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FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS OF DEGREE TWO

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Our purpose is to prove the following

THEOREM. *Let k be an even integer ≥ 6. Let*

$$f(Z) = \sum a(T)e(\operatorname{tr} TZ)$$

be a Siegel cusp form of degree two, weight k. Then we have

$$a(T) = O(|T|^{k/2-1/4+\varepsilon}) \qquad \text{for any } \varepsilon > 0.$$

This was announced in [3] where we put an assumption on estimates of generalized Kloosterman sums. Here, we give a complete proof with a proof of that assumption.

Every cusp form of degree two, weight $k \geq 6$ ($k \equiv 0 \bmod 2$) is a linear combination of Poincaré series [1,4]. Using their rather formal Fourier expansion given in [1], we prove our theorem.

Notation. By $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ we denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, repsectively. H denotes the upper half-plane of genus two:

$$H = \{Z = X + iY \in M_2(\boldsymbol{C}) \mid {}^tZ = Z, \operatorname{Im} Z = Y > 0\}.$$

We set $\Gamma = Sp_2(\boldsymbol{Z}) = \{M \in M_4(\boldsymbol{Z}) \mid {}^tMJM = J\}$ where $J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}$, $\Lambda = \{S \in M_2(\boldsymbol{Z}) \mid {}^tS = S\}$, and $\Lambda^* = \{S = (s_{ij}) \in M_2(\boldsymbol{Q}) \mid s_{ii} \in \boldsymbol{Z}, 2s_{12} = 2s_{21} \in \boldsymbol{Z}\}$. $e(Z)$ means $\exp(2\pi iz)$ for a complex number z.

§ 1.

In this section we prepare two arithmetic lemmas.

Let $C \in M_2(\boldsymbol{Z})$, $|C| \neq 0$. For P, $T \in \Lambda^*$, we set

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$$K(P, T; C) = \sum_{D} e(\mathrm{tr}(AC^{-1}P + C^{-1}DT)),$$

where D runs over $\{D \in M_2(Z) \bmod C\Lambda \mid \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma\}$ and $A \in M_2(Z)$ is any matrix such that $\begin{pmatrix} A & * \\ C & D \end{pmatrix} \in \Gamma$. If $D' \equiv D \bmod C\Lambda$, and $\begin{pmatrix} A' & * \\ C & D' \end{pmatrix} \in \Gamma$, then we have $A' \equiv A \bmod \Lambda C$. Hence a generalized Kloosterman sum $K(P, T; C)$ is well-defined. One of our aims in this section is to prove

PROPOSITION 1. *Let $C \in M_2(Z)$, $|C| \neq 0$ and $C = U^{-1}\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} V^{-1}$, $U, V \in GL(2, Z)$, $0 < c_1 | c_2$. Then we have*

$$K(P, T; C) = O(c_1^2 c_2^{1/2+\varepsilon}(c_2, t)^{1/2}) \qquad \text{for } P, T \in \Lambda^*,$$

where ε is any positive number and t is the $(2, 2)$-entry of $T[V]$. Moreover $K(P, T; C) = K(T, P; {}^tC)$ holds.

LEMMA 1. *Let $C = \begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix}$, $c_1 | c_2$, $c_i > 0$ and $C = FH$ where $F = \begin{pmatrix} f_1 & \\ & f_2 \end{pmatrix}$, $H = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}$, $f_1 | f_2$, $h_1 | h_2$, $f_i, h_i > 0$, $(f_2, h_2) = 1$, For integers s, t with $sf_2 + th_2 = 1$, we set $X_1 = sf_2F^{-1}$, $X_2 = th_2H^{-1}$. Then $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ if and only if $\begin{pmatrix} HA & HB - X_1{}^tAD \\ F & X_2D \end{pmatrix}$, $\begin{pmatrix} FA & FB - X_2{}^tAD \\ H & X_1D \end{pmatrix} \in \Gamma$.*

Proof. We note that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(Z)$ is in Γ if and only if ${}^tAD - {}^tCB = 1_2$, and ${}^tAC$, ${}^tBD$ are symmetric. The "only if"-part is proved directly. The "if"-part follows immediately from

$$A = X_2(HA) + X_1(FA),\ D = H(X_2D) + F(X_1D) \text{ and}$$
$$B = 2X_1X_2{}^tAD + X_1(FB - X_2{}^tAD) + X_2(HB - X_1{}^tAD).$$

LEMMA 2. *Let C, F, H, X_1 and X_2 be those in Lemma 1. The mapping $D \bmod C\Lambda \mapsto (X_2D \bmod F\Lambda,\ X_1D \bmod H\Lambda)$ from $\{D \bmod C\Lambda \mid \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma\}$ to $\{D \bmod F\Lambda \mid \begin{pmatrix} * & * \\ F & D \end{pmatrix} \in \Gamma\} \times \{D \bmod H\Lambda \mid \begin{pmatrix} * & * \\ H & D \end{pmatrix} \in \Gamma\}$ is bijective.*

Proof. The mapping is obviously well-defined. Suppose that $X_2D_1 \equiv X_2D_2 \bmod F\Lambda$, $X_1D_1 \equiv X_1D_2 \bmod H\Lambda$, then we have $X_2D \in F\Lambda$, $X_1D \in H\Lambda$ where $D = D_1 - D_2$. Hence $D = H(X_2D) + F(X_1D) \in C\Lambda$ follows from $FH = HF = C$. Conversely suppose $\begin{pmatrix} * & * \\ F & D_1 \end{pmatrix}$, $\begin{pmatrix} * & * \\ H & D_2 \end{pmatrix} \in \Gamma$. We set $D = HD_1 + FD_2$. Then $X_2D - D_1 = F(th_2H^{-1}D_2 - sf_2F^{-1}D_1) \in F\Lambda$ and $X_1D - D_2 = H(sf_2F^{-1}D_1 - th_2H^{-1}D_2) \in H\Lambda$ imply the surjectiveness of the mapping if $\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma$. To show $\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma$ we have only to prove that $C^{-1}D$ is symmetric and $(C, D) \in M_{2,4}(Z)$ is primitive. The first follows from $C^{-1}D = F^{-1}D_1 + H^{-1}D_2$.

If a prime p does not divide $c_2 = f_2h_2$, then C is in $GL_2(Z_p)$. If $p|h_2$, then $\mathrm{rk}((C, D) \bmod p) = \mathrm{rk}\left(\left(\begin{pmatrix} f_1h_1 & \\ & 0\end{pmatrix}, \begin{pmatrix} h_1 & \\ & 0\end{pmatrix}D_1 + FD_2\right) \bmod p\right) = \mathrm{rk}\left(\left(\begin{pmatrix} h_1 & \\ & 0\end{pmatrix}, FD_2\right) \bmod p\right) = \mathrm{rk}((H, D_2) \bmod p) = 2$. Similarly for $p|f_2$, we have $\mathrm{rk}((C, D) \bmod p) = 2$. Thus (C, D) is locally and hence globally primitive.

LEMMA 3. *Let C, F, H, X_1 and X_2 be those in Lemma 1. Then $K(P, T; C) = K(P[X_2], T; F)K(P[X_1], T; H)$ holds.*

Proof. Suppose $\begin{pmatrix} A & * \\ C & D\end{pmatrix} \in \Gamma$. Then we have

$$\begin{aligned}
&\mathrm{tr}(AC^{-1}P + C^{-1}DT)\\
&\quad = \mathrm{tr}((X_2HA + X_1FA)F^{-1}H^{-1}P + F^{-1}H^{-1}(HX_2D + FX_1D)T)\\
&\quad = \mathrm{tr}(X_2HAF^{-1}H^{-1}P + F^{-1}X_2DT) + \mathrm{tr}(X_1FAF^{-1}H^{-1}P + H^{-1}X_1DT)\\
&\quad = \mathrm{tr}(X_2HAF^{-1}(X_1F + X_2H)H^{-1}P + F^{-1}X_2DT)\\
&\qquad + \mathrm{tr}(X_1FAF^{-1}(X_1F + X_2H)H^{-1}P + H^{-1}X_1DT)\\
&\quad = \mathrm{tr}(HAF^{-1}P[X_2] + F^{-1}X_2DT) + \mathrm{tr}(FAH^{-1}P[X_1] + H^{-1}X_1DT)\\
&\qquad + \mathrm{tr}(X_2HAX_1H^{-1}P + X_1FAF^{-1}X_2P).
\end{aligned}$$

Moreover we have $X_2HAX_1H^{-1} = th_2AX_1H^{-1} = AX_1X_2 = sf_2th_2AC^{-1} \in \Lambda$ and

$$X_1FAF^{-1}X_2 = sf_2AF^{-1}X_2 = AX_1X_2 \in \Lambda.$$

Thus $\mathrm{tr}(AC^{-1}P + C^{-1}DT) \equiv \mathrm{tr}(HAF^{-1}P[X_2] + F^{-1}X_2DT) + \mathrm{tr}(FAH^{-1}P[X_1] + H^{-1}X_1DT) \bmod 1$ follows for $P, T \in \Lambda^*$, and then Lemmas 1, 2 complete the proof.

LEMMA 4. *Let p be a prime and $0 \leq e_1 \leq e_2$. Then we have*

$$K\left(P, T; \begin{pmatrix} p^{e_1} & \\ & p^{e_2}\end{pmatrix}\right) = O(p^{2e_1+e_2/2}(p^{e_2}, t)^{1/2})$$

where t is the (2,2)-entry of T.

*Proof.**) Put $C = \begin{pmatrix} p^{e_1} & \\ & p^{e_2}\end{pmatrix}$, $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4\end{pmatrix}$. Since $\begin{pmatrix} * & * \\ C & D\end{pmatrix} \in \Gamma$ if and only if $C^{-1}D$ is symmetric and (C, D) is primitive, $\begin{pmatrix} * & * \\ C & D\end{pmatrix} \in \Gamma$ if and only if $d_3 = p^{e_2-e_1}d_2$ and (i) $e_1 = e_2 = 0$, (ii) $e_1 = 0$, $e_2 > 0$, $p \nmid d_4$, (iii) $0 < e_1 < e_2$, $p \nmid d_1d_4$ or (iv) $0 < e_1 = e_2$, $p \nmid (d_1d_4 - d_2^2)$.

$D \bmod C\Lambda$ is equivalent to d_1, $d_2 \bmod p^{e_1}$, $d_4 \bmod p^{e_2}$. Suppose that

*) This proof was suggested by Prof. Y.-N. Nakai.

