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ON NON-LOCAL PROBLEMS FOR PARABOLIC EQUATIONS

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The main purposes of this paper are to investigate the existence and the uniqueness of a non-local problem for a linear parabolic equation

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t)\frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = f(x, t) \tag{1}$$

in a cylinder $D = \Omega \times (0, T]$. Given functions β_i $(i = 1, \cdots, N)$ on Ω and numbers $T_i \in (0, T]$ $(i = 1, \cdots, N)$, the problem in question is to find a solution u of (1) satisfying the following conditions

$$u(x, t) = \phi(x, t) \quad \text{on} \quad \Gamma, \tag{2}$$

$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) = \Psi(x) \quad \text{on} \quad \Omega, \tag{3}$$

where f, ϕ and Ψ are given functions and Γ denotes the lateral surface of D, i.e., $\Gamma = \partial\Omega \times [0, T]$.

In Section 1 we establish the maximum principle associated with the problem described by (1), (2) and (3). Theorem 1 leads immediately to the uniqueness of solution of the problem (1), (2) and (3) as well as to an estimate of the solution in terms of f, ϕ and Ψ. We also briefly discuss certain properties of the solutions related to the behaviour of the coefficients β_i $(i = 1, \cdots, N)$. In Theorem 5 of Section 2 we establish the existence of the solution in a bounded cylinder. The results are then applied to derive the existence and the uniqueness of solution of the non-local problem in an unbounded cylinder (Section 3). In Section 4 we establish an integral representation of solutions and give a construction of the solution of a non-local problem in $R_n \times (0, T]$ with $\Psi \in L^2(R_n)$. In the last section we modify the condition (3) by replacing a finite sum by an infinite series and briefly discuss the uniqueness and the existence of solution of the resulting problem. Theorems of Sections 1 and 2 of this

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paper extend and improve earlier results obtained by Kerefov [3] and Vabishchevich [6], where historical references can be found. They only considered the case $N = 1$.

1. Let $D = \Omega \times (0, T]$, where Ω is a bounded domain in R_n. By Γ we denoted the lateral surface of D, i.e., $\Gamma = \partial\Omega \times [0, T]$.

Throughout this section we make the following assumption

(A) The coefficients a_{ij}, b_i and c are continuous on D and moreover

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j > 0$$

for all vectors $\xi \neq 0$ and $(x, t) \in D$.

By $C^{2,1}(D)$ we denote the set of functions u continuous on D with their derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_i$ $(i, j = 1, \cdots, n)$ and $\partial u/\partial t$ (at $t = T$ the derivative $\partial u/\partial t$ is understood as the left-hand derivative).

LEMMA 1. *Let $u \in C^{2,1}(D) \cap C(\bar{D})$. Suppose that $c(x, t) \leq 0$ on D and $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ on Ω and $\beta_i(x) \leq 0$ on Ω $(i = 1, \cdots, N)$. If $Lu \leq 0$ in D, $u(x, t) \geq 0$ on Γ and $u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) \geq 0$ on Ω, then $u(x, t) \geq 0$ on $\bar{D}$.*

Proof. Assume that $u < 0$ at some point of $\bar{D}$, then there exists a point $(x_0, t_0) \in \bar{D}$ such that $u(x_0, t_0) = \min_{\bar{D}} u(x, t) < 0$. By the strong maximum principle $(x_0, t_0) = (x_0, 0)$ with $x_0 \in \Omega$ (see Friedman [2] Chap. 2 or Protter and Weinberger [5] Chap. 3). Thus, we find that

$$0 \leq u(x_0, 0) + \sum_{i=1}^{N} \beta_i(x_0)u(x_0, T_i) \leq u(x_0, 0)\left[1 + \sum_{i=1}^{N} \beta_i(x_0)\right].$$

Hence $u(x_0, 0) \geq 0$ provided $1 + \sum_{i=1}^{N} \beta_i(x_0) > 0$ and we get a contradiction.

In the case $\sum_{i=1}^{N} \beta_i(x_0) = -1$ we put $u(x_0, T_k) = \min_{i=1,\ldots,N} u(x_0, T_i)$, then

$$\begin{aligned} u(x_0, 0) - u(x_0, T_k) &= u(x_0, 0) + u(x_0, T_k) \sum_{i=1}^{N} \beta_i(x_0) \\ &\geq u(x_0, 0) + \sum_{i=1}^{N} \beta_i(x)u(x_0, T_i) \geq 0 . \end{aligned}$$

Hence u takes on a negative minimum at $(x_0, T_k) \in D$. This contradiction completes the proof.

COROLLARY. *Suppose that the assumptions of Lemma 1 hold. If $L \geq 0$ in D, $u(x, t) \leq 0$ an Γ and $u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) \leq 0$ on Ω, then $u(x, t) \leq 0$ on $\bar{D}$.*

Now we can state the main result of this section.

THEOREM 1. *Let* $u \in C^{2,1}(D) \cap C(\bar{D})$ *be a solution of the problem* (1), (2) *and* (3) *with* f, ϕ *and* Ψ *continuous on* $\bar{D}$, Γ *and* $\bar{\Omega}$ *respectively. Suppose that* $c(x, t) \leq -c_0$ *in* D, *where* c_0 *is a positive constant. Assume further that* $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ *and* $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ *on* Ω. *Then*

$$
|u(x, t)| \leq \frac{2}{c_0} e^{(c_0/2)T} \sup_D |f(x, t)| + e^{(c_0/2)T} \sup_\Gamma |\phi(x, t)| + (1 - e^{-(c_0/2)T_k})^{-1} \sup_\Omega |\Psi(x)| \tag{4}
$$

for all $(x, t) \in \bar{D}$, *where* $T_k = \min_{i=1,\dots,N} T_i$.

Proof. We first suppose that $-1 < -\beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ on Ω, where β_0 is a positive constant. Let $M = \sup_D |f(x, t)|$, $M_1 = \sup_\Gamma |\phi(x, t)|$, $M_2 = \sup_\Omega |\Psi(x)|$ and put

$$
v(x, t) = u(x, t) - \frac{M}{c_0} - M_1 - \frac{M_2}{1 - \beta_0}.
$$

Then

$$
Lv = f - \frac{c}{c_0} M - cM_1 - \frac{cM_2}{1 - \beta_0} \geq c_0 M_1 + \frac{c_0}{1 - \beta_0} M_2 > 0
$$

in D, $v(x, t) \leq 0$ on Γ and

$$
\begin{aligned}
v(x, 0) + \sum_{i=1}^{N} \beta_i(x) v(x, T_i) &= \Psi(x) - \frac{M}{c_0} - M_1 - \frac{M_2}{1 - \beta_0} \\
&\quad - \left(\frac{M}{c_0} + M_1 + \frac{M_2}{1 - \beta_0}\right) \sum_{i=1}^{N} \beta_i(x) \leq \left(\frac{M}{c_0} + M_1\right)(\beta_0 - 1) \\
&\quad + M_2\left(1 - \frac{1}{1 - \beta_0} + \frac{\beta_0}{1 - \beta_0}\right) < 0
\end{aligned}
$$

on Ω. It follows from Lemma 1 that $v \leq 0$ on D. Similarly we can establish the inequality $u(x, t) \geq -(M/c_0) - M_1 - M_2/(1 - \beta_0)$ for $(x, t) \in \bar{D}$ considering the auxiliary function

$$
w(x, t) = u(x, t) + \frac{M}{c_0} + M_1 + \frac{M_2}{1 - \beta_0}
$$

In the general case we put $u(x, t) = e^{-(c_0/2)t} z(x, t)$. Then z satisfies the equation

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 z}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x, t)\frac{\partial z}{\partial x_i} + \left(c(x, t) + \frac{c_0}{2}\right)z - \frac{\partial z}{\partial t} = e^{(c_0/2)t}f(x, t) \tag{5}$$

in D with $c(x, t) + c_0/2 \leq -(c_0/2)$ in D,

$$z(x, t) = e^{(c_0/2)t}\phi(x, t) \quad \text{on } \Gamma$$

and

$$z(x, 0) + \sum_{i=1}^{N} \beta_i(x)e^{-(c_0/2)T_i}z(x, T_i) = \Psi(x) \quad \text{on } \Omega .$$

It is clear that $-e^{-(c_0/2)T_k} \leq \sum_{i=1}^{N} \beta_i(x)e^{-(c_0/2)T_i} \leq 0$ on Ω and the estimate easily follows.

