for every m, n in N ; in such a case the dimension k of N is at most equal to g . Moreover, fixing k , we have that the number of totally isotropic subspaces of dimension k is given by

$$\mu_k = \sum_{i=1}^k (2^{2(g-i+1)} - 1) / (2^{k-i+1} - 1)$$

cf. [7].

We say that a coset Nm in $(F_2)^{2g}$ is even or odd according as $e(ma) = 1$ or -1 for every a in N . If such a "pure coset" exists than N is necessarily totally isotropic. Furthermore if k is the dimension of N as

above the number of pure coset is $2^{2(g-k)}$ and the number of even cosets is

$$\nu_k = 2^{g-k-1}(2^{g-k} + 1)$$

cf. [4] p. 213.

Let m_1, \dots, m_r be a sequence of (column) vectors in Z^{2g} , we say that it is essentially independent if for any choice of $1 \leq i_1 < \dots < i_{2k} \leq r$, where $k \geq 1$ we have

$$\sum_{j=1}^{2k} m_{i_j} \equiv 0 \pmod{2}.$$

The point is that $e(m)$, $e(m_1, m_2, m_3)$ or $e(m_1, m_2)$ and the property of being essentially independent are $Sp(2g, Z)$ -invariant and they characterize the $Sp(2g, Z)$ -orbits in $(F_2)^{2g}$ in the following sense:

LEMMA 1. *Let $m_1, \dots, m_r \pmod{2}$ and $n_1, \dots, n_r \pmod{2}$ denote two sequences in $(F_2)^{2g}$ both with r terms, then they are conjugate under the action of $Sp(2g, F_2)$ if and only if essentially independent subsequences correspond to each other and further*

$$e(m_i) = e(n_i) \quad \text{and} \quad e(m_i, m_j, m_k) = e(n_i, n_j, n_k)$$

for all $1 \leq i \leq r$ and $1 \leq i < j < k \leq r$.

A proof of this lemma can be found in [4] p. 212.

Using this lemma we remark if $r = 4$ and $e(m_i) = 1$ for $i = 1 \dots 4$ we get exactly up to permutations 4, 8, 9 orbits when g is 1, resp 2, resp ≥ 3 . We shall enumerate them from 1 to 9 and give below a characterization and a representative of each orbit

1	$m_1 = m_2 = m_3 = m_4$	$m_i = {}^t(0 \dots 0) \quad i = 1 \dots 4$
2	$m_1 = m_2 = m_3 \neq m_4$	$(m_1, m_4) = {}^t \begin{pmatrix} 0 \dots 0 \\ 10 \dots 0 \end{pmatrix}$
3	$m_1 = m_2 \neq m_3 = m_4$	$(m_1, m_3) = {}^t \begin{pmatrix} 0 \dots 0 \\ 10 \dots 0 \end{pmatrix}$
4	$m_1 = m_2, m_i \neq m_j$ for $2 \leq i < j \leq 4$ $e(m_2, m_3, m_4) = -1$	$(m_2, m_3, m_4) = {}^t \left(\begin{array}{ccc ccc} 0 & \dots & 0 & 0 & \dots & 0 \\ 10 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 10 & \dots & 0 \end{array} \right)$
5	$m_1 = m_2, m_i \neq m_j$ for $2 \leq i < j \leq 4$ $e(m_2, m_3, m_4) = 1$	$(m_2, m_3, m_4) = {}^t \begin{pmatrix} 0 & \dots & 0 \\ 10 & \dots & 0 \\ 010 & \dots & 0 \end{pmatrix}$

<p>6</p> <p>$m_i \neq m_j$ for $1 \leq i < j \leq 4$ $m_1 + m_2 + m_3 + m_4 \equiv 0 \pmod{2}$</p>	$(m_1, m_2, m_3, m_4) = \begin{pmatrix} {}^t 0 \dots 0 \\ 10 \dots 0 \\ 010 \dots 0 \\ 110 \dots 0 \end{pmatrix}$
<p>7</p> <p>$m_i \neq m_j$ for $1 \leq i < j \leq 4$ $m_1 + m_2 + m_3 + m_4 \not\equiv 0 \pmod{2}$ $e(m_i, m_j, m_k) = -1$ for $1 \leq i < j < k \leq 4$</p>	$(m_1, m_2, m_3, m_4) = \begin{pmatrix} {}^t 0 \dots 0 & 0 \dots 0 \\ 10 \dots 0 & 0 \dots 0 \\ 0 \dots 0 & 10 \dots 0 \\ 110 \dots 0 & 110 \dots 0 \end{pmatrix}$
<p>8</p> <p>$m_i \neq m_j$ for $1 \leq i < j \leq 4$ $m_1 + m_2 + m_3 + m_4 \not\equiv 0 \pmod{2}$ $e(m_1, m_2, m_3) = e(m_1, m_2, m_4) = -1$ $e(m_1, m_3, m_4) = e(m_2, m_3, m_4) = +1$</p>	$(m_1, m_2, m_3, m_4) = \begin{pmatrix} {}^t 0 \dots 0 & 0 \dots 0 \\ 10 \dots 0 & 0 \dots 0 \\ 0 \dots 0 & 10 \dots 0 \\ 0 \dots 0 & 110 \dots 0 \end{pmatrix}$
<p>9</p> <p>$m_i \neq m_j$ for $1 \leq i < j \leq 4$ $m_1 + m_2 + m_3 + m_4 \not\equiv 0 \pmod{2}$ $e(m_i, m_j, m_k) = 1$ for $1 \leq i < j < k \leq 4$</p>	$(m_1, m_2, m_3, m_4) = \begin{pmatrix} {}^t 0 \dots 0 \\ 10 \dots 0 \\ 010 \dots 0 \\ 0010 \dots 0 \end{pmatrix}$

