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INTEGRABLE DERIVATIONS IN RINGS OF UNEQUAL CHARACTERISTIC

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In this paper we will investigate integrability of derivations in rings of unequal characteristic. Generally speaking this case is more difficult than the case of characteristic p which was studied in [4]. New interesting phenomena arise, cf. e.g. Theorem 1 and Example 3. Although we obtain useful sufficient conditions for integrability only in very restricted cases, we also give several examples to illuminate the situation.

In this work all rings are assumed to be commutative with a unit element. A local ring will mean a noetherian local ring. Let A be a ring. The set of all derivations of A into itself is denoted by $\mathrm{Der}(A)$. If k is a subring of A, the submodule of $\mathrm{Der}(A)$ consisting of the derivations which vanish on k is denoted by $\mathrm{Der}_k(A)$.

DEFINITION 1. A differentiation $\underline{D}$ of A is a sequence $\underline{D} = \{D_0 = 1, D_1, D_2, \cdots\}$ of additive endomorphisms $D_i : A \to A$ such that

$$D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b) \qquad \text{for every } n\,.$$

The set of differentiations of A is denoted by $HS(A)$. Let t be an indeterminate and put

$$E(a) = \sum_{n=0}^{\infty} D_n(a)t^n \in A[[t]]\,.$$

Then E is a ring homomorphism of A into $A[[t]]$ such that $a \equiv E(a) \bmod t$ for every $a \in A$, and conversely any such homomorphism E comes from a differentiation of A. Sometimes we identify E with $\underline{D}$.

DEFINITION 2. Let k be a subring of A. A differentiation $\underline{D}$ of A is called a differentiation of A over k if $D_i(a) = 0$ for all $i > 0$ and for all

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$a \in k$. The set of such differentiations is denoted by $HS_k(A)$; this is a subgroup of $HS(A)$, the group law in $HS(A)$ being defined as follows. If $\underline{D} = \{1, D_1, D_2, \cdots\}$ and $\underline{D}' = \{1, D_1', D_2', \cdots\}$, then

$$\underline{D}\cdot\underline{D}' = \{1, D_1 + D_1', D_2 + D_1D_1' + D_2', \cdots, \sum_{i=1}^{n} D_iD'_{n-i}, \cdots\}.$$

DEFINITION 3. A differentiation $\underline{D}$ is said to be *iterative* if:

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j} \qquad \text{for all } i, j.$$

DEFINITION 4. Two differentiations $\underline{D} = \{1, D_1, D_2, \cdots\}$ and $\underline{D}' = \{1, D_1', D_2', \cdots\}$ of A are said to commute with each other if $D_mD_n' = D_n'D_m$ for all m, n.

If $\underline{D}$ and $\underline{D}'$ are iterative and commute with each other, then their product $\underline{D}\cdot\underline{D}'$ is again iterative ([4, § 1]).

DEFINITION 5. We say that a derivation $D \in \mathrm{Der}(A)$ is *integrable* if there exists a differentiation $\underline{D} = \{1, D_1, D_2, \cdots\}$ of A with $D_1 = D$. We also say that $\underline{D}$ lifts D. We denote the set of the integrable derivations by $\mathrm{Ider}(A)$.

DEFINITION 6. Let k be a subring of A. A derivation D of A is said to be integrable over k if it can be lifted to some $\underline{D} \in HS_k(A)$.

The set of the derivations of A which are integrable over k is denoted by $\mathrm{Ider}_k(A)$; this is not necessarily equal to $\mathrm{Der}_k(A) \cap \mathrm{Ider}(A)$.

DEFINITION 7. We say that a derivation D is *strongly integrable* if there exists an iterative differentiation $\underline{D}$ with $D_1 = D$.

If the ring A is of characteristic zero and every non-zero integer is a non-zero-divisor in A, then an iterative differentiation $\underline{D}$ must satisfy $D_n = D_1^n/n!$ (all $n > 0$). Thus, if a derivation D is strongly integrable the iterative differentiation which lifts D is unique, although there are (in general) many non-iterative ones which lift D. If the ring A contains the rational number field $\boldsymbol{Q}$, then every derivation D of A is strongly integrable: the differentiation

$$\underline{D} = \{1, D, D^2/2!, D^3/3!, \cdots, D^n/n!, \cdots\}$$

is iterative and lifts D.