Theorem 1 and a classical maximum principle for solutions of parabolic equations allow us to compare a solution of the problem (1), (2) and (3) with a solution of an initial boundary value problem.

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *hold. Let* $u \in C^{2,1}(D) \cap C(\bar{D})$ *be a solution of the problem* (1), (2) *and* (3), *and* $v \in C^{2,1}(D) \cap C(\bar{D})$ *a solution of* (1) *satisfying the initial boundary value conditions* $v(x, t) = \phi(x, t)$ *on* Γ *and* $v(x, 0) = \Psi(x)$ *on* Ω. *Then*

$$|u(x, t) - v(x, t)| \leq \sup_{\Omega} \sum_{i=1}^{N} |\beta_i(x)| \Big[\frac{2}{c_0} e^{(c_0/2)T} \sup_{D} |f(x, t)| + e^{(c_0/2)T} \sup_{\Gamma} |\phi(x, t)| + (1 - e^{-(c_0/2)T_k})^{-1} \sup_{\Omega} |\Psi(x)|\Big]$$

for all $(x, t) \in \bar{D}$.

In particular if $\beta_i = \beta_\nu^i(x)$ $(i = 1, \cdots, N)$ where $\beta_\nu^i \to 0$ uniformly as $\nu \to \infty$ for all i, then the corresponding sequence u_ν of solutions of the problem (1), (2) and (3) converges uniformly to v in $\bar{D}$.

THEOREM 3. *Let* $c(x, t) \leq 0$ *in* D *and assume that* $-1 \leq \sum_{i=1}^{N} \beta_i^j(x) \leq 0$ $(j = 1, 2)$ *and that* $\beta_i^1(x) \leq \beta_i^2(x) \leq 0$ $(i = 1, \cdots, N)$ *on* Ω. *Suppose further that* $f \leq 0$, $\phi \geq 0$ *and* $\Psi \geq 0$ *on* D, Γ *and* $\bar{\Omega}$ *respectively. If* u_1 *and* u_2 *are solutions belonging to* $C^{2,1}(D) \cap C(\bar{D})$ *of the problem* (1), (2) *and* (3) *with* $\beta_i = \beta_i^1(x)$ $(i = 1, \cdots, N)$ *and* $\beta_i = \beta_i^2(x)$ $(i = 1, \cdots, N)$ *respectively, then* $u_1(x, t) \geq u_2(x, t)$ *on* $\bar{D}$.

Proof. We put $w(x, t) = u_1(x, t) - u_2(x, t)$, then $Lw = 0$ in D, $w(x, t) = 0$ on Γ and

$$w(x,0) + \sum_{i=1}^{N} \beta_i^1(x) w(,x, T_i) = \sum_{i=1}^{N} (\beta_i^2(x) - \beta_i^1(x)) u_2(x, T_i) \quad \text{on } \Omega .$$

Since $u_2(x, t) \geq 0$ on $\bar{D}$, it follows from Lemma 1, that $w(x, t) \geq 0$ for all $(x, t) \in \bar{D}$.

Lemma 1 yields the uniqueness of solutions of the problem (1), (2) and (3) under the assumptions that $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ and $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ on Ω. Vabishchevich [6] pointed out, without giving any proof, that in the case $N = 1$ the uniqueness can be proved under the assumption $|\beta(x)| \leq 1$ on Ω. For the sake of completeness we include the proof of uniqueness under the assumption $\sum_{i=0}^{N} |\beta_i(x)| \leq 1$ on Ω.

THEOREM 4. *Suppose that $c(x, t) \leq 0$ on D and $\sum_{i=1}^{N} |\beta_i(x)| \leq 1$ on Ω. Then the problem* (1), (2) *and* (3) *has at most one solution in $C^{2,1}(D) \cap C(\bar{D})$.*

Proof. Let u be a solution of the homogeneous problem

$$Lu = 0 \quad \text{in } D$$
$$u(x, t) = 0 \quad \text{on } \Gamma$$

and

$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x) u(x, T_i) = 0 \quad \text{on } \Omega .$$

Suppose that $u \not\equiv 0$. We also many assume that there exists a point in $(x_0, t_0) \in \bar{D}$ such that $u(x_0, t_0) = \min_{\bar{D}} u(x, t) < 0$. It is clear that $(x_0, t_0) = (x_0, 0)$ with $x_0 \in \Omega$. We can assume that $|u(x_0, T_1)| = \max_{i=1,\dots,N} |u(x_0, T_i)| > 0$, since otherwise there is nothing to prove. Obviously,

$$|u(x_0, 0)| \leq |u(x_0, T_1)| \sum_{i=1}^{N} |\beta_i(x_0)| \leq |u(x_0, T_1)| .$$

If $u(x_0, T_1) < 0$ then $u(x_0, T_1) \leq u(x_0, 0)$. Hence u attains its negative minimum at (x_0, T_1) and we get a contradiction, therefore $u(x_0, T_1) > 0$. Thus there exists a point $(x_1, t_1) \in \bar{D}$ such that $u(x_1, t_1) = \max_{\bar{D}} u(x, t) > 0$. Again $(x_1, t_1) = (x_1, 0)$ with $x_1 \in \Omega$. Put $|u(x_1, T_s)| = \max_{i=1,\dots,N} |u(x_1, T_i)|$. We may assume that $|u(x_1, T_s)| > 0$, since otherwise there is nothing to prove. Now we must distinguish two cases

$$|u(x_0, 0)| < u(x_1, 0) \quad \text{or} \quad u(x_1, 0) \leq |u(x_0, 0)| .$$

In the first case we have

$$|u(x_0, 0)| < u(x_1, 0) \leq |u(x_1, T_s)| \sum_{i=1}^{N} |\beta_i(x)| \leq |u(x_1, T_s)| ,$$

consequently if $u(x_1, T_s) < 0$ then $u(x_0, 0) > u(x, T_s)$. Hence u takes on a positive minimum at $(x_1, T_s) \in D$ and we get a contradiction. On the other hand if $u(x_1, T_s) > 0$ we have $u(x_1, 0) \leq u(x_1, T_s)$. Hence u attains a positive maximum at (x_1, T_s) and we arrive at a contradiction. Similarly in the second case we obtain

$$u(x_1, 0) \leq |u(x_0, 0)| \leq u(x_0, T_1) \sum_{i=1}^{N} |\beta_i(x_0)| \leq u(x_0, T_1)$$

and u takes on a positive maximum at $(x_0, T_1) \in D$. This contradiction completes the proof.