We have denoted by $(|)$ the division of the first entry vector from the second.

The dimension of $C[\theta_m]_4$

We know that $Sp(2g, \mathbf{R})$ acts on $\mathfrak{S}g \times C^g$ as

$$\sigma \cdot (\tau, z) = ((a\tau + b)(c\tau + d)^{-1}, {}^t(c\tau + d)^{-1}z).$$

We shall recall the transformation formula of theta functions

$$\theta_{\sigma \cdot m}(\sigma \cdot (\tau, z)) = k(\sigma) \det(c\tau + d)^{1/2} e(\phi_m(\sigma) + \frac{1}{2} {}^t z(c\tau + d)^{-1} cz) \theta_m(\tau, z)$$

in which

$$\phi_m(\sigma) = -\frac{1}{2} ({}^t m' {}^t b d m' + {}^t m'' {}^t a c m'' - 2 {}^t m' {}^t b c m - (a {}^t b)_0 (d m' - c m''))$$

We take $M = (m_1, \dots, m_r)$ arbitrarily from Z^{2g} and put

$$P(M) = \theta_{m_1} \dots \theta_{m_r}.$$

We say $P(M)$ a monomial of degree r in the Thetanullwerte. Then, if for any σ in $Sp(2g, Z)$ we put

$$\sigma \cdot M = (\sigma \cdot m_1, \dots, \sigma \cdot m_r),$$

we get holomorphic functions $P(M)$ on $\mathfrak{S}g$ satisfying

$$P(\sigma \cdot M)(\sigma \cdot \tau) = k(\sigma)^r \det(c\tau + d)^{r/2} e\left(\sum_{i=1}^r \phi_{m_i}(\sigma)\right) P(M)(\tau).$$

We know $P(M) \neq 0$ if and only if no m is odd $1 \leq i \leq r$. We recall that if m, a, b are arbitrary element of Z^{2g} then we have the Riemann's theta-formula for the Thetanullwerte:

$$\theta_m \theta_{m+a} \theta_{m+b} \theta_{m-a-b} = 2^{-g} \sum_{n \pmod 2} e(m, n) \theta_n \theta_{n+a} \theta_{n+b} \theta_{n-a-b}$$

cf. [4] p. 141.

We shall convert the formula into one with elements of $(F_2)^{2g}$ as subscripts of θ . For that purpose we shall use the symbol

$$(a, b, c) = e\left\{\frac{1}{2} \sum_{i=1}^g (a'_i b'_i c'_i + a''_i b'_i c'_i + a''_i b''_i c'_i)\right\}$$

in which a, b, c are elements of $(Z)^{2g}$ and a'_i resp a''_i are the i -th coefficients of a' and a'' . Then the theta-formula can be restated as follows

LEMMA 2. *If m, a, b are in $(F_2)^{2g}$ then we have*

$$\begin{aligned} &(m, a, a)(m, a, b)(m, b, b) \theta_m \theta_{na} \theta_{mb} \theta_{mab} \\ &= 2^{-g} \sum_{n \in (F_2)^{2g}} e(m, n) (n, a, a)(n, a, b)(n, b, b) \theta_n \theta_{na} \theta_{nb} \theta_{nab} \end{aligned}$$

A proof of this lemma can be found in [5].

From now on we shall assume $e(m) = 1$. All relation of degree k between the Thetanullwerte can be written as

$$\sum_{j=1}^n c_j P(M_j) = 0 \tag{1}$$

where $M_j = (m_1^j, \dots, m_k^j) j = 1 \dots n$.