$\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma$ and ${}^tAC$ is symmetric, $(A\,{}^tD - 1)C^{-1} \in M_2(Z)$ for $A \in M_2(Z)$. Then we have $\begin{pmatrix} A & (A\,{}^tD - 1)C^{-1} \\ C & D \end{pmatrix} \in \Gamma$ since $A\,{}^tD - \{(A\,{}^tD - 1)C^{-1}\}\,{}^tC = 1_2$, $A\,{}^t\{(A\,{}^tD - 1)C^{-1}\}$ and $C\,{}^tD$ are symmetric. Set $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix}$, $T = \begin{pmatrix} t_1 & t_2/2 \\ t_2/2 & t_4 \end{pmatrix} \in \Lambda^*$. When $e_1 = e_2 = 0$, the lemma is obvious. Suppose $e_1 = 0 < e_2$. Then we may suppose $D = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ $(d \bmod p^{e_2}, p \nmid d)$. Denoting by $\bar{d}$ an integer n which satisfies $nd \equiv 1 \bmod p^{e_2}$, we can take $\begin{pmatrix} 0 & 0 \\ 0 & \bar{d} \end{pmatrix}$ as A. Thus $K(P, T; C) = \sum_{\substack{d \bmod p^{e_2} \\ p \nmid d}} e((\bar{d}p_4 + dt_4)p^{-e_2})$ is an ordinary Kloosterman sum and the lemma holds in this case. Suppose $0 < e_1 < e_2$. Let d be an integer such that $d(d_1d_4 - p^{e_2-e_1}d_2^2) \equiv 1 \bmod p^{e_2}$ and set $A = d\begin{pmatrix} d_4 & -p^{e_2-e_1}d_2 \\ -d_2 & d_1 \end{pmatrix}$. Then ${}^tAC$ is symmetric and $(A\,{}^tD - 1)C^{-1} \in M_2(Z)$. Hence we have

$$\begin{aligned} K(P, T; C) = \sum_{\substack{d_1, d_2 \bmod p^{e_1} \\ d_4 \bmod p^{e_2} \\ p \nmid d_1 d_4}} & e(d(d_4p_1p^{-e_1} - d_2p_2p^{-e_1} + d_1p_4p^{-e_2}) \\ & + d_1t_1p^{-e_1} + d_2t_2p^{-e_1} + d_4t_4p^{-e_2}). \end{aligned}$$

Set $\delta = d_1d_4 - p^{e_2-e_1}d_2^2$, then $d\delta \equiv 1 \bmod p^{e_2}$ and $d_4 \equiv \bar{d}_1\delta + p^{e_2-e_1}\bar{d}_1d_2^2 \bmod p^{e_2}$ where $d_1\bar{d}_1 \equiv 1 \bmod p^{e_2}$. Then $K(P, T; C)$ equals

$$\begin{aligned} & \sum_{\substack{d_1, d_2 \bmod p^{e_1} \\ p \nmid d_1}} e(d_1t_1p^{-e_1} + d_2t_2p^{-e_1} + \bar{d}_1p_1p^{-e_1} + \bar{d}_1d_2^2t_4p^{-e_1}). \\ & \sum_{\substack{\delta \bmod p^{e_2} \\ p \nmid \delta}} e(\{(d_1p_4 - p^{e_2-e_1}d_2p_2 + p^{2e_2-2e_1}\bar{d}_1d_2^2p_1)d + \bar{d}_1t_4\delta\}p^{-e_2}) \\ & \quad = O(p^{2e_1+e_2/2}(t_4, p^{e_2})^{1/2}), \end{aligned}$$

since the last sum on δ is an ordinary Kloosterman sum.

Suppose $0 < e_1 = e_2 = e$. Set $\delta = d_1d_4 - d_2^2$ and let d be an integer such that $d\delta \equiv 1 \bmod p^e$. Then we can take $d\begin{pmatrix} d_4 & -d_2 \\ -d_2 & d_1 \end{pmatrix}$ as A. Thus $K(P, T; C)$ equals

$$\sum_{\substack{d_1, d_2, d_4 (p^e) \\ p \nmid \delta}} e(\{d(d_4p_1 - d_2p_2 + d_1p_4) + d_1t_1 + d_2t_2 + d_4t_4\}p^{-e}) = \Sigma_1 + \Sigma_2,$$

where d_2 in Σ_1 is supposed to be $p \mid d_2$ and d_2 in Σ_2 is supposed to be $p \nmid d_2$. We have $\Sigma_1 = O(p^{2e-1+e/2}(t_4, p^e)^{1/2})$ quite similarly to the case (iii). Now we estimate Σ_2. We define integers δ_1, δ_4, $\bar{\delta}$ by $d_1 \equiv d_2\delta_1$, $d_4 \equiv d_2\delta_4$, $\bar{\delta}(\delta_1\delta_4 - 1) \equiv 1 \bmod p^e$; then $\bar{\delta} \equiv d_2^2 d \bmod p^e$, and Σ_2 equals

$$\sum_{\substack{d_2 (p^e) \\ p \nmid d_2}} \sum_{\substack{\delta_1, \delta_4 (p^e) \\ p \nmid (\delta_1\delta_4 - 1)}} e(\{\bar{\delta}\bar{d}_2(\delta_4p_1 - p_2 + \delta_1p_4) + (\delta_1t_1 + t_2 + \delta_4t_4)d_2\}p^{-e}),$$

where $\bar{d}_2$ is an integer such that $d_2\bar{d}_2 \equiv 1 \bmod p^e$. Hence we have

$$\begin{aligned}\Sigma_2 &= O\Big(\sum_{\substack{\delta_1,\delta_4(p^e)\\ p\nmid(\delta_1\delta_4-1)}} p^{e/2}(\delta_1 t_1 + t_2 + \delta_4 t_4, p^e)^{1/2}\Big)\\ &= O\left(p^{e/2}\sum_{x(p^e)}(x,p^e)^{1/2}\#\left\{\delta_1,\delta_4(p^e)\,\middle|\, \begin{matrix}\delta_1\delta_4 \not\equiv 1 \bmod p^e\\ x \equiv \delta_1 t_1 + t_2 + \delta_4 t_4 \bmod p^e\end{matrix}\right\}\right).\end{aligned}$$

Set $p^s = (t_4, p^e)$. If $s = e$, then the lemma holds trivially. Hence we assume $s < e$. Set $t_4 = up^s$, $(u,p) = 1$. Since $x \equiv \delta_1 t_1 + t_2 + \delta_4 t_4 \bmod p^e$ implies $x \equiv \delta_1 t_1 + t_2 \bmod p^s$, we have

$$\begin{aligned}\Sigma_2 &= O\left(p^{e/2}\sum_{x(p^e)}(x,p^e)^{1/2}\#\left\{\delta_1,\delta_4(p^e)\,\middle|\, \begin{matrix}x \equiv \delta_1 t_1 + t_2 \bmod p^s\\ u\delta_4 \equiv (x-\delta_1 t_1 - t_2)p^{-s} \bmod p^{e-s}\end{matrix}\right\}\right)\\ &= O\Big(p^{e/2}\sum_{0\le i\le e} p^{(e-i)/2}\sum_{\substack{v(p^i)\\ p\nmid v}} p^s\#\{\delta_1 \bmod p^e \mid \delta_1 t_1 + t_2 \equiv vp^{e-i} \bmod p^s\}\Big)\\ &= O\Big(p^{e+s}\sum_{0\le i\le e} p^{-i/2}\#\{\delta_1 \bmod p^e,\ v \bmod p^i \mid p\nmid v,\ \delta_1 t_1 + t_2 \equiv vp^{e-i} \bmod p^s\}\Big).\end{aligned}$$

If $\operatorname{ord}_p t_1$, $\operatorname{ord}_p t_2 \ge s$, then we have

$$\Sigma_2 = O\Big(p^{e+s}\sum_{\substack{0\le i\le e\\ e-i\ge s}} p^{-i/2+e+i}\Big) = O(p^{5e/2+s/2}).$$

If $\operatorname{ord}_p t_1 \ge s$, $a_2 = \operatorname{ord}_p t_2 < s$, then $\delta_1 t_1 + t_2 \equiv vp^{e-i} \bmod p^s$ implies $a_2 = e - i$ and $v \equiv t_2 p^{-a_2} \bmod p^{s-a_2}$, and hence

$$\begin{aligned}\Sigma_2 &= O(p^{e+s-(e-a_2)/2+e+(e-a_2)-(s-a_2)})\\ &= O(p^{5e/2+a_2/2}) = O(p^{5e/2+s/2}).\end{aligned}$$

If $a_1 = \operatorname{ord}_p t_1 < s$, $a_2 = \operatorname{ord}_p t_2 < a_1$, then $\delta_1 t_1 + t_2 \equiv vp^{e-i} \bmod p^s$ implies $a_2 = e - i$ and $t_2 p^{-a_2} \equiv v \bmod p^{a_1-a_2}$, and hence

$$\begin{aligned}\Sigma_2 &= O(p^{e+s-(e-a_2)/2+e-a_2-(a_1-a_2)+e-(s-a_1)})\\ &= O(p^{5e/2+a_2/2}) = O(p^{5e/2+s/2}).\end{aligned}$$

Suppose $a_1 < s$, $a_2 \ge a_1$, then $\delta_1 t_1 + t_2 \equiv vp^{e-i} \bmod p^s$, $p\nmid v$ imply $e - i \ge a_1$ and $\delta_1(t_1 p^{-a_1}) \equiv (vp^{e-i} - t_2)p^{-a_1} \bmod p^{s-a_1}$. Hence we have

$$\begin{aligned}\Sigma_2 &= O\Big(p^{e+s}\sum_{0\le i\le e-a_1} p^{-i/2+i+e-(s-a_1)}\Big)\\ &= O(p^{e+s+(e-a_1)/2+e-s+a_1}) = O(p^{5e/2+a_1/2}) = O(p^{5e/2+s/2}).\end{aligned}$$

Thus we have completed a proof of Lemma 4.

The former of Proposition 1 follows easily from Lemmas 3, 4 and $K(P, T; U^{-1}CV^{-1}) = K(P[{}^tU], T[V]; C)$.