2. For the existence theorem we shall need the following assumptions

(A_1) There exist positive constants λ_0 and λ_1 such that, for any vector $\xi \in R_n$

$$\lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq \lambda_1 |\xi|^2$$

for all $(x, t) \in D$.

(A_2) The coefficients a_{ij}, b_i $(i, j = 1, \cdots, n)$, c and f are Hölder continuous in D (exponent α).

(A_3) The functions ϕ, Ψ and β_i $(i = 1, \cdots, N)$ are continuous respectively on Γ, $\bar{\Omega}$ and $\bar{\Omega}$ and, in addition,

$$\Psi(x) = \phi(x, 0) + \sum_{i=1}^{N} \beta_1(x)\phi(x, T_i)$$

for all $x \in \partial\Omega$.

Moreover we assume that $\partial\Omega \in C^{2+\alpha}$.

THEOREM 5. *Let $c(x, t) \leq -c_0$, where c_0 is a positive constant and assume that $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ and $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ on $\bar{\Omega}$. Then there exists a unique solution in $C^{2,1}(D) \cap C(\bar{D})$ of the problem* (1), (2) *and* (3).

Proof. We first assume that $\phi \equiv 0$ on Γ, then by the condition (A_3) $\Psi(x) = 0$ on $\partial\Omega$. We try to find a solution in the form

$$u(x, t) = \int_{\Omega} G(x, t; y, 0)u(y, 0)dy - \int_0^t \int_{\Omega} G(x, t; y, \tau)f(y, \tau)dyd\tau, \tag{6}$$

where $u(y, 0)$ is to be determined and G denotes the Green function for the operator L. The condition (3) leads to the Fredholm integral equation of the second kind

$$(7)\qquad \begin{aligned} &u(x, 0) + \sum_{i=1}^{N} \beta_i(x) \int_{\Omega} G(x, T_i; y, 0) u(y, 0) dy \\ &\quad = \Psi(x) + \sum_{i=1}^{N} \beta_i(x) \int_0^{T_i} G(x, T_i; y, \tau) f(y, \tau) dy d\tau . \end{aligned}$$

Applying Theorem 4 it is easy to show that the corresponding homogeneous equation only has a trivial solution in $L^2(\Omega)$. Hence there exists a unique solution $u(\cdot, 0)$ in $L^2(\Omega)$ of the equation (7). Since $\Psi(x) = 0$ on $\partial\Omega$, it follows from the properties of the Green function that $u(\cdot, 0) \in C(\bar{\Omega})$ and $u(x, 0) = 0$ on $\partial\Omega$. Consequently the formula (6) gives a solution in this case.

Suppose next $\phi \not\equiv 0$, but assume that there exists a function $\Phi \in \bar{C}^{2+\alpha}(D)$ such that $\Phi = \phi$ on Γ. Introducing $v = u - \Phi$ we then immediately obtain, by the previous result, the existence of a solution v to $Lv = f - L\Phi$ which vanishes on Γ and satisfies the condition

$$v(x, 0) + \sum_{i=1}^{N} \beta_i(x) v(x, T_i) = \Psi(x) - \Phi(x, 0) - \sum_{i=1}^{N} \beta_i(x) \Phi(x, T_i)$$

for all $x \in \Omega$. Then assertions for u then follow.

We finally consider the general case, where ϕ is only assumed to be continuous. By Theorem 2 in Friedman [2] (p. 60) and the Weierstrass approximation theorem there exists a sequenece of polynomials Φ_m on $\bar{D}$ which approximates ϕ uniformly on Γ. Now we define a function Ψ_m on $\partial\Omega$ by the following formula

$$\Psi_m(x) = \Phi_m(x, 0) + \sum_{i=1}^{N} \beta_i(x) \Phi_m(x, T_i)$$

for $x \in \partial\Omega$. Since $\lim_{m\to\infty} \Psi_m = \Psi$ uniformly on $\partial\Omega$, one can construct a sequence of functions $\{\tilde{\Psi}_m\}$ in $C(\bar{\Omega})$ such that $\lim_{m\to\infty} \tilde{\Psi}_m = \Psi$ uniformly on $\bar{\Omega}$ and $\tilde{\Psi}_m = \Psi_m$ on $\partial\Omega$ for all m. By what we have already proved there exist solutions to the problem

$$\begin{aligned} Lu_m &= f \quad \text{in } D, \\ u_m(x, t) &= \Phi_m(x, t) \quad \text{on } \Gamma , \end{aligned}$$

and

$$u_m(x, 0) + \sum_{i=1}^{N} \beta_i(x) u_m(x, T_i) = \tilde{\Psi}_m(x) \quad \text{on } \Omega .$$

By Theorem 1 (the inequality (4)) the sequence $u_m(x, t)$ is uniformly convergent on $\bar{D}$ to a function u. It is clear that u satisfies the conditions (2) and (3). Using Friedman-Schauder interior estimates (Friedman [2], Theorem 5 p. 64) one can easily prove that u satisfies the equation (1).

Remark. In the above proof we followed the argument used in the proof of Theorem 9 in Friedman [2] (p. 70–71). For the definition of the space $\bar{C}^{2+\alpha}(D)$ see Friedman [2] (p. 61–62).

THEOREM 6. *Suppose that* $\sum_{i=1}^{N}|\beta_i(x)| \leq 1$ *on* Ω, $c(x, t) \leq 0$ *on* D *and* $\phi \equiv 0$ *on* Γ. *Then the problem* (1), (2) *and* (3) *has a unique solution in* $C^{2,1}(D) \cap C(\bar{D})$.

Proof. A solution to this problem is given by the formula

$$u(x, t) = \int_\Omega G(x, t; y, 0)u(y, 0)dy - \int_0^t \int_\Omega G(x, t; y, \tau)f(y, \tau)dyd\tau ,$$

where $u(x, 0)$ is a solution of the Fredholm integral equation of the second kind

$$\begin{aligned} u(x, 0) &+ \sum_{i=1}^{N} \beta_i(x) \int_\Omega G(x, T_i; y, 0)u(y, 0)dy \\ &= \Psi(x) + \sum_{i=1}^{N} \beta_i(x) \int_0^{T_i} \int_\Omega G(x, T_i; y, \tau)f(y, \tau)dyd\tau . \end{aligned}$$

3. In this section we investigate the existence of a solution of the problem (1), (2) and (3) in an unbounded cylinder. Let $D = \Omega \times (0, T]$, where Ω is an unbounded domain in R_n.

In the next theorem we give a general method of constructing a solution. We shall need the following assumptions

(B_1) The coefficients a_{ij}, b_i $(i, j = 1, \cdots, n)$ and c are continuous on D and moreover

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j > 0$$

for every $(x, t) \in D$ and any vector $\xi \neq 0$, $a_{ij} = a_{ji}$ $(i, j = 1, \cdots, n)$.