If the characteristic of A is a prime number p, a differentiation $\underline{D} =$

$\{1, D_1, D_2, \cdots\}$ is iterative iff, putting $\delta_0 = D_1$, $\delta_1 = D_p, \cdots, \delta_i = D_{p^i}, \cdots$, we have

i) $\delta_i\delta_j = \delta_j\delta_i$ for all i, j,

ii) $\delta_i^p = 0$ for all i, and

iii) for every $n > 0$, $D_n = \delta_0^{n_0}\delta_1^{n_1}\cdots\delta_r^{n_r}/(n_0!n_1!\cdots n_r!)$ where $n = n_0 + n_1p + \cdots + n_rp^r$ $(0 \leqslant n_i < p)$ is the p-adic expansion of n.

Thus, in order that a derivation D of A is strongly integrable, it is necessary that $D^p = 0$. This condition is also sufficient if A is a field ([4, Th. 7]), but not in general. Moreover, if D is strongly integrable the iterative differentiations which lift D are usually not unique.

The unequal characteristic case is more complicated. Let A be a ring and p be a prime number. Assume

$$(*)\quad \begin{cases} (1) & p \text{ is neither zero nor unit in } A, \text{ and} \\ (2) & \text{all prime numbers other than } p \text{ are units in } A. \end{cases}$$

(The most important example is the case where A is a local ring of characteristic zero with residue field of characteristic p.) Put $\overline{A} = A/pA$. Since every derivation (or differentiation) D of A is trivial on the prime subring, it induces a derivation (resp. a differentiation) $\overline{D}$ of $\overline{A}$. Thus

$$\begin{array}{ccc} D \text{ is integrable} & \Longrightarrow & \overline{D} \text{ is integrable} \\ \Uparrow & & \Uparrow \\ D \text{ is strongly integrable} & \Longrightarrow & \overline{D} \text{ is strongly integrable.} \end{array}$$

There are examples of non-integrable derivations. In fact, in Example 2 of [4] both D and $\overline{D}$ are non-integrable. So far we have not succeeded in finding examples of non-integrable D such that $\overline{D}$ is integrable. But we have the following examples:

Example 1. Let (R, pR, k) be a valuation ring of characteristic zero and let $A = R[[X]]$, $\overline{A} = A/pA = k[[X]]$. Let $D_0 := d/dX \in \mathrm{Der}_R(A)$, $D := XD_0$, and let $\overline{D}_0$, $\overline{D}$ be the induced derivations of $k[[X]]$ over k. Then $\overline{D}(X) = X$, hence $\overline{D}^p \neq 0$. Therefore $\overline{D}$ is not strongly integrable, hence D is not strongly integrable. On the other hand, D is integrable because D_0 is so. $\{1, D_0, D_0^2/2!, D_0^3/3!, \cdots\}$ is the iterative differentiation which lifts D_0, therefore $\{1, D, X^2D_0^2/2!, X^3D_0^3/3!, \cdots\}$ is a differentiation which lifts D, but it is not iterative.

Example 2. Let $R = Z[X, Y]$ and $A = R_{2R}$. Thus A is a DVR with 2 as prime element. Therefore $\overline{A} = A/2A$ is a field of characteristic 2.

Consider the derivation $D \in \mathrm{Der}(A)$ defined by $D(X) = Y^2$ and $D(Y) = X^2$. Then $\bar{D}^2 = 0$, hence $\bar{D}$ is strongly integrable by [4, Th. 7]. But D is not so, because $D^4(X) = 4Y^5 + 20X^3Y^2$ is not divisible by $4!$ in A. On the other hand, A is smooth over Z and every derivation of A is integrable.

THEOREM 1. *Let A be a ring of characteristic zero satisfying the condition* (*) *stated above. Assume that p is not a zero-divisor in A. Then every derivation D of A such that $D(A) \subseteq pA$ is strongly integrable.*

Proof. Since the positive integers are non-zero-divisors in A by assumption, the total ring of quotients K of A contains Q. Every derivation D of A is naturally extended to a derivation of K, and as such it can be lifted to the iterative differentiation $D = \{1, D, D^2/2!, \cdots, D^n/n!, \cdots\}$ of K. Suppose that D maps A into pA. Then $D^n(A) \subseteq p^nA$. On the other hand, denoting the normalized p-adic valuation by $v_p(\ )$, we have

$$\begin{aligned} v_p(n!) &= [n/p] + [n/p^2] + [n/p^3] + \cdots \\ &< n(p^{-1} + p^{-2} + \cdots) = np^{-1}/(1 - p^{-1}) = n/(p-1) \leqslant n\,. \end{aligned}$$

Thus $D^n/n!$ maps A into A, and so D (as a derivation of A) is strongly integrable. Q.E.D.