The latter is proved as follows:

$$K(P, T; C) = \sum_{D \bmod CΛ} e(\mathrm{tr}(AC^{-1}P + C^{-1}DT))$$
$$= \sum_{D \bmod CΛ} e(\mathrm{tr}({}^tD\,{}^tC^{-1}T + {}^tC^{-1}\,{}^tAP))\,,$$

and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ if and only if $\begin{pmatrix} {}^tD & {}^tB \\ {}^tC & {}^tA \end{pmatrix} \in \Gamma$. Suppose $\begin{pmatrix} A_i & B_i \\ C & D_i \end{pmatrix} \in \Gamma$ $(i = 1, 2)$ and $D_1 \equiv D_2 \bmod CΛ$, then we set $D_1 = D_2 + CS$, $S \in Λ$. $\begin{pmatrix} A_2 & * \\ C & D_2 \end{pmatrix}\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_2 & * \\ C & D_1 \end{pmatrix}$ implies $A_2 = A_1 + \bar{S}C$ for some $\bar{S} \in Λ$. Thus $D_1 \equiv D_2 \bmod CΛ$ implies ${}^tA_1 \equiv {}^tA_2 \bmod {}^tCΛ$. Hence we have

$$K(P, T; C) = \sum_{{}^tA \bmod {}^tCΛ} e(\mathrm{tr}({}^tD\,{}^tC^{-1}T + {}^tC^{-1}\,{}^tAP))$$
$$= K(T, P; {}^tC)\,.$$

For $G = (g_{ij}) \in Λ^*$ we set $e(G) = (g_{11}, g_{22}, 2g_{12})$. Set

$$S = \left\{ \begin{pmatrix} b \\ d \end{pmatrix} \middle| \, b, d \in Z, (b, d) = 1 \right\}.$$

For a fixed natural number n we define an equivalence relation $\sim$ in S by the following:

$$\begin{pmatrix} b \\ d \end{pmatrix} \sim \begin{pmatrix} b' \\ d' \end{pmatrix} \text{ iff } \begin{pmatrix} b \\ d \end{pmatrix} \equiv w\begin{pmatrix} b' \\ d' \end{pmatrix} \bmod n$$

for an integer w prime to n. Set $S(n) = S/\sim$, then another aim in this section is to prove

PROPOSITION 2. *For $P \in Λ^*$ we have*

$$\sum_{x \in S(n)} (P[x], n)^{1/2} = O(n^{1+\varepsilon}(e(P), n)^{1/2}) \qquad \textit{for any } \varepsilon > 0\,.$$

LEMMA 5. *Let m, n be relatively prime natural numbers and $P \in Λ^*$. Then we have*

$$\sum_{x \in S(mn)} (P[x], mn)^{1/2} \leq \Big(\sum_{x \in S(m)} (P[x], m)^{1/2}\Big)\Big(\sum_{y \in S(n)} (P[y], n)^{1/2}\Big)\,.$$

Proof. The mapping $x \in S(mn) \mapsto (x \in S(m), x \in S(n))$ is injective. From $(P[x], mn) = (P[x], m)(P[x], n)$ follows the lemma.

Hence we have only to prove Proposition 2 when n is a power of a prime p.

LEMMA 6. *Let p be a prime and e a natural number.*
Put $S' = \left\{ \begin{pmatrix} b \\ d \end{pmatrix} \middle| \, b, d \in Z, (b, d, p) = 1 \right\}$ *and*

$$\binom{b}{d} \approx \binom{b'}{d'} \textit{ iff } \binom{b}{d} \equiv w\binom{b'}{d'} \bmod p^e$$

for an integer $w(\not\equiv 0 \bmod p)$. *Then we have*

$$\sum_{x\in S(p^e)} (P[x],\ p^e)^{1/2} = \sum_{x\in S'/\approx} (P[x], p^e)^{1/2}.$$

Proof. The lemma follows immediately from the following fact: if $(b, d, p) = 1$, then there exist $B, D \in Z$ such that $B \equiv b \bmod p^e$, $D \equiv d \bmod p^e$ and $(B, D) = 1$.

If $V \in M_2(Z)$, $p \nmid |V|$, then we have $V(S'/\approx) = S'/\approx$. Hence we may assume

(i) $P = \begin{pmatrix} up^{a_1} & \\ & uvp^{a_2}\end{pmatrix}$, $0 \le a_1 \le a_2$, $p \nmid uv$,

(ii) $P = 2^a\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1\end{pmatrix}$, $a \ge 0$ $(p = 2)$, or

(iii) $P = 2^a\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0\end{pmatrix}$, $a \ge 0$ $(p = 2)$.

It is easy to see that we can take as $S'/\approx$

$$\binom{n}{1}(n \bmod p^e),\ \binom{n}{p^t}(p \nmid n,\ n \bmod p^{e-t},\ t = 1, 2, \cdots, e).$$

Set $\mathfrak{S}(P, p^e) = \sum_{x\in S(p^e)} (P[x], p^e)^{1/2}$ and $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4\end{pmatrix}$. Now we prove Proposition 2.

(1) Suppose that P is of type (i) and $a_1 \ge e$.
In this case $\mathfrak{S}(P, p^e) \le p^{e/2}(p^e + \varphi(p^{e-1}) + \cdots + \varphi(1)) = O(p^{3e/2})$.

(2) Suppose that P is of type (i) and $a_1 < e$.

$$\begin{aligned}\mathfrak{S}(P, p^e) &= p^{a_1/2} \sum_{\binom{x_1}{x_2}\in S(p^e)} (x_1^2 + vx_2^2p^{a_2-a_1}, p^{e-a_1})^{1/2} \\ &= p^{a_1/2} \sum_{\substack{n(p^e)\\ p\nmid n}} (n^2 + vp^{a_2-a_1}, p^{e-a_1})^{1/2} \\ &\quad + p^{a_1/2} \sum_{n(p^{e-1})} (p^2n^2 + vp^{a_2-a_1}, p^{e-a_1})^{1/2} \\ &\quad + p^{a_1/2} \sum_{1\le t\le e} \sum_{\substack{n(p^{e-t})\\ p\nmid n}} (n^2 + vp^{a_2-a_1+2t}, p^{e-a_1})^{1/2}.\end{aligned}$$

If $a_1 = a_2$, then

$$\begin{aligned}\mathfrak{S}(P, p^e) &= p^{a_1/2} \sum_{\substack{n(p^e)\\ p\nmid n}} (n^2 + v, p^{e-a_1})^{1/2} + p^{a_1/2+e-1} + p^{a_1/2} \sum_{1\le t\le e} \varphi(p^{e-t}) \\ &= p^{a_1/2} \sum_{\substack{n(p^e)\\ p\nmid n}} (n^2 + v, p^{e-a_1})^{1/2} + 2p^{a_1/2+e-1}.\end{aligned}$$

If $a_1 < a_2$, then

$$\mathfrak{S}(P, p^e) = p^{a_1/2}\varphi(p^e) + p^{a_1/2} \sum_{n(p^{e-1})} (p^2n^2 + vp^{a_2-a_1}, p^{e-a_1})^{1/2} + p^{a_1/2+e-1} .$$

Hence we have only to prove the following lemmas.

LEMMA 7. $\sum_{\substack{n(p^e)\\ p\nmid n}} (n^2 + v, p^{e-a_1})^{1/2} = O(ep^e) = O(p^{e(1+\varepsilon)})$ *if* $a_1 < e$.

LEMMA 8. $\sum_{n(p^{e-1})} (p^2n^2 + vp^{a_2-a_1}, p^{e-a_1})^{1/2} = O(p^{e(1+\varepsilon)})$ *if* $a_1 < a_2$, $a_1 < e$.

Proof of Lemma 7. If p is odd and $(-v/p) = -1$, then Lemma 7 is trivial. Suppose that p is odd and $(-v/p) = 1$, then there exists an integer $g \in Z_p$ such that $g^2 + v = 0$. If $n^2 + v \equiv 0 \bmod p$, then there exist $m \in Z_p^\times$, $s \geq 1$ such that $n = \pm g + mp^s$. Then we have $p^s \| (n^2 + v)$ since $n^2 + v = p^s(\pm 2gm + m^2p^s)$. Thus we have

$$\begin{aligned}\sum_{\substack{n(p^e)\\ p\nmid n}} (n^2 + v, p^{e-a_1})^{1/2} &= \sum_{\substack{n(p^e)\\ p\nmid n, n^2+v\equiv O(p)}} (n^2 + v, p^{e-a_1})^{1/2} + \sum_{\substack{n(p^e)\\ p\nmid n, n^2+v\not\equiv O(p)}} 1 \\ &\leq 2 \sum_{1\leq s\leq e} \sum_{\substack{m \bmod p^{e-s}\\ p\nmid m}} (p^s, p^{e-a_1})^{1/2} + p^e \\ &= 2 \sum_{1\leq s\leq e-a_1} p^{s/2}\varphi(p^{e-s}) + 2 \sum_{e-a_1<s\leq e} p^{(e-a_1)/2}\varphi(p^{e-s}) + p^e \\ &= O(ep^e) .\end{aligned}$$

Suppose $p = 2$. If $v \not\equiv 7 \bmod 8$, $n^2 + v \not\equiv 0 \bmod 8$ for odd n, and so Lemma 7 is obvious. Assume $v \equiv 7 \bmod 8$ and take an integer $g \in Z_2^\times$ such that $g^2 + v = 0$. Let n be an odd integer and $n = g + 2^rm$ $(r \geq 1, 2 \nmid m)$. Since $n^2 + v = 2^{r+1}(gm + 2^{r-1}m^2)$, we have

$$\begin{aligned}\sum_{\substack{n(2^e)\\ 2\nmid n}} (n^2 + v, 2^{e-a_1})^{1/2} &= \sum_{\substack{m(2^{e-1})\\ 2\nmid m}} (2^2(gm + m^2), 2^{e-a_1})^{1/2} + \sum_{2\leq r\leq e} \sum_{\substack{m(2^{e-r})\\ 2\nmid m}} (2^{r+1}, 2^{e-a_1})^{1/2} \\ &= \sum_{\substack{n(2^{e-1})\\ 2|n}} (2^2n, 2^{e-a_1})^{1/2} + \sum_{2\leq r\leq e} 2^{e-r-1}(2^{r+1}, 2^{e-a_1})^{1/2} \\ &= \sum_{1\leq r\leq e-1} 2^{e-2-r}(2^{2+r}, 2^{e-a_1})^{1/2} + \sum_{2\leq r\leq e} 2^{e-r-1}(2^{r+1}, 2^{e-a_1})^{1/2} \\ &= O(e2^e) .\end{aligned}$$

Proof of Lemma 8. Suppose $a_2 \geq e$, then we have

$$\begin{aligned}\sum_{n(p^{e-1})} (p^2n^2 + vp^{a_2-a_1}, p^{e-a_1})^{1/2} &= \sum_{n(p^{e-1})} (p^2n^2, p^{e-a_1})^{1/2} \\ &= \sum_{0\leq r\leq e-1} \varphi(p^{e-1-r})(p^{2+2r}, p^{e-a_1})^{1/2} \\ &= O(ep^e) .\end{aligned}$$

Suppose $a_2 < e$, then

$$\sum_{n(p^{e}-1)} (p^2n^2 + vp^{a_2-a_1}, p^{e-a_1})^{1/2}$$
$$= \sum_{0\le r<(a_2-a_1-2)/2} \varphi(p^{e-1-r})p^{1+r} + \sum_{\substack{r=(a_2-a_1-2)/2\\ m(p^{e-1-r})\\ p\nmid m}} p^{(a_2-a_1)/2} \cdot$$
$$(m^2 + v, p^{e-a_2})^{1/2} + \sum_{(a_2-a_1-2)/2<r\le e-1} \varphi(p^{e-1-r})p^{(a_2-a_1)/2}$$
$$= O(ep^e) + \sum_{\substack{r=(a_2-a_1-2)/2\\ m(p^{e-1-r})\\ p\nmid m}} p^{(a_2-a_1)/2}(m^2 + v, p^{e-a_2})^{1/2}$$
$$= O(ep^e) + p^{(a_2-a_1)/2}O(ep^{e-1-r}) \text{ by Lemma 7 } (e-1-r > e-a_2)$$
$$= O(ep^e) = O(p^{e(1+\varepsilon)}).$$