(B_2) There exists a family of positive function $H(x, \delta)$ $(0 < \delta \leq \delta_0)$ defined on Ω with properties:

(i) $H \in C^2(\Omega) \cap C(\bar{\Omega})$ for $0 < \delta \leq \delta_0$ and $LH \leq -c_0 H$ for all $(x, t) \in D$ and $0 < \delta \leq \delta_0$, where c_0 is a positive constant,

(ii) $\lim_{|x| \to \infty} \dfrac{H(x, \delta_1)}{H(x, \delta_2)} = 0$ for $0 < \delta_1 < \delta_2 \leq \delta_0$,

(iii) there exists a positive constant κ such that

$$H(x, \delta_1) \leq \kappa H(x, \delta_2)$$

for all $x \in \Omega$ and $0 < \delta_1 < \delta_2 \leq \delta_0$.

For a sequence $\{R_p\}$ of positive numbers we define

$$\Omega_p = \Omega \cap \{x : |x| < R_p\}, \quad \Gamma_p = \partial\Omega_p \times [0, T] \quad \text{and} \quad D_p = \Omega_p \times (0, T].$$

(B_3) There exists a sequence of positive numbers R_p converging to ∞ as $p \to \infty$ such that the problem (1), (2) and (3) is solvable on every D_p, i.e. for every bounded and Hölder continuous function f on D_p and all continuous functions ϕ and Ψ on Γ_p and $\bar{\Omega}_p$ respectively, and satisfying the condition

$$\Psi(x) = \phi(x, 0) + \sum_{i=1}^{N} \beta_i(x)\phi(x, T_i) \quad \text{on } \partial\Omega_p,$$

the problem

$$Lu = f \quad \text{in } D_p,$$
$$u(x, t) = \phi(x, t) \quad \text{on } \Gamma_p$$

and

$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) = \Psi(x) \quad \text{on } \Omega_p$$

has a unique solution in $C^{2,1}(D_p) \cap C(\bar{D}_p)$.

We shall say that a function u defined on D belongs to $E_H(D)$ if there exist positive constants M and $\delta < \delta_0$ such that $|u(x, t)| \leq MH(x, \delta)$ for all $(x, t) \in D$.

We shall say that a function v defined on Ω belongs to $E_H(\Omega)$ if there exist positive constants M and $\delta < \delta_0$ such that $|v(x)| \leq MH(x, \delta)$ for all $x \in \Omega$.

We are now in a position to construct a solution of the problem (1), (2) and (3). The construction given in the proof of Theorem 7 below is a modification of the method used by Krzyżański [4] to solve the Cauchy problem for parabolic equations.

THEOREM 7. *Suppose that the assumptions* (B_1), (B_2) *and* (B_3) *hold. Let* $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ *and* $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ *on* Ω. *Assume that* $f \in E_H(D)$ *is an Hölder continuous function, that* $\phi \in E_H(D)$ *and* $\Psi \in E_H(\Omega)$ *are continuous functions on* $\bar{D}$ *and* $\bar{\Omega}$ *respectively and moreover that*

$$\Psi(x) = \phi(x, 0) + \sum_{i=1}^{N} \beta_i(x)\phi(x, T_i) \quad on\ \partial\Omega_p$$

$p = 1, 2, \cdots$. *Then the problem* (1), (2) *and* (3) *has a unique solution in* $C^{2,1}(D) \cap C(\bar{D}) \cap E_H(D)$,

Proof. It is clear that there exist positive constants M and $\delta \leq \delta_0$ such that

$$|\phi(x, t)| \leq MH(x, \delta), \qquad |f(x, t)| \leq MH(x, \delta) \quad \text{on } D,$$
$$|\Psi(x)| \leq MH(x, \delta) \quad \text{on } \Omega.$$

By the assumption (B_3) for every p there exists a unique solution u_p in $C^{2,1}(D) \cap C(\bar{D})$ of the problem

$$Lu_p = f \quad \text{on } D_p,$$
$$u_p(x, t) = \phi(x, t) \quad \text{on } \Gamma_p,$$

and

$$u_p(x, 0) + \sum_{i=1}^{N} \beta_i(x)u_p(x, T_i) = \Psi(x) \quad \text{on } \bar{\Omega}_p.$$

Put

$$u_p(x, t) = v_p(x, t)H(x, \delta) \quad p = 1, 2, \cdots$$

for $(x, t) \in D_p$. Then for every p $|v_p(x, t)| \leq M$ on Γ_p,

$$\left| v_p(x, 0) + \sum_{i=1}^{N} \beta_i(x)v_p(x, T_i)\right| \leq \frac{|\Psi(x)|}{H(x, \delta)} \leq M \quad \text{on } \Omega_p$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2 v_p}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left(b_i(x, t) + \frac{2}{H(x, \delta)} \sum_{j=1}^{n} a_{ij}(x, t)\frac{\partial H}{\partial x_j}\right)\frac{\partial v_p}{\partial x_i} + \frac{LH}{H}v_p - \frac{\partial v_p}{\partial t} = \frac{f(x, t)}{H(x, \delta)} \tag{8}$$

in D_p. It follows from the assumption (B_2 i) and Theorem 1 that

$$|v_p(x, t)| \leq \left[\frac{2}{c_0} e^{(c_0/2)T} + e^{(c_0/2)T} + (1 - e^{-(c_0/2)T_k})^{-1}\right]M = M_1$$

for all $(x, t) \in D_p$, $p = 1, 2, \cdots$, where $T_k = \min_i T_i$. Let $\delta < \delta_1 < \delta_0$ and put

$$u_p(x, t) = \bar{v}_p(x, t)H(x, \delta_1) \qquad p = 1, 2, \cdots$$

and

$$u_{pq}(x, t) = u_p(x, t) - u_q(x, t) = H(x, \delta_1)[\bar{v}_p(x, t) - \bar{v}_q(x, t)] = H(x, \delta_1)\bar{v}_{pq}(x, t)$$

for $p < q$. The function $\bar{v}_{pq}$ satisfies the homogeneous equation of the form (7) with $H(x, \delta)$ replaced by $H(x, \delta_1)$ and

$$\bar{v}_{pq}(x, 0) + \sum_{i=1}^{N} \beta_i(x)\bar{v}_{pq}(x, T_i) = 0$$

on Ω_p. Moreover

$$\bar{v}_{pq}(x, t) = 0 \quad \text{on } (\partial\Omega_p \cap \partial\Omega) \times (0, T]$$

and

$$\bar{v}_{pq}(x, t) = \frac{\phi_p(x, t)}{H(x, \delta_1)} - \frac{u_q(x, t)}{H(x, \delta_1)} \quad \text{on } \Gamma_p \cap D,$$

consequently

$$|\bar{v}_{pq}(x, t)| \leq (M + M_1) \sup_{\partial\Omega_p - \partial\Omega} \frac{H(x, \delta)}{H(x, \delta_1)} \quad \text{on } \Gamma_p .$$

Let

$$\varepsilon_p = (M + M_1) \sup_{\partial\Omega_p - \partial\Omega} \frac{H(x, \delta)}{H(x, \delta_1)} .$$

Thus by Theorem 1 we have

$$|\bar{v}_{pq}(x, t)| \leq \varepsilon_p e^{(c_0/2)T}$$

on $\bar{D}_p$. By the assumption (B_2 ii) $\lim_{p\to\infty} \varepsilon_p = 0$, hence $\bar{v}_p$ converges uniformly on every $\bar{D}_s$ to a function $\bar{v}$. Put $u(x, t) = \bar{v}(x, t)H(x, \delta_1)$ for $(x, t) \in \bar{D}$. Clearly $u \in E_H(D)$ is continuous on $\bar{D}$ and satisfies (2) and (3). To show that u satisfies (1), fix an arbitrary index p and consider the problem

$$Lz = f \quad \text{in } D_p ,$$
$$z(x, t) = u(x, t) \quad \text{on } \Gamma_p ,$$
$$z(x, 0) + \sum_{i=1}^{N} \beta_i(x)z(x, T_i) = \Psi(x) \quad \text{on } \Omega_p .$$