In the case of regular local rings of unequal characteristic, we can derive integrability of derivations from smoothness just as in [4, § 3]. We need the following lemma.

LEMMA 1 ([5, Th. 8.4]). *Let (A, m) be a local ring, k be a DVR (discrete valuation ring) with prime element u, and $f: k \to A$ be a local homomorphism. Assume that A/m is separable over k/uk. Then A is formally smooth over k (with respect to the maximal ideal) iff* (1) *f is injective,* (2) *A is regular, and* (3) *$f(u) \notin m^2$.*

Proof. This is an easy consequence of [3, (19.7.1)]. For a direct proof, see [5].

In the following we use the homology and the cohomology of André [1].

LEMMA 2. *Let (A, m) be a regular local ring of characteristic zero and k be a subring which is a DVR with prime element u such that $m \cap k = uk$, $u \notin m^2$. Assume moreover that the residue field k/uk is perfect. Then:*

(i) *for every $P \in \mathrm{Spec}(A)$ the local ring A_P is formally smooth over k;*

(ii) *$H_i(k, A, M) = 0$ for all A-modules M and for all $i > 0$;*

(iii) $H_0(k, A, A) = \Omega_{A/k}$ *is A-flat;*

(iv) $H^i(k, A, M) = \mathrm{Ext}^i_A(H_0(k, A, A), M)$ *for all A-modules M and for all* $i \geqslant 0$.

Proof. (i) The local ring A_P is regular, and $u \notin P^2 A_P$. Therefore, if $u \in P$ then A_P is formally smooth over k by Lemma 1. If $u \notin P$, then A_P contains the quotient field K of k. Since K is of characteristic zero, the regular local ring A_P is formally smooth over K.

(ii), (iii), (iv) are derived from (i) as in the proof of [4, Lemma 5].

THEOREM 2. *Let (A, m) be a regular ring of characteristic zero with residue field of characteristic p, and let k be a subring of A. Assume that*

(1) *k is a DVR with prime element u,*

(2) $u \in m$, $u \notin m^2$,

(3) *the residue field k/uk of k is perfect, and*

(4) $\Omega_{A/k}$ *is a finite A-module.*

Then A is smooth (i.e. formally smooth with respect to the discrete topology) over k; consequently, we have

$$\mathrm{Der}_k(A) = \mathrm{Ider}_k(A)\,.$$

Proof. The module of differentials $\Omega_{A/k}$ is A-flat by Lemma 2 and finite by assumption. Hence it is free. Therefore $H^1(k, A, M) = \mathrm{Ext}^1_A(\Omega_{A/k}, M) = 0$ for every A-module M. This means smoothness ([1] p. 223 Prop. 17 and p.222 Def. 14). The last assertion follows from this by [4, Th. 8].

COROLLARY. *Let (A, m) be a regular local ring of characteristic zero with residue field of characteristic p. Assume that A is unramified (i.e. $p \notin m^2$) and that Ω_A is finite over A. Then we have* $\mathrm{Der}(A) = \mathrm{Ider}(A)$.

Proof. Put $k := Z_{pZ}$ in the theorem.

THEOREM 3. *Let (A, m) be a complete regular local ring of characteristic zero with residue field of characteristic p, and let k be a subring satisfying* (1), (2) *of the preceding theorem. Assume that A/m is separable over k/uk. Then we have*

$$\mathrm{Der}_k(A) = \mathrm{Ider}_k(A)\,.$$

Proof. Since A is formally smooth over k by Lemma 1 and complete by assumption, the proof of [4, Th. 10] applies to this case without change. (Cf. also [2] p. 104 Prop. 25.2.)

COROLLARY. *Let (A, m) be a regular local ring of unequal characteristic which is complete and unramified. Then* $\mathrm{Der}(A) = \mathrm{Ider}(A)$.

Remark. The assumption of unramifiedness cannot be omitted, as the following example shows.

EXAMPLE 3. Let $C = (Z_{pZ})^{\wedge}$ be the ring of p-adic integers and x be an indeterminate. Put $S = C[[x]]$, $R = S[[U]]/(U^p - p(x+1)) = S[u]$. Then $R/(u, x) = C[[x, U]]/(x, U, p) = C/pC$, hence R is a complete regular local ring. The derivation $U^{p-1}\partial/\partial x + \partial/\partial U$ of $C[[x, U]]$ induces a derivation D of R such that $D(x) = u^{p-1}$, $D(u) = 1$. The derivation $\bar{D}$ of R/pR is not integrable since $\bar{u}^p = 0$ in R/pR. Therefore D is not integrable either.

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