(3) Suppose that P is of type (ii).
If $a \ge e$, then $\mathfrak{S}(P, 2^e) = O(2^{3e/2})$ follows as in case of (1).
Suppose $a < e$. Then we have

$$\mathfrak{S}(P, 2^e) = \sum_{x\in S(2^e)} \left(2^a\begin{pmatrix}1 & 1/2\\ 1/2 & 1\end{pmatrix}[x], 2^e\right)^{1/2}$$
$$= 2^{a/2} \sum_{\binom{x_1}{x_2}\in S(2^e)} (x_1^2 + x_1x_2 + x_2^2, 2^{e-a})^{1/2}$$
$$= 2^{a/2} \# S(2^e) = O(2^{a/2+e}) .$$

(4) Suppose that P is of type (iii).
Similarly to the above we may suppose $a < e$, then we have

$$\mathfrak{S}(P, 2^e) = \sum_{\binom{x_1}{x_2}\in S(2^e)} (2^a x_1x_2, 2^e)^{1/2}$$
$$= \sum_{n(2^e)} (2^a n, 2^e)^{1/2} + \sum_{1\le t\le e} \sum_{\substack{n(2^{e-t})\\ 2\nmid n}} (2^{a+t}n, 2^e)^{1/2}$$
$$- \sum_{0\le t\le e} \varphi(2^{e-t})(2^{a+t}, 2^e)^{1/2} + \sum_{1\le t\le e-1} 2^{e-t-1}(2^{a+t}, 2^e)^{1/2} + (2^{a+e}, 2^e)^{1/2}$$
$$= O(e2^{a/2+e}) .$$

Thus we have completed a proof of Proposition 2.

§ 2.

In this section we give a formal Fourier expansion of Poincaré series [1]. Let k be an even integer ≥ 6, and $Q \in \Lambda^*$, $Q > 0$. We set

$$j(M, Z) = |CZ + D| \text{ for } M = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \Gamma = Sp_2(Z),\ Z \in H,$$

and
$$\Gamma_1(\infty) = \left\{\begin{pmatrix}1_2 & S\\ 0 & 1_2\end{pmatrix} \middle| S \in \Lambda\right\} \subset \Gamma .$$

We define Poincaré series $g(Z, Q)$ by

$$\sum_{M \in \Gamma_1(\infty)\backslash\Gamma} e(\operatorname{tr} QM\langle Z\rangle) j((M, Z)^{-k}, \quad Z \in H.$$

It is known ([1], [4]) that any cusp form is a linear combination of Poincaré series. Hence we have only to prove our theorem for Poincaré series. Let $\mathfrak{h}$ be a complete system of representatives of $\Gamma_1(\infty)\backslash\Gamma/\Gamma_1(\infty)$, $\theta(M) = \left\{S \in \Lambda \,|\, M\begin{pmatrix} 1_2 & S \\ & 1_2 \end{pmatrix} M^{-1} \in \Gamma_1(\infty)\right\}$ for $M \in \Gamma$.

LEMMA 1. $\Gamma_1(\infty)M\Gamma_1(\infty) = \bigcup_{S \in \Lambda/\theta(M)} \Gamma_1(\infty)M\begin{pmatrix} 1_2 & S \\ & 1_2 \end{pmatrix}$ *(disjoint).*

Proof. It is obvious.

Thus we have

$$g(Z, Q) = \sum_{M \in \mathfrak{h}} \sum_{S \in \Lambda/\theta(M)} e(\operatorname{tr} Q \cdot M\langle Z + S\rangle) j(M, Z + S)^{-k}.$$

Setting
$$\begin{aligned} H(M, Z) &= \sum_{S \in \Lambda/\theta(M)} e(\operatorname{tr} Q \cdot M\langle Z + S\rangle) j(M, Z + S)^{-k} \\ &= \sum_{\Lambda^* \ni T \geq 0} h(M, T) e(\operatorname{tr} TZ), \end{aligned}$$

we have

$$h(M, T) = \int_{X \bmod 1} H(M, Z) e(-\operatorname{tr} TZ) dX,$$

where $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_4 \end{pmatrix}$ is the real part of Z and $dX = dx_1 dx_2 dx_4$. If we set $g(Z, Q) = \sum_{\Lambda^* \ni T > 0} a(T) e(\operatorname{tr} TZ)$, then we have

$$a(T) = \sum_{M \in \mathfrak{h}} h(M, T) \qquad \text{for } 0 < T \in \Lambda^*.$$

Now we determine $\mathfrak{h}$, $\theta(M)$ explicitly.

LEMMA 2. *$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{h}$ is parametrized by C and $D \bmod C\Lambda$.*

Proof. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $S_1, S_2 \in \Lambda$, we have

$$\begin{pmatrix} 1_2 & S_1 \\ & 1_2 \end{pmatrix} M \begin{pmatrix} 1_2 & S_2 \\ & 1_2 \end{pmatrix} = \begin{pmatrix} * & * \\ C & CS_2 + D \end{pmatrix}.$$

This implies immediately Lemma 2.

LEMMA 3. *As $\left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{h} \,|\, C = 0\right\}$ we can choose $\left\{\begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix} \,\middle|\, U \in GL(2, Z)\right\}$ and $\theta(M) = \Lambda$.*

Proof. It is trivial.

LEMMA 4. *As* $\left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{h} \,|\, \mathrm{rk}\, C = 1\right\}$ *we can choose*

$$\left\{ M = \begin{pmatrix} * & * \\ U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} {}^tV & U^{-1}\begin{pmatrix} d_1 & d_2 \\ 0 & d_4 \end{pmatrix} V^{-1} \end{pmatrix} \in \Gamma \;\middle|\; \begin{array}{l} U \in \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2, Z)\right\} \backslash GL(2, Z), \quad V \in GL(2, Z) / \left\{\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(2, Z)\right\} \\ c_1 \geq 1,\ d_4 = \pm 1,\ (c_1, d_1) = 1, \quad d_1, d_2 \bmod c_1 \end{array} \right\}$$

and $\theta(M) = \left\{S \in \Lambda \,|\, S[V] = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}\right\}$ for the above specialized M.

Proof. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\mathrm{rk}\, C = 1$. Set $C = U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} {}^tV$, $U, V \in GL(2, Z)$, $c_1 \geq 1$. We can take $U \in \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in GL(2, Z)\right\} \backslash GL(2, Z)$ and $V \in GL(2, Z) / \left\{\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(2, Z)\right\}$ since $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} = \pm \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}$. Set $D = U^{-1}\begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} V^{-1}$. Since $C\,{}^tD$ is symmetric, we have $d_3 = 0$. The primitiveness of (C, D) implies that $\begin{pmatrix} c_1 & 0 & d_1 & d_2 \\ 0 & 0 & 0 & d_4 \end{pmatrix}$ is primitive. Hence $d_4 = \pm 1$, and $(c_1, d_1) = 1$ hold. $D \bmod C\Lambda$ is equivalent to $d_1, d_2 \bmod c_1$ since

$$C\Lambda = U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} {}^tV\Lambda = U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} \Lambda V^{-1} = U^{-1}\left\{\begin{pmatrix} c_1 s_1 & c_1 s_2 \\ 0 & 0 \end{pmatrix} \,\middle|\, s_i \in Z\right\} V^{-1}.$$

From $M^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$ follows $M\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}M^{-1} = \begin{pmatrix} * & * \\ -CS\,{}^tC & 1+CS\,{}^tA \end{pmatrix}$. Thus $\theta(M) \ni S$ is equivalent to $CS(-\,{}^tC, {}^tA) = 0$ and so $CS = 0$. Since $CS = 0$ means $\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} S[V] = 0$, we have completed a proof of Lemma 4 except the uniqueness of U, V, c_1, d_i.

Suppose $C = U_1^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} {}^tV_1 = U_2^{-1}\begin{pmatrix} c_1' & 0 \\ 0 & 0 \end{pmatrix} {}^tV_2$, $D = U_1^{-1}\begin{pmatrix} d_1 & d_2 \\ & d_4 \end{pmatrix} V_1^{-1} = U_2^{-1}\begin{pmatrix} d_1' & d_2' \\ & d_4' \end{pmatrix} V_2^{-1}$ where U_i, V_i, $\cdots$ are supposed to be representatives. Comparing elementary divisors of C, we have $c_1 = c_1'$. Set $U = U_2 U_1^{-1}$, ${}^tV = {}^tV_2\,{}^tV_1^{-1}$, then $U\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix} {}^tV$ holds and this implies $U = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and hence $U_1 = U_2$, $U = 1_2$. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} {}^tV$ implies $V = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and then $V_1 = V_2$. Thus $d_i = d_i'$ holds.

LEMMA 5. *As* $\left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{h} \,|\, |C| \neq 0\right\}$ *we can choose*

$$\left\{\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma \,\middle|\, |C| \neq 0,\ D \bmod C\Lambda\right\}$$

and $\theta(M) = \{0\}$.

Proof. The former follows from Lemma 2. The latter follows from

$$M\begin{pmatrix} 1_2 & S \\ & 1_2 \end{pmatrix}M^{-1} = \begin{pmatrix} * & * \\ -CS\,{}^tC & * \end{pmatrix} \text{ for } M = \begin{pmatrix} * & * \\ C & * \end{pmatrix}.$$

§ 3.

Hereafter we fix $0 < Q$, $T \in \Lambda^*$, and we assume that T is Minkowski-reduced without loss of generality since $a(T) = a(T[U])$ for $U \in GL(2, Z)$ ($a(T)$ is a Fourier coefficient of $g(Z, Q)$). In this section we estimate $\sum_{\substack{M = \begin{pmatrix} * & * \\ C & * \end{pmatrix} \in \mathfrak{h} \\ \mathrm{rk}\, C \leq 1}} h(M, T)$.

First, suppose $M = \begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix}$, $U \in GL(2, Z)$, then we have

$$H(M, Z) = e(\mathrm{tr}\, Q \cdot M\langle Z \rangle) = e(\mathrm{tr}\, Q[{}^tU]Z)$$

by Lemma 3 in Section 2. This yields $\sum_{M = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathfrak{h}} h(M, T) = O(1)$. Next we consider the case of $\mathrm{rk}\, C = 1$.

LEMMA 1. *Let* $M = \begin{pmatrix} * & * \\ U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}{}^tV & U^{-1}\begin{pmatrix} d_1 & d_2 \\ 0 & d_4 \end{pmatrix}V^{-1} \end{pmatrix} \in \Gamma$, *where* $U, V \in GL(2, Z)$, $d_4 = \pm 1$, $c_1 > 0$.