Since u satisfies the condition (3), it is clear that

$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) = \Psi(x) \quad \text{on } \partial\Omega_p .$$

By the assumption (B_3) this problem has a unique solution z. Since $u_q \to u$

as $q \to \infty$ uniformly on $\bar{D}_p$, given $\varepsilon > 0$ we can find q_0 such that $|u_q(x, t) - u(x, t)| < \varepsilon$ for all $(x, t) \in \Gamma_p$ and $q \geq q_0$. Put

$$u_q(x, t) - z(x, t) = w_q(x, t)H(x, \delta)$$

for $(x, t) \in \bar{D}_p$, $q \geq q_0$. Then w_q satisfies the homogeneous equation (8) in D_p and the following conditions

$$|w_q(x, t)| \leq \varepsilon \sup_{\Gamma_p} H(x, \delta)^{-1} \quad \text{on } \Gamma_p$$

and

$$w_q(x, 0) + \sum_{i=1}^{N} \beta_i(x) w_q(x, T_i) = 0 \quad \text{on } \Omega_p .$$

By Theorem 1

$$|w_q(x, t)| \leq \varepsilon e^{(c_0/2)T} \sup_{\Gamma_p} H(x, \delta)^{-1}$$

for all $(x, t) \in \bar{D}_p$. Letting $\varepsilon \to 0$ we obtain $u \equiv z$ on D_p and the result follows. To establish uniqueness, let $u \in C^{2,1}(D) \cap C(\bar{D}) \cap E_H(D)$ be a solution of the problem (1), (2) and (3) with $f \equiv 0$, $\phi \equiv 0$ and $\Psi \equiv 0$. There exist positive constants M and $\delta < \delta_0$ such that $|u(x, t)| \leq MH(x, \delta)$ in D. Choose $\delta < \delta_1 < \delta_0$ and put

$$u(x, t) = v(x, t)H(x, \delta_1) \quad \text{on } D .$$

By (ii) (the assumption (B_2)) given $\varepsilon > 0$ we can find a positive number R such that

$$|v(x, t)| \leq \varepsilon \quad \text{for } (x, t) \in \Omega \cap (|x| \geq R) \times (0, T] .$$

By Theorem 1

$$|v(x, t)| \leq \varepsilon e^{(c_0/2)T}$$

for all $(x, t) \in \bar{\Omega} \cap (|x| \leq R) \times [0, T]$ and the uniqueness easily follows.

To apply Theorem 7 we introduce the following assumptions

(C_1) The coefficients a_{ij}, b_i $(i, j = 1, \cdots, n)$ and c are bounded on $R_n \times [0, T]$ and Hölder continuous (with exponent α) on every compact subset in $R_n \times [0, T]$ and moreover

$$c(x, t) \leq -c_0 \quad \text{for all } (x, t) \in R_n \times [0, T] ,$$

where c_0 is a positive constant.

(C_2) There exists positive constants λ_0 and λ_1 such that for any vector $\xi \in R_n$

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq \lambda_1|\xi|^2$$

for all $(x, t) \in R_n \times (0, T]$, $a_{ij} = a_{ji}$ $(i, j = 1, \cdots, n)$.

As an application of Theorem 7 we shall prove the existence of a solution u of the equation (1) in $R_n \times (0, T]$ satisfying the condition

$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) = \Psi(x) \quad \text{on } R_n . \tag{9}$$

It is clear that the function $H(x, \delta) = \prod_{i=1}^{n} \cosh \delta x_i$ has properties (i), (ii) and (iii) of the assumption (B_2) (with $\Omega = R_n$) provided $0 < \delta < \delta_0$, where δ_0 is sufficiently small.

In this situation

$$\begin{aligned} E_H(R_n \times (0, T]) = \{u;\ & u \text{ defined on } R_n \times (0, T] \text{ and } |u(x, t)| \leq Me^{\delta|x|} \\ & \text{for all } (x, t) \in R_n \times (0, T] \text{ and certain } M > 0 \text{ and } 0 < \delta < \delta_0\}, \end{aligned}$$

similarly

$$\begin{aligned} E_H(R_n) = \{v;\ & v \text{ defined on } R_n \text{ and } |v(x)| \leq Me^{\delta|x|} \\ & \text{for all } x \in R_n \text{ and certain } M > 0 \text{ and } 0 < \delta < \delta_0\}. \end{aligned}$$

THEOREM 8. *Suppose that the assumptions* (C_1) *and* (C_2) *holds. Let* $\beta_i \in C(R_n)$, $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ *and* $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ *on* R_n. *If* $f \in E_H(R_n \times (0, T])$ *is a Hölder continuous function on every compact subset of* $R_n \times [0, T]$ *and* $\Psi \in E_H(R_n) \cap C(R_n)$, *then the problem* (1), (9) *has a unique solution in* $E_H(R_n \times (0, T]) \cap C^{2,1}(R_n \times (0, T] \cap C(R_n \times [0, T])$.

Proof. Let ϕ be a continuous function belonging to $E_H(R_n \times (0, T])$ such that $\phi(x, 0) = \Psi(x)$ on R_n and $\phi(x, t) = 0$ on $R_n \times [T_0, T]$, where $T_0 = \min_{i=1,\cdots,N} T_i$. By Theorem 5 the problem (1), (2) and (3) has a unique solution on every D_p. Applying Theorem 7 the result easily follows.

In the sequel we shall need the following result.

LEMMA 2. *Suppose that the assumptions* (C_1) *and* (C_2) *hold in* $R_n \times (0, T]$. *Let* $\beta_i \in C(R_n)$ $(i = 1, \cdots, N)$, $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ *and* $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ *on* R_n. *Then for any bounded function* f *on* $R_n \times [0, T]$ *and Hölder continuous on every compact subset of* $R_n \times [0, T]$ *and for any continuous and bounded function* Ψ *on* R_n *there exists a unique solution* u *of the problem* (1), (9) *in* $E_H(R_n \times (0, T]) \cap C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T])$ *such that*

$$|u(x, t)| \leq \frac{2}{c_0} e^{(c_0/2)T} \sup_{R_n \times [0,T]} |f(x, t)| + (1 - e^{-(c_0/2)T_k})^{-1} \sup_{R_n} |\Psi(x)|$$

for all $(x, t) \in R_n \times [0, T]$, *where* $T_k = \min_i T_i$.

Proof. We start with the following observation, the proof of which is routine,

$$\text{if } u \in C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T]) \cap E_H(R_n \times (0, T])$$

and

$$Lu \leq 0 \quad \text{in } R_n \times (0, T],$$
$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x) u(x, T_i) \geq 0 \quad \text{on } R_n$$

then $u \geq 0$ on $R_n \times [0, T]$.