We say the relation (1) irreducible if

- a) $c_j \neq 0$ for each j .
- b) If $\sum_{i=1}^n c_{j_i} P(M_{j_i}) = 0$ with $1 \leq j_i \leq n$ then $j_i \neq j_k$ for $i \neq k$ and $n = \bar{n}$.
- c) $P(M_j) \neq P(M_k)$ if $j \neq k$.

Clearly we are interested only in studying irreducible relations. With this notations we have the following

LEMMA 3. *Let $\sum_{j=1}^n c_j P(M_j)$ denote a irreducible relation, the integral symmetric matrices of degree $2g$ $M_j {}^t M_j$ and $M_i {}^t M_i$ are related as follows:*

$$\begin{aligned} M_j {}^t M_j &\equiv M_i {}^t M_i \pmod 2 \\ (M_j {}^t M_j)_0 &\equiv (M_i {}^t M_i)_0 \pmod 4. \end{aligned} \tag{2}$$

Moreover (2) is preserved under the action of $Sp(2g, \mathbb{Z})$.

The proof of this lemma is similar to Lemma 2 in [3] and Lemma 2 in [6].

Now we are in the conditions of studying the quartic relations between the Thetanullwerte. They are completely described by the following

LEMMA 4. *There are not irreducible quartic relations in which occur monomials $P(M)$ with $m_1 + m_2 + m_3 + m_4 \not\equiv 0 \pmod{2}$. Moreover the quartic relations are of the form:*

- (i) $\sum_{m \text{ even}} c_m \theta_m^4$
- (ii) $\sum_{m \text{ even}} c_m \theta_m^2 \theta_{mn}^2 \quad n \neq 0 \text{ fixed} \tag{3}$
- (iii) $\sum_{m \text{ even}} c_m \theta_m \theta_{ma} \theta_{mb} \theta_{mab} \quad a \neq 0, \quad b \neq 0, \quad ab \neq 0 \text{ a, b fixed.}$

Proof. Let us assume $\sum_{j=1}^n c_j P(M_j) = 0$ and $m_1^1 + m_2^1 + m_3^1 + m_4^1 \not\equiv 0 \pmod{2}$ Applying the above lemma we can assume M_1 equal to one of the representative of the orbits 2, 4, 5, 7, 8, 9. We study only case 8, because all the others are similar.

Thus we can assume

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 \\ \hline \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Then

$$M_1 {}^t M_1 \equiv \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 1 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & \cdot & 0 \end{pmatrix} \pmod{2} \text{ and } (M_1 {}^t M_1)_0 \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \hline 2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{4} .$$

This implies that if M_1 and M_j satisfy (2) then $M_1 = M_j$, up to a permutation of the columns, thus $P(M_1) = P(M_j)$. The second part of the lemma can be proved in the same way.

We remark that the relations in (3) can be written as

$$\sum_{N^m \text{ even}} c_m \prod_{n \in Nm} \theta_n^e$$

where N is a fixed totally isotropic subspace of dimension $S = 0, 1, 2$ and $2^s e = 4$.

We want to compute the dimension of the C -vector space $C[\theta_m]_4$. Let T_g be the C -vector space spanned by the θ_m^4 , we know

$$\dim_C T_g = (2^g + 1)(2^{g-1} + 1)/3 = \pi_0 \quad \text{cf. [8].}$$

Always from [8] we deduce that the dimension of the vector space spanned by $\theta_m^2 \theta_{ma}^2$, a fixed and $a \neq 0$, is $(2^{g-1} + 1) \cdot (2^{g-2} + 1)/3 = \pi_1$ and the dimension of the vector space spanned by $\theta_m \theta_{ma} \theta_{mb} \theta_{mab}$, a, b fixed, $a \neq b$, $a \neq 0$, $b \neq 0$ is $(2^{g-2} + 1) \cdot (2^{g-3} + 1)/3 = \pi_2$. We recall we have $\mu_i \nu_i$ monomials of the form $\theta_m^2 \theta_{ma}^2$, $a \neq 0$ such that $e(m) = e(ma) = 1$, thus the dimension of the vector space spanned by these monomials is

$$\mu_1(2^{g-1} + 1)(2^{g-2} + 1)/3 = \mu_1 \pi_1.$$

Moreover the monomials of the form $\theta_m \theta_{ma} \theta_{mb} \theta_{mab}$, $a \neq 0$, $b \neq 0$, $ab \neq 0$ such that $e(m) = e(ma) = e(mb) = e(mab) = 1$ are $\mu_2 \nu_2$, thus the dimension of the vector space spanned by these monomials is

$$\mu_2(2^{g-2} + 1)(2^{g-3} + 1) = \mu_2 \pi_2.$$

Finally we know that all monomials $P(M)$ with $m_1 + m_2 + m_3 + m_4 \not\equiv 0 \pmod{2}$ are linearly independent and they span a vector space of dimension

$$\binom{2^{g-1}(2^g + 1) + 3}{4} - \sum_{i=0}^2 \mu_i \nu_i.$$

If ℓ is a positive integer we shall denote by $\Gamma_g(\ell)$ the normal subgroup of $Sp(2g, Z)$ defined by $a \equiv d \equiv 1_g$, $b \equiv c \equiv 0 \pmod{\ell}$; we shall further denote by $\Gamma_g(\ell, 2\ell)$ the subgroup of $\Gamma_g(\ell)$ defined by $(a^t b)_0 \equiv (c^t d)_0 \equiv 0 \pmod{2\ell}$.