Set $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4 \end{pmatrix} = Q[{}^tU]$, $S = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4 \end{pmatrix} = T[{}^tV^{-1}]$ *and* a_1 *denotes an integer such that* $a_1 d_1 \equiv 1 \bmod c_1$. *Then we have*

$$\begin{aligned} h(M, T) = {} & (-1)^{k/2}\sqrt{2}\,\pi\,|Q|^{3/4-k/2}\delta_{p_4, s_4}|T|^{k/2-3/4}s_4^{-1/2}c_1^{-3/2} \\ & \times e(\{a_1 s_4 d_2^2 - (a_1 d_4 p_2 - s_2)d_2\}/c_1 + (a_1 p_1 + d_1 s_1)/c_1 - d_4 p_2 s_2/(2c_1 s_4)) \\ & \times J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1 s_4)\,, \end{aligned}$$

where δ *is the Kronecker's delta function and* J *is the ordinary Bessel function.*

Proof. At first, we suppose $M = \begin{pmatrix} {}^tU & \\ & U^{-1} \end{pmatrix} M_0 \begin{pmatrix} {}^tV & \\ & V^{-1} \end{pmatrix}$, where $M_0 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $ad - bc = 1$, $c > 0$. Then $h(M, T)$ equals

$$\begin{aligned}
&\int_{X \bmod 1} H(M, Z)e(-\operatorname{tr} TZ)dX \\
&\quad = \int_{X \bmod 1} \sum_{S \in \Lambda/\theta(M)} e(\operatorname{tr} Q\cdot M\langle Z+S\rangle)j(M, Z+S)^{-k}e(-\operatorname{tr} TZ)dX \\
&\quad = \int_{X \bmod 1} \sum_{S \in \Lambda/\theta(M)} e(\operatorname{tr} Q[{}^tU]\cdot M_0\langle Z[V]+S[V]\rangle)j(M_0, Z[V]+S[V])^{-k} \\
&\qquad \times e(-\operatorname{tr} TZ)dX\,.
\end{aligned}$$

Setting $W = X + i\operatorname{Im} Z[V]$, $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_4 \end{pmatrix}$, we have by virtue of Lemma 4 in Section 2,

$$h(M, T) = \int_{\substack{x_1, x_2 \in R \\ x_4 \bmod 1}} e(\operatorname{tr} Q[{}^tU]\cdot M_0\langle W\rangle)j(M_0, W)^{-k}e(-\operatorname{tr} T[{}^tV^{-1}]W)dX\,.$$

Since $M_0\langle W\rangle = \begin{pmatrix} (aw_1+b)(cw_1+d)^{-1} & * \\ w_2(cw_1+d)^{-1} & -(cw_1+d)^{-1}cw_2^2+w_4 \end{pmatrix}$, $j(M_0, W) = cw_1 + d$ where $W = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_4 \end{pmatrix}$, we have

$$\begin{aligned}
h(M, T) &= \int_{\substack{x_1, x_2 \in R \\ x_4 \bmod 1}} e(p_1\{a/c - c^{-2}(w_1+c^{-1}d)^{-1}\} + p_2w_2c^{-1}(w_1+c^{-1}d)^{-1} + p_4w_4 \\
&\qquad - p_4(w_1+c^{-1}d)^{-1}w_2^2 - s_1w_1 - s_2w_2 - s_2w_4)(cw_1+d)^{-k}dx_1dx_2dx_4 \\
&= \delta_{p_4, s_4}e(p_1a/c)\int_{x_1 \in R} e(-p_1c^{-2}(w_1+c^{-1}d)^{-1} - s_1w_1)(cw_1+d)^{-k}dx_1 \\
&\qquad \times \int_{x_2 \in R} e(-p_4(w_1+c^{-1}d)^{-1}w_2^2 + \{p_2c^{-1}(w_1+c^{-1}d)^{-1} - s_2\}w_2)dx_2\,.
\end{aligned}$$

Since we know

$$\int_{x_2 \in R} e(\alpha w_2^2 + \beta w_2)dx_2 = e(-\beta^2/4\alpha)\sqrt{2}^{-1}\sqrt{i/\alpha} \text{ for } \alpha, \beta \in C \ (\operatorname{Im}\alpha > 0)\,,$$

setting $\alpha = -p_4(w_1+c^{-1}d)^{-1}$, $\beta = p_2c^{-1}(w_1+c^{-1}d)^{-1} - s_2$, we have

$$\begin{aligned}
h(M, T) &= \sqrt{2}^{-1}\delta_{p_4, s_4}s_4^{-1/2}c^{-k}e(-s_2p_2/(2cs_4))e((p_1a+s_1d)/c)(-1)^{k/2} \\
&\qquad \times \int_{x_1 \in R} e(-s_4^{-1}|T|w_1 - s_4^{-1}c^{-2}|Q|w_1^{-1})(w_1/i)^{-k+1/2}dx_1\,.
\end{aligned}$$

It is easy to see

$$\int_{x_1 \in R} e(-aw_1 - bw_1^{-1})(w_1/i)^{-k+1/2}dx_1 = 2\pi(b/a)^{3/4-k/2}J_{k-3/2}(4\pi\sqrt{ab})$$

$$\text{for } a, b > 0.$$

Thus we have

$$h(M, T) = \sqrt{2}\,\pi|Q|^{3/4-k/2}\delta_{p_4,s_4}|T|^{k/2-3/4}s_4^{-1/2}c^{-3/2}e(-p_2s_2/(2cs_4))(-1)^{k/2}$$
$$\times e((p_1a + s_1d)/c)J_{k-3/2}(4\pi\sqrt{|T||Q|}/cs_4)\,.$$

Now we come back to the general case.

Let $C = U^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & 0\end{pmatrix}{}^tV$, $D = U^{-1}\begin{pmatrix} d_1 & d_2 \\ & d_4\end{pmatrix}V^{-1}$. Then $C = \left(\begin{pmatrix} 1 & -d_2d_4 \\ & d_4\end{pmatrix}U\right)^{-1}$ $\times\begin{pmatrix} c_1 & 0 \\ 0 & 0\end{pmatrix}{}^tV$, $D = \left(\begin{pmatrix} 1 & -d_2d_4 \\ & d_4\end{pmatrix}U\right)^{-1}\begin{pmatrix} d_1 & 0 \\ 0 & 1\end{pmatrix}V^{-1}$ hold. If we set $P = \begin{pmatrix} p_1 & p_2/2 \\ p_2/2 & p_4\end{pmatrix}$ $= Q[{}^tU]$, $S = \begin{pmatrix} s_1 & s_2/2 \\ s_2/2 & s_4\end{pmatrix} = T[{}^tV^{-1}]$ as in the statement of Lemma 1, then we have $Q\left[{}^t\left(\begin{pmatrix} 1 & -d_2d_4 \\ & d_4\end{pmatrix}U\right)\right] = \begin{pmatrix} p_1 - p_2d_2d_4 + p_4d_2^2 & * \\ p_2d_4/2 - p_4d_2 & p_4\end{pmatrix}$.

Applying the former, we have

$$\begin{aligned} h(M, T) &= \sqrt{2}\,\pi|Q|^{3/4-k/2}\delta_{p_4,s_4}|T|^{k/2-3/4}s_4^{-1/2}c_1^{-3/2}(-1)^{k/2} \\ &\quad\times e(-(p_2d_4 - 2p_4d_2)s_2/2c_1s_4)e((p_1 - p_2d_2d_4 + p_4d_2^2)a_1 + s_1d_1)/c_1) \\ &\quad\times J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1s_4) \\ &= (-1)^{k/2}\sqrt{2}\,\pi|Q|^{3/4-k/2}\delta_{p_4,s_4}|T|^{k/2-3/4}s_4^{-1/2}c_1^{-3/2} \\ &\quad\times \{e(\{a_1s_4d_2^2 - (a_1d_4p_2 - s_2)d_2\}/c_1 + (a_1p_1 + d_1s_1)/c_1 - d_4p_2s_2/2c_1s_4) \\ &\quad\times J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1s_4)\,. \end{aligned}$$

Hereafter $M \in \mathfrak{h}$ is supposed to be parametrized by U, V, c_1, d_1, d_2, d_4 as in Lemma 4 of Section 2. From Lemma 1 follows

$$\Big|\sum_{d_2 \bmod c_1} h(M, T)\Big| \ll \delta_{p_4,s_4}|T|^{k/2-3/4}s_4^{-1/2}(s_4, c_1)^{1/2}c_1^{-1}|J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1s_4)|\,,$$

since $\sum_{n \bmod c} e((an^2 + bn)/c) = O((a, c)^{1/2}c^{1/2})$.

Since U is parametrized by the second row up to sign, we have

$$\begin{aligned} \sum_U \sum_{\substack{d_1 \bmod c_1 \\ (d_1,c_1)=1 \\ d_4=\pm 1}} \Big|\sum_{d_2 \bmod c_1} h(M, T)\Big| &\ll \sum_{u=\binom{u_3}{u_4}} \delta_{Q[u],s_4}|T|^{k/2-3/4}s_4^{-1/2}(s_4, c_1)^{1/2} \\ &\quad\times |J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1s_4)|\,, \end{aligned}$$

where we set $U = \begin{pmatrix} * & * \\ u_3 & u_4\end{pmatrix}$,

$$\ll |T|^{k/2-3/4}s_4^{-1/2+\varepsilon}(s_4, c_1)^{1/2}|J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1s_4)|\,,$$

since the number of solutions u to $Q[u] = s_4$ is $O(s_4^\varepsilon)$. V is parametrized by the first column and $s_4 = T\begin{bmatrix} -v_3 \\ v_1\end{bmatrix}$ for $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4\end{pmatrix}$. Thus we have

$$\sum_{U,V}\sum_{\substack{d_1 \bmod c_1\\(d_1,c_1)=1\\d_4=\pm1}}\Big|\sum_{d_2 \bmod c_1} h(M,T)\Big|$$
$$\ll |T|^{k/2-3/4}\sum_{m=1}^{\infty}A(m,T)m^{-1/2+\varepsilon}(m,c_1)^{1/2}|J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1m)|,$$

where $A(m, T) = \#\left\{\begin{pmatrix}v_1\\v_2\end{pmatrix}\middle|(v_1, v_2) = 1,\ T\begin{bmatrix}v_1\\v_2\end{bmatrix} = m\right\}$.

We prepare the following

LEMMA 2. *Let t, m be natural numbers. Then*

$$\sum_{1\leq c\leq t/m}(m,c)^{1/2}c^{1/2} = O(t^{3/2}m^{-3/2+\varepsilon})$$
$$\sum_{c>t/m}(m,c)^{1/2}c^{3/2-k} = O(t^{5/2-k}m^{k-5/2+\varepsilon})$$

and $J_{k-3/2}(x) = O(\min(x^{k-3/2}, 1/\sqrt{x}))$ *for* $x > 0$.