We first suppose that $-1 < -\beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ on R_n, where β_0 is a positive constant. Put

$$v(x, t) = u(x,t) - \frac{M}{c_0} - \frac{M_1}{1 - \beta_0},$$

where

$$M = \sup_{R_n \times [0,T]} |f(x, t)| \quad \text{and} \quad M_1 = \sup_{R_n} |\Psi(x)|.$$

Then

$$Lv = f - \frac{c}{c_0} M - \frac{cM_1}{1 - \beta_0} \geq \frac{c_0 M_1}{1 - \beta_0} > 0$$

in $R_n \times (0, T]$ and

$$\begin{aligned} v(x, 0) + \sum_{i=1}^{N} \beta_i(x) v(x, T_i) &= \Psi(x) - \frac{M}{c_0} - \frac{M_1}{1 - \beta_0} - \left(\frac{M}{c_0} + \frac{M_1}{1 - \beta_0}\right) \sum_{i=1}^{N} \beta_i(x) \\ &\leq \frac{M}{c_0}(\beta_0 - 1) + M_1\left(1 - \frac{1}{1 - \beta_0} + \frac{\beta_0}{1 - \beta_0}\right) < 0 \end{aligned}$$

on R_n. By the preceding remark

$$u \leq \frac{M}{c_0} + \frac{M_1}{1 - \beta_0} \quad \text{on } R_n \times [0, T].$$

Similarly using

$$w(x, t) = u(x, t) + \frac{M}{c_0} + \frac{M_1}{1 - \beta_0}$$

as a comparison function we deduce the inequality

$$u \geq -\frac{M}{c_0} - \frac{M_1}{1-\beta_0} \quad \text{on } R_n \times [0, T].$$

In the general case we use the transformation $u(x, t) = v(x, t)e^{-(c_0/2)t}$.

4. In this section we derive an integral representation of the problem (1), (2) and (3) in an infinite strip and in a bounded cylinder.

THEOREM 9. *Suppose that the assumptions* (C_1) *and* (C_2) *hold in* $R_n \times (0, T]$. *Let* β_i $(i = 1, \cdots, N)$ *and* Ψ *be a continuous and bounded functions on* R_n. *Assume further that*

$$-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \quad \textit{and} \quad \beta_i(x) \leq 0 \ (i = 1, \cdots, N) \quad \textit{on } R_n.$$

Then the unique solution in $C^{2,1}(R_n \times (0, T]) \cap C(R_n[0, T]) \cap E_H(R_n \times (0, T])$ *of the problem* (1), (9) *with* $f \equiv 0$ *is given by*

$$u(x, t) = \int_{R_n} P(x, t, y)\Psi(y)dy, \tag{10}$$

for $(x, t) \in R_n \times (0, T]$, *where* $P(x, t, y)$ *as a function of* (x, t) *satisfies the equation* $LP = 0$ *in* $R_n \times (0, T]$ *for almost all* $y \in R_n$. *Moreover* P *satisfies the equation*

$$P(x, t, y) = -\int_{R_n} \Gamma(x, t; z, 0) \sum_{i=1}^{N} \beta_i(z)P(z, T_i, y)dz + \Gamma(x, t; y, 0) \tag{11}$$

for all $(x, t) \in R_n \times (0, T]$ *and almost all* $y \in R_n$, *where* $\Gamma(x, t, y, 0)$ *is the fundamental solution of* $Lu = 0$.

Proof. Let Ψ be a continuous and bounded function in $L^2(R_n)$. By Lemma 2 the unique solution of the problem (1), (9) in $C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T]) \cap E_H(R_n \times (0, T])$ is bounded on $R_n \times [0, T]$. We first prove that for each $\delta > 0$ there exists a positive constant $C(\delta)$ such that

$$|u(x, t)| \leq C(\delta)\left[\int_{R_n} \Psi(y)^2 dy\right]^{1/2} \tag{12}$$

on $R_n \times [\delta, T]$. To prove (12) we first assume that $-1 < \beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ on R_n, where β_0 is a positive constant. Consider the Cauchy problem for the homogeneous equation (1) with the initial condition

$$z(x, 0) = -\sum_{i=1}^{N} \beta_i(x)u(x, T_i) + \Psi(x)$$

on R_n. The unique solution z in $E_H(R_n \times (0, T])$ is given by

$$z(x, t) = -\int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_i(y) u(y, T_i) dy + \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy$$

for all $(x, t) \in R_n \times (0, T]$ (Friedman [2], p. 26). Since u is a solution of the same problem we obtain

$$u(x, t) = -\int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_i(y) u(y, T_i) dy + \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy \tag{13}$$

for all $(x, t) \in R_n \times (0, T]$. Now it is well known that

$$\int_{R_n} \Gamma(x, t; y, 0) dy \leq 1 \tag{14}$$

for all $(x, t) \in R_n \times (0, T]$ and

$$0 < \Gamma(x, t; y, 0) \leq C_1 t^{-(n/2)} e^{-\mathscr{H}(|x-y|^2)/t} \tag{15}$$

for all $(x, t) \in R_n \times (0, T]$ and $y \in R_n$, where C_1 and $\mathscr{H}$ are positive constants (Friedman [2], p. 24). Applying the Hölder inequality we derive from (13), (14) and (15) that

$$\max_{i=1,\cdots,N} \sup_{R_n} |u(x, T_i)| \leq \frac{C_1}{1-\beta_0} T_k^{-(n/4)} \left[\int_{R_n} e^{-2\mathscr{H}|x|^2} dx\right]^{1/2} \left[\int_{R_n} \Psi(x)^2 dx\right]^{1/2}, \tag{16}$$

where $T_k = \min_{i=1,\cdots,N} T_i$. Using again the representation (13) and the estimates (14), (15) and (16) we obtain

$$|u(x, t)| \leq \left[\frac{\beta_0}{1-\beta_0} C_1 C_2 + C_1 C_3 t^{-(n/4)}\right] \left[\int_{R_n} \Psi(x)^2 dx\right]^{1/2} \tag{17}$$

for all $(x, t) \in R_n \times (0, T]$, where

$$C_2 = T_k^{-(n/4)} \left[\int_{R_n} e^{-2\mathscr{H}|x|^2} dx\right]^{1/2} \quad \text{and} \quad C_3 = \left[\int_{R_n} e^{-2\mathscr{H}|x|^2} dx\right]^{1/2},$$

and the estimate (12) easily follows. In the general case we use the transformation $u(x, t) = v(x, y) e^{-(c_0/2)t}$. By (12) the mapping $\Psi \to u(x, t)$ defines a linear functional on $C_b(R_n) \cap L^2(R_n)$ continuous in L^2-norm. Here $C_b(R_n)$ denotes the space of continuous and bounded functions on R_n. Consequently the representation (10) follows from the Riesz representation theorem of a linear continuous functional on $L^2(R_n)$. To derive (11) observe that by (10) and (13) we have for every continuous bounded function Ψ

$$\int_{R_n} P(x, t, y)\Psi(y)dy = -\int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_i(y)\left[\int_{R_n} P(y, T_i, z)\Psi(z)dz\right]dy + \int_{R_n} \Gamma(x, t; y, 0)\Psi(y)dy$$

for $(x, t) \in R_n \times (0, T]$. Consequently if we fix $(x, t) \in R_n \times (0, T]$, applying Fubini's theorem, we obtain the identity (11) for almost all $y \in R_n$. Now choose $y \in R_n$ such that

$$\int_{R_n} \Gamma(x, T; z, 0) \sum_{j=1}^{N} \beta_j(z)P(z, T_j, y)dz$$

is finite. Then by Theorem 1 in Watson [6] the integral

$$\int_{R_n} \Gamma(x, t, z, 0) \sum_{j=1}^{N} \beta_j(z)P(z, T_j, y)dz$$

is finite for all $(x, t) \in R_n \times (0, T]$ and represents a solution of the equation $Lv = 0$ in $R_n \times (0, T]$ and the last assertion of the theorem easily follows.