It is a well known fact that

$$\theta_m(\sigma \cdot \tau) = \det(c\tau + d)^{1/2} \theta_m(\tau)$$

for every σ in $\Gamma_g(4, 8)$, then

$$P(M)(\sigma \cdot \tau) = \det(c\tau + d)^2 P(M)(\tau) \quad (4)$$

for $M = (m_1, m_2, m_3, m_4)$.

We have learned from J. Igusa [2] the following facts:

(a) (4) holds for every σ in $\Gamma_g(4)$ if and only if

$$(M^t M)_0 \equiv 0 \pmod{2}$$

(b) (4) holds for every σ in $\Gamma_g(2, 4)$ if and only if

$$(M^t M) \equiv 0 \pmod{2}$$

(c) (4) holds for every σ in $\Gamma_g(2)$ if and only if M satisfies (2).

Now let Γ be a subgroup of $Sp(2g, \mathbf{Z})$ such that $\Gamma \supseteq \Gamma_g(4, 8)$ then we can define $(C[\theta_m]_4)^\Gamma$ as the vector space spanned by the quartic polynomials invariant with respect to Γ . Moreover, always in [2] is proved that if $\Gamma \subseteq \Gamma_g(2)$ then a polynomial is invariant with respect to Γ if each of the monomials occurring in the sum is invariant too. Thus we proved the following

THEOREM 1. (i) $\dim_{\mathbf{C}} (C[\theta_m]_4)^{\Gamma_g(2)} = \pi_0$

(ii) $\dim_{\mathbf{C}} (C[\theta_m]_4)^{\Gamma_g(2,4)} = \pi_0 + \mu_1 \pi_1$

(iii) $\dim_{\mathbf{C}} (C[\theta_m]_4)^{\Gamma_g(4)} = \pi_0 + \mu_1 \pi_1 + \mu_2 \pi_2$

(iv) $\dim_{\mathbf{C}} C[\theta_m]_4 = q - \sum_{i=0}^2 \mu_i (\nu_i - \pi_i)$

where $q = \binom{2^{g-1}(2^g + 1) + 3}{4}$.

We remark that $(C[\theta_m]_4)^{\Gamma_g(1)} = 0$ since up to a constant factor an element of this vector space has to be the symmetrization of θ_0^4 that is 0.

Now we want to write all quartic relations using the Riemann's theta formula. Let us introduce the symmetric 2^{2g} by 2^{2g} matrix of signs

$$K_g = \begin{pmatrix} K_g^+ & L \\ {}^t L & K_g^- \end{pmatrix} = e(m_i, m_j) \quad i, j = 1 \dots 2^{2g}$$

formed with respect to any fixed ordering of m_i in $(\mathbf{F}_2)^{2g}$ such that $K_g^+(K_g^-)$ is the matrix for $e(\cdot, \cdot)$ on the subset of the even (odd) characteristics.

We learned from [1] the following facts.

LEMMA 5. K_g has two eigenspaces of dimension $l_g^\pm = 2^{g-1}(2^g \pm 1)$ with eigenvalues $\pm 2^g$ respectively, while K^\pm has eigenspaces of dimension $(2^g \pm 1)(2^{g-1} \pm 1)/3$ and $(2^{2g} - 1)/3$ with eigenvalues $\pm 2^g$ and $\mp 2^{g-1}$ respectively.

If $K_g^+ v = -2^{g-1} v$ for $v = (v_i)$ in $C^{l_g^+}$, then from the Riemann's theta formula we get

$$\sum_{m \in K_g^+} v_m(m, a, a)(m, a, b)(m, b, b)\theta_m\theta_{m_a}\theta_{m_b}\theta_{m_{ab}} = 0.$$

Clearly the coefficients v are defined on the even cosets Nm with $N = \{a, a, b, ab\}$. Thus we can think of v_m as a -2^{g-k-1} eigenvector of K_{g-k}^+ , where $k = \dim N$.

Now counting the relations that we obtain, we can see that these are all quartic relations between the Thetanullwerte. Thus we get

PROPOSITION 1. *All the quartic relations between the Thetanullwerte are consequence of the Riemann's theta formula.*

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