Proof.

$$\sum_{1\leq c\leq t/m}(m,c)^{1/2}c^{1/2} \ll \sum_{r|m}\sum_{s\leq t/mr}r^{1/2}(sr)^{1/2}$$
$$= \sum_{r|m}r\sum_{s\leq t/mr}s^{1/2} \ll \sum_{r|m}r(t/mr)^{3/2} \ll (t/m)^{3/2}\sum_{r|m}r^{-1/2} = O((t/m)^{3/2}m^{\varepsilon}).$$
$$\sum_{c>t/m}(m,c)^{1/2}c^{3/2-k} \ll \sum_{r|m}\sum_{s>t/mr}r^{1/2}(rs)^{3/2-k}$$
$$= \sum_{r|m}r^{2-k}\sum_{s>t/mr}s^{3/2-k} \ll \sum_{r|m}r^{2-k}(t/mr)^{5/2-k} \ll (t/m)^{5/2-k}\sum_{r|m}r^{-1/2}$$
$$= O((t/m)^{5/2-k}m^{\varepsilon}).$$

The estimates for the Bessel function is well known.

From Lemma 2 follows

$$\sum_{c_1\geq 1}(m, c_1)^{1/2}|J_{k-3/2}(4\pi\sqrt{|T||Q|}/c_1m)|$$
$$\ll \sum_{c_1<\sqrt{|T|}/m}(m,c_1)^{1/2}(c_1m/\sqrt{|T|})^{1/2} + \sum_{c_1>\sqrt{|T|}/m}(m,c_1)^{1/2}(\sqrt{|T|}/c_1m)^{k-3/2}$$
$$\ll (m/\sqrt{|T|})^{1/2}|T|^{3/4}m^{-3/2+\varepsilon} + (\sqrt{|T|}/m)^{k-3/2}|T|^{5/4-k/2}m^{k-5/2+\varepsilon}$$
$$\ll |T|^{1/2}m^{-1+\varepsilon}.$$

Thus we have $\sum_{\substack{M\in\mathfrak{h}\\ \mathrm{rk}\, C=1}} h(M,T)| \ll |T|^{k/2-1/4}\sum_{m=1}^{\infty}A(m,T)m^{-3/2+2\varepsilon}$. We assumed that T is Minkowski-reduced, then $T \gg m(T)1_2$ holds where $m(T) = \min_{0\neq u\in Z^2} T[u]$. Hence we have

$$\sum_{m=1}^{\infty}A(m,T)m^{-3/2+2\varepsilon} \leq \sum_{0\neq u\in Z^2}T[u]^{-3/2+2\varepsilon} \ll m(T)^{-3/2+2\varepsilon}\sum_{\substack{(u_1,u_2)\in Z\\(u_1,u_2)\neq(0,0)}}(u_1^2+u_2^2)^{-3/2+2\varepsilon}$$
$$\ll m(T)^{-3/2+2\varepsilon} = O(1).$$

Hence we have

$$|\sum_{\substack{M\in\mathfrak{h}\\ \mathrm{rk}\, C=1}} h(M, T)| = O(|T|^{k/2-1/4}).$$

§ 4.

In this section we estimate $\sum_{\substack{M\in\mathfrak{h}\\ |C|\neq 0}} h(M, T)$.

LEMMA 1. *Set* $P(n) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL(2, Z) \mid b\equiv 0 \bmod n\right\}$. *Then*

$$\{C\in M_2(Z) \mid |C|\neq 0\} = \left\{U^{-1}\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}V^{-1} \;\middle|\; \begin{matrix} U\in GL(2, Z),\ V\in GL(2, Z)/P(c_2/c_1),\\ 0< c_1|c_2\end{matrix}\right\}$$

and $GL(2, Z)/P(c_2/c_1)$ *corresponds bijectively to* $S(c_2/c_1)$ *in Section* 1, *by the mapping* $V\mapsto$ *the second column of* V.

Proof. Set $V=\begin{pmatrix} v_1 & v_2\\ v_3 & v_4\end{pmatrix}\in GL(2, Z)$. Then

$$\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}V^{-1} = |V|\begin{pmatrix} v_4 & -v_2c_1/c_2\\ -v_3c_2/c_1 & v_1\end{pmatrix}\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}$$

holds, and so $\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}V^{-1}\in GL(2, Z)\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}$ if and only if $V\in P(c_2/c_1)$. Suppose that $C=U_1^{-1}\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}V_1^{-1}=U_2^{-1}\begin{pmatrix} c_1' & \\ & c_2'\end{pmatrix}V_2^{-1}$, $U_i, V_i\in GL(2, Z)$, $0<c_1|c_2$, $0<c_1'|c_2'$ and that V_1, V_2 are representatives in $GL(2, Z)/P(c_2/c_1)$. Comparing elementrary divisors, we have

$$c_i = c_i' \ (i=1, 2) \quad\text{and}\quad U_2U_1^{-1}\begin{pmatrix} 1 & \\ & c_2/c_1\end{pmatrix} = \begin{pmatrix} 1 & \\ & c_2/c_1\end{pmatrix}V_2^{-1}V_1 .$$

This implies $V_2^{-1}V_1\in P(C_2/C_1)$. Hence $V_1=V_2$ and so $U_1=U_2$ hold. The second assertion is obvious.

Let $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}$, $|C|\neq 0$. By Lemma 5 in Section 2 we have

$$\begin{aligned} h(M, T) &= \int_{X \bmod 1} H(M, Z)e(-\operatorname{tr} TZ)dX\\ &= \int_{X \bmod 1}\sum_{S\in A} e(\operatorname{tr} Q\cdot M\langle Z+S\rangle)j(M, Z+S)^{-k}e(-\operatorname{tr} TZ)dX\\ &= |C|^{-k}e(\operatorname{tr}(QAC^{-1}+TC^{-1}D))\\ &\quad\times\int_{X \bmod 1}\sum_{S\in A} e(-\operatorname{tr}\{Q^tC^{-1}(Z+S+C^{-1}D)^{-1}C^{-1}\\ &\quad + T(Z+C^{-1}D)\})|Z+S+C^{-1}D|^{-k}dX \end{aligned}$$

$$\begin{aligned}(\text{since } M\langle Z\rangle = AC^{-1} - {}^tC^{-1}(Z + C^{-1}D)^{-1}C^{-1})\\ = |C|^{-k}e(\operatorname{tr}(QAC^{-1} + TC^{-1}D))\\ \times \int_X e(-\operatorname{tr}(Q[{}^tC^{-1}]Z^{-1} + TZ))|Z|^{-k}dX.\end{aligned}$$

For positive definite matrices P, $S \in GL(2, \boldsymbol{R})$ we set

$$J(P, S) = \int_X e(-\operatorname{tr}(PZ^{-1} + SZ))|Z|^{-k}dX.$$

Then it is known ([1]) that

$J(P, S)$ does not depend on $\operatorname{Im} Z$, and
$J(P, S) = \|R\|^{3-2k}J(P[R^{-1}], S[{}^tR])$ for $R \in GL(2, \boldsymbol{R})$.

For a positive definite matrix P, we denote by $\sqrt{P}$ a matrix A such that $A^2 = P$, $A > 0$. Then we have, for $P, S > 0$,

$$\begin{aligned}J(P, S) &= |P|^{3/2-k}J(1_2, S[\sqrt{P}])\\ &= |P|^{3/2-k}|S[\sqrt{P}]|^{k/2-3/4}J(\sqrt{S[\sqrt{P}]}, \sqrt{S[\sqrt{P}]})\\ &= |P|^{3/4-k/2}|S|^{k/2-3/4}\tilde{J}(\sqrt{S[\sqrt{P}]}),\end{aligned}$$

where we set $\tilde{J}(P) = J(P, P)$ for $0 < P \in GL(2, \boldsymbol{R})$.

Since $\tilde{J}(P[F]) = \tilde{J}(P)$ for every orthogonal matrix $F \in GL(2, \boldsymbol{R})$, $\tilde{J}(P)$ is determined by eigen-values of P.

It is easy to see that for $0 < S \in GL(2, \boldsymbol{R})$

$$\tilde{J}((4\pi)^{-1}S) = 2(2\pi)^{-3}|2^{-1}S|^{k-3/2}A_{k-3/2}(4^{-1}S^2),$$

where $A_\delta(M)$ is a generalized Bessel function defined in [2], and it is known

$$A_{k-3/2}(4^{-1}S^2)|2^{-1}S|^{k-3/2} = \frac{2}{\pi}\int_0^1 J_{k-3/2}(s_1t)J_{k-3/2}(s_2t)t(1-t^2)^{-1/2}dt$$

for $S = \begin{pmatrix} s_1 & \\ & s_2\end{pmatrix} > 0$.

Thus we have, for $s_1, s_2 > 0$,

$$\tilde{J}\left(\begin{pmatrix} s_1 & \\ & s_2\end{pmatrix}\right) = 2^{-1}\pi^{-4}\int_0^1 \prod_{i=1,2} J_{k-3/2}(4\pi s_i t)t(1-t^2)^{-1/2}dt.$$

Hence we have, for $M = \begin{pmatrix} * & * \\ C & D\end{pmatrix} \in \Gamma$, $|C| \neq 0$,

$$h(M, T) = 2^{-1}\pi^{-4}|Q|^{3/4-k/2}|T|^{k/2-3/4}\|C\|^{-3/2}e(\mathrm{tr}(AC^{-1}Q + C^{-1}DT))$$
$$\times \int_0^1 \prod_{i=1,2} J_{k-3/2}(4\pi s_i t)t(1-t^2)^{-1/2}dt\,,$$

where s_1, s_2 are eigen-values of $\sqrt{T[\sqrt{Q}[{}^tC^{-1}]]}$, and so

$$\sum_{D \bmod CA} h(M, T) = \kappa|T|^{k/2-3/4}\|C\|^{-3/2}K(Q, T; C)$$
$$\times \int_0^1 \prod_{i=1,2} J_{k-3/2}(4\pi s_i t)t(1-t^2)^{-1/2}dt\,,$$