Similarly in the case of a bounded cylinder one can prove

THEOREM 10. *Suppose the assumptions of Theorem* 5 *hold. Let u be a solution of the problem* (1), (2) *and* (3) *with $\phi \equiv 0$ and $f \equiv 0$. Then*

$$u(x, t) = \int_{\Omega} p(x, t, y)\Psi(y)dy$$

for all $(x, t) \in D$, where $p(x, t, y)$ as a function of (x, t) satisfies the equation $Lp = 0$ for almost all $y \in \Omega$. Moreover

$$p(x, t, y) = -\int_{\Omega} G(x, t; z, 0) \sum_{i=1}^{N} \beta_i(z)p(z, T_i, y)dz + G(x, t; y, 0) \tag{18}$$

for all$(x, t) \in D$ and almost all $y \in \Omega$, where $G(x, t; y, 0)$ is the Green function for the operator L.

In the following theorem we shall show that p and P tend to infinity at the same rate as $t^{-(n/2)}$.

THEOREM 11. *Let the assumptions of Theorem* 9 *hold and let $D = \Omega \times (0, T]$ be a bounded cylinder with $\partial\Omega \in C^{2+\alpha}$. Then there exists a positive constant C such that*

$$p(x, t, y) \leq C\int_{\Omega} G(x, t; z, 0)dz + G(x, t; y, 0) \tag{19}$$

for all $(x, t) \in D$ *and almost all* $y \in \Omega$, *and moreover*

$$P(x, t, y) \leq C\int_{R_n} \Gamma(x, t; z, 0)dz + \Gamma(x, t; y, 0) \tag{20}$$

for all $(x, t) \in R_n \times (0, T]$ *and almost all* $y \in R_n$, *where* C *depends on* C_1 *and* n.

Proof. We first assume that $-1 < \beta_0 \leq \sum_{i=1}^N \beta_i(x) \leq 0$ on Ω, where β_0 is a positive constant.

Let Ψ be a continuous and non-negative function on R_n with compact support in Ω. It follows from Theorem 9, 10 and the maximum principle that

$$\int_\Omega p(x, t, y)\Psi(y)dy \leq \int_{R_n} P(x, t, y)\Psi(y)dy$$

for all $(x, t) \in D$. Since Ψ is an arbitrary non-negative function we deduce from the last inequality

$$p(x, t, y) \leq P(x, t, y)$$

for all $(x, t) \in D$ and almost all $y \in \Omega$. Fix y in Ω such that the last inequality holds. Since $P(x, T_i, y)$ is continuous as a function of x we get

$$p(x, T_i, y) \leq \sup_{z \in \bar{\Omega}} P(z, T_i, y) < \infty \quad (i = 1, \cdots, N)$$

Using the identity (18), the estimate (15) and the obvious inequality $G(x, t; y, 0) \leq \Gamma(x, t; y, 0)$ for all $(x, t) \in R_n \times (0, T]$ and $y \in R_n$ we derive the estimate

$$\max_{i=1,\cdots,N} \sup_{x \in \Omega} p(x, T_i, y) \leq \frac{C_1 T_k^{-(n/2)}}{1-\beta_0}, \qquad \text{where } T_k = \min_{i=1,\cdots,N} T_i .$$

Now applying again the identity (18) we obtain

$$p(x, t, y) \leq \frac{C_1 T_k^{-(n/2)}\beta_0}{1-\beta_0}\int_\Omega G(x, t; z, 0)dz + G(x, t; y, 0)$$

for all $(x, t) \in D$ and almost all $y \in \Omega$. In the general case we use the transformation $u(x, t) = v(x, t)e^{-(c_0/2)t}$.

To prove (20) put $D_m(|x| < m) \times (0, T]$ and denote by $G_m(x, t; y, 0)$ the Green function for the operator L. By the preceding result we have for every m

$$p_m(x, t; y) \leq C \int_{|z|<m} G(x, t; z, 0)dz + G_m(x, t; y, 0)$$

for all $(x, t) \in D_m$ and almost all $y \in \{|x| < m\}$, where p_m denotes "p-function" for the problem (1), (2) and (3) in D_m. By a standard argument one can prove that $\{G_m\}$ and $\{p_m\}$ are increasing sequences converging to G and p respectively and the result easily follows.

It follows from the proof of Theorem 9 (the inequality (12)) that the problem (1), (9) can be solved for $\Psi \in L^2(R_n)$, but this requires a new formulation of the condition (9).

We shall say that a function $u(x, t)$ defined on $R_n \times (0, T]$ has a parabolic limit at x_0 if there exists a number b such that for all $\gamma > 0$, we have

$$\lim_{\substack{(x,t)\to(x_0,0) \\ |x-x_0|<\gamma\sqrt{t}}} u(x, t) = b\,.$$

We express this briefly by writing $p - \lim_{(x,t)\to(x_0,0)} u(x, t) = b$ (see Chaborowski [1] p. 257).

Let $\Psi \in L^2(R_n)$. We shall say that a function u belonging to $C^{2,1}(R_n \times (0, T])$ is a solution of the problem (1), (9) if it satisfies the equation (1) in $R_n \times (0, T]$ and

$$p - \lim_{(x,t)\to(y,0)} u(x, t) = -\sum_{i=1}^{N} \beta_i(y)u(y, T_i) + \Psi(y)$$

for almost all $y \in R_n$.

THEOREM 12. *Suppose that the assumptions* (C_1) *and* (C_2) *hold in* $R_n \times (0, T]$. *Let* $\beta_i \in C(R_n)$ $(i = 1, \cdots, N)$ $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ *and* $\beta_i(x) \leq 0$ $(i = 1, \cdots, N)$ *on* R_n. *Assume that* $\Psi \in L^2(R_n)$ *and that* f *is a bounded function on* $R_n \times [0, T]$ *and Hölder continuous on every compact subset of* $R_n \times [0, T]$. *Then there exists a solution of the problem* (1), (9).

Proof. Let $\{\Psi_r\}$ be a sequence of functions in $C(R_n)$ with compact supports which converges to Ψ in $L^2(R_n)$. By Theorem 9 there exists a unique bounded solution u_r in $C^{2,1}(R_n \times (0, T] \cap C(R_n \times [0, T])$ to the problem

$$Lu_r = f \quad \text{in } R_n \times (0, T]$$

and

$$u_r(x, 0) + \sum_{i=1}^{N} \beta_i(x)u_r(x, T_i) = \Psi_r(x) \quad \text{on } R_n\,.$$

It follows from (12) that

$$|u_r(x, t) - u_s(x, t)| \leq C(\delta)\left\{\int_{R_n} [\Psi_r(x) - \Psi_s(x)]^2 dx\right\}^{1/2}$$

for all $(x, t) \in R_n \times [\delta, T]$. Hence $u_r(x, t)$ converges uniformly on $R_n \times [\delta, T]$ for every $\delta > 0$ to a continuous function $u(x, t)$ on $R_n \times (0, T]$. As in the proof of Theorem 9 it is easy to establish the representation

$$\begin{aligned} u_r(x, t) = &-\int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_i(y) u_r(y, T_i) dy \\ &+ \int_{R_n} \Gamma(x, t; y, 0) \Psi_r(y) dy - \int_0^t \int_{R_n} \Gamma(x, t; y, \tau) f(y, \tau) dy d\tau \end{aligned}$$

for all $(x, t) \in R_n \times (0, T]$. Letting $r \to \infty$ we obtain

$$\begin{aligned} u(x, t) = &-\int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_i(y) u(y, T_i) dy \\ &+ \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy - \int_0^t \int_{R_n} \Gamma(x, t; y, \tau) f(y, \tau) dy d\tau \end{aligned}$$

for $(x, t) \in R_n \times (0, T]$. Since $u(x, T_i)$ are bounded on R_n it is easy to see that $u(x, t)$ satisfies the equation (1) in $R_n \times (0, T]$. It follows from Theorem 3.1 in Chabrowski [1] that

$$p - \lim_{(x,t)\to(y,0)} u(x, t) = -\sum_{i=1}^{N} \beta_i(y) u(y, T_i) + \Psi(y)$$

for almost all $y \in R_n$.