where $\kappa = 2^{-1}\pi^{-4}|Q|^{3/4-k/2}$ and $K(Q, T; C)$ is a generalized Kloosterman sum defined in Section 1 and s_1, s_2 are positive numbers such that s_1^2, s_2^2 are eigen-values of $T\cdot Q[{}^tC^{-1}]$. Since $T = \begin{pmatrix} t_1 & * \\ * & t_2 \end{pmatrix}$ is supposed to be Minkowski-reduced, we have $T \asymp \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$, that is, there are constants, κ_1, κ_2 such that $T > \kappa_1\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$, $T < \kappa_2\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$. If $A > 0$, $B \geqq B_1 > 0$, then $\mathrm{tr}\,AB = \mathrm{tr}\sqrt{A}B\sqrt{A} \geqq \mathrm{tr}\sqrt{A}B_1\sqrt{A} = \mathrm{tr}\,AB_1$ holds. Hence we have $\mathrm{tr}\,T\cdot Q[{}^tC^{-1}] = \mathrm{tr}\,T[C^{-1}]\cdot Q \asymp \mathrm{tr}\,T[C^{-1}] \asymp \mathrm{tr}\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}[C^{-1}]$ and $|T\cdot Q[{}^tC^{-1}]| \asymp \left|\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}[C^{-1}]\right|$. From these follow $s_1^2 \asymp s_1'$, $s_2^2 \asymp s_2'$ where s_1', s_2' are eigen-values of $\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}[C^{-1}]$. Set $P = T\cdot Q[{}^tC^{-1}]$, then $\mathrm{tr}\,P < 1$ implies $s_1^2 + s_2^2 < 1$ and $s_1^2 \ll 1$, $s_2^2 \ll 1$. $\mathrm{tr}\,P < 2|P|$ implies $(s_1^2 + s_2^2)/s_1^2s_2^2 < 2$ and $s_1^2 \gg 1$, $s_2^2 \gg 1$. If $\mathrm{tr}\,P \geqq 1$ and $\mathrm{tr}\,P \geqq 2|P|$, then we have either $s_1^2 \geqq 2/3$, $s_2^2 \leqq 2$ or $s_1^2 < 2/3$, $s_2^2 > 1/3$. Since $J_{k-3/2}(x) = O(\min(x^{k-3/2}, 1/\sqrt{x}))$, we have

$$\left|\int_0^1 \prod_{i=1,2} J_{k-3/2}(4\pi s_i t)t(1-t^2)^{-1/2}dt\right| \ll \begin{cases} |P|^{k/2-3/4} & \text{if } \mathrm{tr}\,P < 1\,, \\ |P|^{-1/4} & \text{if } \mathrm{tr}\,P < 2|P|\,, \\ |P|^{k/2-3/4}(\mathrm{tr}\,P)^{(1-k)/2} & \text{otherwise}\,. \end{cases}$$

Thus we have

$$\Big|\sum_{D \bmod CA} h(M, T)\Big| \ll |T|^{k/2-3/4}c_1^{1/2}c_2^{-1+\varepsilon}(c_2, T[v])^{1/2}$$
$$\times \begin{cases} |P|^{k/2-3/4} & \text{if } \mathrm{tr}\,P < 1\,, \\ |P|^{-1/4} & \text{if } \mathrm{tr}\,P < 2|P|\,, \\ |P|^{k/2-3/4}(\mathrm{tr}\,P)^{(1-k)/2} & \text{otherwise}\,. \end{cases}$$

where $C = U^{-1}\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix}V^{-1}$, U, $V \in GL(2, Z)$, $0 < c_1|c_2$ and $P = T\cdot Q[{}^tC^{-1}]$, and v is the second column of V.

Fix $0 < c_1|c_2$ and $V \in GL(2, Z)$ and let v be the second column of V.

We suppose that $A = T\left[V\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}^{-1} U_1\right]$ is Minkowski-reduced for $U_1 \in GL(2, Z)$. Set $C = U^{-1}U_1^{-1}\begin{pmatrix} c_1 & \\ & c_2\end{pmatrix}V^{-1}$, then $|T\cdot Q[{}^tC^{-1}]| = |Q||A| \asymp |A|$ and $\operatorname{tr}(T\cdot Q[{}^tC^{-1}] \asymp \operatorname{tr}(T\cdot 1_2[{}^tC^{-1}]) = \operatorname{tr} A[U]$. Thus we have

$$\sum_{U\in GL(2,Z)} \Big|\sum_{D \bmod CA} h(M, T)\Big| \ll |T|^{k/2-3/4} c_1^{1/2} c_2^{-1+\varepsilon}(c_2, T[v])^{1/2} f(A)\,,$$

where

$$f(A) = \sum_{\substack{U\in GL(2,Z)\\ \operatorname{tr} A[U]\ll 1}} |A|^{k/2-3/4} + \sum_{\substack{U\in GL(2,Z)\\ \operatorname{tr} A[U]\ll |A|}} |A|^{-1/4} + \sum_{\substack{U\in GL(2,Z)\\ \operatorname{tr} A[U]\gg 1\\ \operatorname{tr} A[U]\gg |A|}} |A|^{k/2-3/4}(\operatorname{tr} A[U])^{(1-k)/2}\,.$$

LEMMA 2. *Let $A^{(2)} > 0$ be Minkowski-reduced. Then we have*

$$f(A) \ll m(A)^{\varepsilon}\max(1, |A|)^{(3-k)/2+\varepsilon}|A|^{k/2-5/4-\varepsilon}\,,$$

where $m(A) = \min_{0\neq x\in Z^2} A[x]$.

Proof. Set $A = \begin{pmatrix} a & b\\ b & c\end{pmatrix}$. Since A in Minkowski-reduced, $A \asymp \begin{pmatrix} a & \\ & c\end{pmatrix}$, $|A| \asymp ac$, $m(A) \asymp a$, $a \leqq c$, and we have only to prove Lemma 2 for $H = \begin{pmatrix} a & \\ & c\end{pmatrix}$ instead of A. First we estimate $\#\{U\in GL(2, Z)|\operatorname{tr} H[U] \ll 1\}$. For $U = \begin{pmatrix} u_1 & u_2\\ u_3 & u_4\end{pmatrix} \in GL(2, Z)$ and $n \in Z$, it is easy to see

$$\operatorname{tr} H\left[\begin{pmatrix} 1 & n\\ & 1\end{pmatrix}U\right] = a(u_3^2 + u_4^2)\{n + (u_1u_3 + u_2u_4)(u_3^2 + u_4^2)^{-1}\}^2 + c(u_3^2 + u_4^2) + a(u_3^2 + u_4^2)^{-1}\,.$$

Hence $\operatorname{tr} H(U) \ll 1$ implies $c \leqq c(u_3^2 + u_4^2) \ll 1$ and $a(u_3^2 + u_4^2)(n + *)^2 \ll 1$. For relatively prime numbers u_3, u_4, we fix $U = \begin{pmatrix} * & *\\ u_3 & u_4\end{pmatrix}$, $U' = \begin{pmatrix} * & *\\ u_3 & u_4\end{pmatrix} \in GL(2, Z)$ with $|U| = 1$, $|U'| = -1$. Then any element in $GL(2, Z)$ is uniquely decomposed as $\begin{pmatrix} 1 & n\\ & 1\end{pmatrix}U$ or $\begin{pmatrix} 1 & n\\ & 1\end{pmatrix}U'$ for $n \in Z$. Thus we have

$$\begin{aligned}\#\{U\in GL(2, Z)|\operatorname{tr} H[U] \ll 1\} &\ll \sum_{\substack{(u_3,u_4)=1\\ u_3^2+u_4^2\ll c^{-1}}} \#\{n\in Z|(n + *)^2 \ll a^{-1}(u_3^2 + u_4^2)^{-1}\}\\ &\ll \sum_{\substack{(u_3,u_4)=1\\ u_3^2+u_4^2\ll c^{-1}}} a^{-1/2}(u_3^2 + u_4^2)^{-1/2}\\ &\ll \sum_{m\ll c^{-1}} a^{-1/2}m^{-1/2+\varepsilon} \ll a^{-1/2}c^{-1/2-\varepsilon}\,.\end{aligned}$$

Thus the first sum in $f(A)$ is $O(a^{k/2-5/4}c^{k/2-5/4-\varepsilon})$ if $c \ll 1$, or 0 otherwise. From $(a \leqq)c \ll 1$ follows

$$a^{k/2-5/4}c^{k/2-5/4-\varepsilon}m(A)^{\varepsilon}\max(1,|A|)^{(3-k)/2+\varepsilon}|A|^{k/2-5/4-\varepsilon})^{-1}$$
$$\asymp \max(1,|A|)^{(k-3)/2-\varepsilon} \ll 1\,.$$

Next we estimate $\#\{U \in GL(2,Z) | \operatorname{tr} H[U] \ll |H|\}$. $\operatorname{tr} H[U] \ll |H|$ implies $u_3^2 + u_4^2 \ll a$, $(n + *)^2 \ll c(u_3^2 + u_4^2)^{-1}$. Similarly to the first sum, we have

$$\#\{U \in GL(2,Z) | \operatorname{tr} H[U] \ll |H|\} \ll \sum_{m \ll a} m^{\varepsilon}(c/m)^{1/2} \ll c^{1/2}a^{1/2+\varepsilon}\,.$$

$u_3^2 + u_4^2 \ll a$ implies $1 \ll a \leq c$. Thus the second sum in $f(A)$ is $O(a^{1/4+\varepsilon}c^{1/4})$ if $1 \ll a$ or 0 otherwise. From $1 \ll a \leqq c$ follows

$$a^{1/4+\varepsilon}c^{1/4}(m(A)^{\varepsilon}\max(1,|A|)^{(3-k)/2+\varepsilon}|A|^{k/2-5/4-\varepsilon})^{-1}$$
$$\asymp (ac)^{3/2-k/2+\varepsilon}\max(1,|A|)^{(k-3)/2-\varepsilon} \ll 1\,.$$

Lastly we estimate the third sum in $f(A)$. Set

$$X = \sum_{\substack{U \in GL(2,\mathbf{Z}) \\ \operatorname{tr} A[U] \gg 1 \\ \operatorname{tr} A[U] \gg |A|}} |A|^{k/2-3/4}(\operatorname{tr} A[U])^{(1-k)/2}\,.$$

Then

$$X \ll (ac)^{k/2-3/4} \sum_{\substack{U \in GL(2,\mathbf{Z}) \\ \operatorname{tr} H[U] \gg \max(1,|H|)}} (\operatorname{tr} H[U])^{(1-k)/2}$$
$$\ll a^{-1/4}c^{k/2-3/4} \sum_{\substack{U \in GL(2,\mathbf{Z}) \\ \operatorname{tr} B[U] \gg \max(a^{-1},c)}} (\operatorname{tr} B[U])^{(1-k)/2}\,,$$

where we set $B = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$, $d = c/a$ $(\geqq 1)$. Hence we have

$$X \ll a^{-1/4}c^{k/2-3/4} \sum_{(u_3,u_4)=1} (u_3^2 + u_4^2)^{(1-k)/2} \sum_{u_1,u_2} \sum_{n} g(n,u_1,u_2,u_3,u_4)^{(1-k)/2}\,,$$

where $g(n,u_1,u_2,u_3,u_4) = \{n + (u_1u_3 + u_2u_4)(u_3^2 + u_4^2)^{-1}\}^2 + (u_3^2 + u_4^2)^{-2} + d$, u_3, u_4 run over relatively prime integers and for given u_3, u_4 we take integers u_1, u_2 such that $u_1u_4 - u_2u_3 = \pm 1$ and $|(u_1u_3 + u_2u_4)(u_3^2 + u_4^2)^{-1}| \leqq 1/2$, and n runs over integerr such that $g(n,u_1,u_2,u_3,u_4) \gg \max(a^{-1},c)(u_3^2 + u_4^2)^{-1}$. ($GL(2,Z)$ is parametrized by u_1, u_2, u_3, u_4 and n.) It is easy to see

$$g(n,u_1,u_2,u_3,u_4) \asymp n^2 + d\,,$$

since $|(u_1u_3 + u_2u_4)(u_3^2 + u_4^2)^{-1}| \leqq 1/2$ and $d \geqq 1$. Hence we have

$$X \ll a^{-1/4}c^{k/2-3/4} \sum_{(u_3,u_4)=1} (u_3^2 + u_4^2)^{(1-k)/2} \sum_{u_1,u_2} \sum_{n} g(n,u_1,u_2,u_3,u_4)^{(1-k)/2}$$
$$\ll a^{-1/4}c^{k/2-3/4} \sum_{(u_3,u_4)=1} (u_3^2 + u_4^2)^{(1-k)/2} \sum (n^2 + d)^{(1-k)/2}\,,$$

where $n \in \boldsymbol{Z}$ must statisfy $n^2 + d \gg \max(a^{-1}, c)(u_3^2 + u_4^2)^{-1}$. Hence we have

$$X \ll a^{-1/4}c^{k/2-3/4} \sum_{m \geq 1} m^{(1-k)/2+\varepsilon} \sum_{n^2+d \gg \max(a^{-1},c)/m} (n^2 + d)^{(1-k)/2}.$$

We prove

$$Y = \sum_{\substack{n \in \boldsymbol{Z} \\ n^2+d \geqq \alpha (>0)}} (n^2 + d)^{(1-k)/2} = O(\alpha^{1-k/2}).$$

If $d \geqq \alpha$, then

$$\begin{aligned} Y = \sum_n (n + d)^{(1-k)/2} &\ll d^{(1-k)/2} + \sum_{n \geqq 1} (n^2 + d)^{(1-k)/2} \\ &\ll d^{(1-k)/2} + \int_0^\infty (x^2 + d)^{(1-k)/2} dx \ll d^{(1-k)/2} + d^{1-k/2} \\ &\ll d^{1-k/2} \leqq \alpha^{1-k/2}. \end{aligned}$$

If $d < \alpha$, then, denoting by m the least positive integer n that satisfies $n^2 + d \geqq \alpha$, then we have

$$\begin{aligned} Y = 2 \sum_{n \geqq m} (n^2 + d)^{(1-k)/2} &= 2 \sum_{n \geqq 0} \{(n + m)^2 + d\}^{(1-k)/2} \\ &\ll \sum_{n \geqq 0} (n^2 + \alpha)^{(1-k)/2} \ll \alpha^{1-k/2}. \end{aligned}$$

We note $\max(a^{-1}, c)/\max(a, c^{-1}) = c/a = d$. Hence we have

$$\begin{aligned} X &\ll a^{-1/4}c^{k/2-3/4}\{\sum_{m \geqq \max(a,c^{-1})} m^{(1-k)/2+\varepsilon} d^{1-k/2} \\ &\quad + \sum_{m < \max(a,c^{-1})} m^{(1-k)/2+\varepsilon} (\max(a^{-1}, c)/m)^{1-k/2}\} \\ &\ll a^{-1/4}c^{k/2-3/4}\{\max(a, c^{-1})^{(3-k)/2+\varepsilon} d^{1-k/2} + \max(a^{-1}, c)^{1-k/2} \max(a, c^{-1})^{1/2+\varepsilon}\} \\ &\asymp a^\varepsilon \max(1, ac)^{(3-k)/2+\varepsilon} (ac)^{k/2-5/4-\varepsilon} \\ &\asymp m(A)^\varepsilon \max(1, |A|)^{(3-k)/2+\varepsilon} |A|^{k/2-5/4-\varepsilon}. \end{aligned}$$

Thus we have completed a proof of Lemma 2.

Lemma 2 implies immediately

LEMMA 3.

$$\begin{aligned} \sum_{U \in GL(2,\boldsymbol{Z})} \Big| \sum_{D \bmod CA} h(M, T) \Big| &\ll |T|^{k-2-\varepsilon} m\left(T\left[V\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix}^{-1}\right]\right)^\varepsilon \\ &\quad \times \max(1, |T|(c_1c_2)^{-2})^{(3-k)/2+\varepsilon} c_1^{3-k+2\varepsilon} c_2^{3/2-k+3\varepsilon} (c_2, T[v])^{1/2}, \end{aligned}$$

where $C = U^{-1}\begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix} V^{-1}$, $0 < c_1 | c_2$ *and* v *is the second column of* V.

Now we can prove our theorem.

$$\sum_{c_1|c_2} \sum_{V\in GL(2,\mathbf{Z})/P(c_2/c_1)} \sum_{U\in GL(2,\mathbf{Z})} \Big|\sum_{D \bmod CA} h(M,T)\Big| \ll \sum_{c_1|c_2}\sum_{V\in GL(2,\mathbf{Z})/P(c_2/c_1)} |T|^{k-2-\varepsilon}$$

$$\times\, m\left(T\left[V\begin{pmatrix}c_1 & \\ & c_2\end{pmatrix}^{-1}\right]\right)^{\varepsilon} \max(1, |T|(c_1c_2)^{-2})^{(3-k)/2+\varepsilon} c_1^{3-k+2\varepsilon} c_2^{3/2-k+3\varepsilon}(c_2, T[v])^{1/2}$$

$$\ll |T|^{k-2-\varepsilon/2} \sum_{c_1|c_2}\sum_{V\in GL(2,\mathbf{Z})/P(c_2/c_1)} \max(1, |T|(c_1c_2)^{-2})^{(3-k)/2+\varepsilon}$$

$$\times\, c_1^{3-k+\varepsilon} c_2^{3/2-k+2\varepsilon}(c_2, T[v])^{1/2}\,,$$

where we used $m\left(T\left[V\begin{pmatrix}c_1 & \\ & c_2\end{pmatrix}^{-1}\right]\right) \ll \left|T\left[V\begin{pmatrix}c_1 & \\ & c_2\end{pmatrix}^{-1}\right]\right|^{1/2}$,

$$= \Sigma_1 + \Sigma_2,$$

where Σ_1 (resp. Σ_2) is a partial sum such that $(c_1c_2)^2 \geqq |T|$ (resp. $(c_1c_2)^2 < |T|$).

$$\Sigma_1 \ll |T|^{k-2-\varepsilon/2} \sum_{\substack{c_1|c_2\\ (c_1c_2)^2\geqq |T|}} c_1^{3-k+\varepsilon}c_2^{3/2-k+2\varepsilon} \sum_{V\in GL(2,\mathbf{Z})/P(c_2/c_1)} c_1^{1/2}(c_2/c_1, T[v])^{1/2}$$

$$\ll |T|^{k-2-\varepsilon/2} \sum_{\substack{c_1|c_2\\ (c_1c_2)^2\geqq |T|}} c_1^{7/2-k+\varepsilon}c_2^{3/2-k+2\varepsilon}(c_2/c_1)^{1+\varepsilon}(c_2/c_1, e(T))^{1/2}$$

$$\ll |T|^{k-2-\varepsilon/2} \sum_{\substack{c_1, n\\ c_1^2 n\geqq \sqrt{|T|}}} c_1^{5-2k+3\varepsilon} n^{5/2-k+3\varepsilon}(e(T), n)^{1/2}\,.$$

Since
$$\sum_{n\geqq\alpha} n^{5/2-k+3\varepsilon}(e(T), n)^{1/2} < \sum_{r|e(T)}\sum_{s\geqq\alpha/r}(sr)^{5/2-k+3\varepsilon}r^{1/2}$$
$$= \sum_{r|e(T)} r^{3-k+3\varepsilon}\sum_{s\geqq\alpha/r} s^{5/2-k+3\varepsilon} \ll \sum_{r|e(T)} r^{3-k+3\varepsilon}(\alpha/r)^{7/2-k+3\varepsilon}$$
$$= \alpha^{7/2-k+3\varepsilon}\sum_{r|e(T)} r^{-1/2} = O(e(T)^{\varepsilon}\alpha^{7/2-k+3\varepsilon})\,,$$

we have
$$\Sigma_1 \ll |T|^{k-2-\varepsilon/2}\sum_{c_1} c_1^{5-2k+3\varepsilon} e(T)^{\varepsilon}(\sqrt{|T|}/c_1^2)^{7/2-k+3\varepsilon}$$
$$= |T|^{k/2-1/4+\varepsilon} e(T)^{\varepsilon}\sum_{c_1} c_1^{-2-3\varepsilon} = O(|T|^{k/2-1/4+2\varepsilon})\,.$$

$$\Sigma_2 \ll |T|^{k-2-\varepsilon/2} \sum_{\substack{c_1|c_2\\ (c_1c_2)^2< |T|}} |T|^{(3-k)/2+\varepsilon}c_1^{-\varepsilon}c_2^{-3/2} \sum_{V\in GL(2,\mathbf{Z})/P(c_2/c_1)} (c_2, T[v])^{1/2}$$

$$\ll |T|^{k/2-1/2+\varepsilon/2} \sum_{\substack{c_1|c_2\\ (c_1c_2)^2< |T|}} c_1^{-\varepsilon}c_2^{-3/2}c_1^{1/2}(c_2/c_1)^{1+\varepsilon}(c_2/c_1, e(T))^{1/2}$$

$$= |T|^{k/2-1/2+\varepsilon/2} \sum_{\substack{c_1, n\\ c_1^2 n< \sqrt{|T|}}} c_1^{-1-\varepsilon} n^{-1/2+\varepsilon}(n, e(T))^{1/2}\,.$$

Since
$$\sum_{n<\beta} n^{-1/2+\varepsilon}(n, e(T))^{1/2} < \sum_{r|e(T)}\sum_{s<\beta/r}(sr)^{-1/2+\varepsilon}r^{1/2}$$
$$= \sum_{r|e(T)} r^{\varepsilon}\sum_{s<\beta/r} s^{-1/2+\varepsilon} \ll \sum_{r|e(T)} r^{\varepsilon}(\beta/r)^{1/2+\varepsilon}$$
$$= \beta^{1/2+\varepsilon}\sum_{r|e(T)} r^{-1/2} = O(e(T)^{\varepsilon}\beta^{1/2+\varepsilon})\,,$$

we have
$$\Sigma_2 \ll |T|^{k/2-1/2+\varepsilon/2}\sum_{c_1} c_1^{-1-\varepsilon}e(T)^{\varepsilon}(\sqrt{|T|}/c_1^2)^{1/2+\varepsilon}$$

$$= |T|^{k/2-1/4+\varepsilon}e(T)^{\varepsilon}\sum_{c_1} c_1^{-2-3\varepsilon} = O(|T|^{k/2-1/4+2\varepsilon}).$$

It is easy to see that $\sum h(M, T)$ is absolutely convergent with minor changes. Thus we have completed a proof of our theorem.

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