5. In this section we briefly discuss the extensions of the previous results to the problem (1), (2) and (3*), where

$$\text{(3*)} \qquad u(x, 0) + \sum_{i=1}^{\infty} \beta_i(x) u(x, T_i) = \Psi(x) \quad \text{on } \Omega,$$

with $T_i \in (0, T]$ $i = 1, 2, \cdots$.

Throughout this section it is assumed that $\inf_i T_i > 0$.

We being with the maximum principle.

LEMMA 3. *Suppose that the assumption* (A) *holds in a bounded cylinder* D. *Let* $c(x, t) \leq 0$ *in* D. *Assume that* $-1 \leq \sum_{i=1}^{\infty} \beta_i(x) \leq 0$ *and* $\beta_i(x) \leq 0$ $(i = 1, 2, \cdots)$ *on* Ω. *Let* u *be a function in* $C^{2,1}(D) \cap C(\bar{D})$ *satisfying the following conditions*

$$Lu \leq 0 \quad \text{in } D,$$

$$u(x, t) \geq 0 \quad on\ \Gamma$$

and

$$u(x, 0) + \sum_{i=1}^{\infty} \beta_i(x) u(x, T_i) \geq 0 \quad on\ \bar{\Omega},$$

then $u \geq 0$ *on* $\bar{D}$.

Proof. Assume that $u < 0$ at some point of $\bar{D}$. Then there exists a point $x_0 \in \Omega$ such that $u(x_0, 0) = \min_{\bar{D}} u(x, t) < 0$. Consequently

$$u(x_0, 0)\Bigl(1 + \sum_{i=1}^{\infty} \beta_i(x_0)\Bigr) \geq 0 .$$

Hence $u(x_0, 0) \geq 0$ provided $\sum_{i=1}^{\infty} \beta_i(x_0) + 1 > 0$ and we get a contradiction.

It remains to consider the case $\sum_{i=1}^{\infty} \beta_i(x_0) = -1$. Let $T_0 = \inf_i T_i$. There exists $S \in [T_0, T]$ such that $u(x_0, S) = \min_{T_0 \leq t \leq T} u(x_0, t)$. Hence

$$u(x_0, 0) \geq -\sum_{i=1}^{\infty} \beta_i(x_0) u(x_0, T_i) \geq -u(x_0, S) \sum_{i=1}^{\infty} \beta_i(x_0) = u(x_0, S)$$

and we get a contradiction.

THEOREM 13. *Suppose that the assumption* (A) *holds in a bounded cylinder. Let* $c(x, t) \leq 0$ *on* D *and* $\sum_{i=1}^{\infty} |\beta_i(x)| \leq 1$ *on* Ω. *Then the problem* (1), (2) *and* (3*) *has at most one solution in* $C^{2,1}(D) \cap C(\bar{D})$.

Proof. Let u be a solution of the homogeneous problem

$$Lu = 0 \quad \text{in } D,$$
$$u(x, t) = 0 \quad \text{on } \Gamma$$

and

$$u(x, 0) + \sum_{i=1}^{\infty} \beta_i(x) u(x, T_i) = 0 \quad \text{on } \Omega .$$

Suppose that $u \not\equiv 0$. As in the proof of Theorem 4 we may assume that there exists a point $x_0 \in \Omega$ such that

$$u(x_0, 0) = \min_{\bar{D}} u(x, t) < 0 . \quad \text{Let } |u(x_0, \kappa)| = \max_{T_0 \leq t \leq T} |u(x_0, T)| ,$$

where $T_0 = \inf_i T_i$ and $\kappa \in [T_0, T]$. Then

$$|u(x_0, 0)| \leq |u(x_0, \kappa)| \sum_{i=1}^{\infty} |\beta_i(x_0)| \leq |u(x_0, \kappa)| .$$

We must assume that $u(x_0, \kappa) > 0$. Hence there exists a point $x_1 \in \Omega$ such that $u(x_1, 0) = \max_{\bar{D}} u(x, t) > 0$. Let $|u(x_1, S)| = \max_{T_0 \leq t \leq T} |u(x_1, S)|$. It is obvious that

$$u(x_1, 0) \leq |u(x_1, S)|.$$

Now considering two cases $u(x_1, 0) \leq |u(x_0, 0)|$ and $|u(x_0, 0)| < u(x_1, 0)$ we arrive at a contradiction (for details see the proof of Theorem 3).

We shall now state analogues of Theorems 5 and 8.

THEOREM 14. *Suppose that the assumptions* (A_1) *and* (A_2) *hold in a bounded cylinder D with $\partial\Omega \in C^{2+\alpha}$. Let $c(x, t) \leq -c_0$ in D, where c_0 is a positive constant and assume that $\beta_i \in C(\bar{\Omega})$ $(i = 1, 2, \cdots)$, $\beta_i(x) \leq 0$ $(i = 1, 2, \cdots)$ and $-1 \leq \sum_{i=1}^{\infty} \beta_i(x) \leq 0$ on Ω and that the series $\sum_{i=1}^{\infty} \beta_i(x)$ is uniformly convergent on $\bar{\Omega}$. Assume finally that f is a Hölder continuous function on D, ϕ and Ψ are continuous function on Γ and $\bar{\Omega}$ respectively and moreover*

$$\phi(x, 0) + \sum_{i=1}^{\infty} \beta_i(x)\phi(x, T_i) = \Psi(x) \quad on\ \partial\Omega\,.$$

Then there exists a unique solution in $C^{2,1}(D) \cap C(\bar{D})$ of the problem (1), (2) *and* (3*).

THEOREM 15. *Let the assumptions* (C_1) *and* (C_2) *hold. Assume that $\beta_i \in C(R_n)$ $(i = 1, 2, \cdots)$, $\beta_i(x) \leq 0$ $(i = 1, 2, \cdots)$ and $-1 \leq \sum_{i=1}^{\infty} \beta_i(x) \leq 0$ on R_n and that the series $\sum_{i=1}^{\infty} \beta_i(x)$ is uniformly convergent on R_n. If f is a bounded on $R_n \times [0, T]$ and Hölder continuous function on every compact subset of $R_n \times [0, T]$ and Ψ is a continuous and bounded function on R_n, then there exists a unique solution in $E_H(R_n \times (0, T]) \cap C^{2,1}(R_n \times (0, T]) \cap C(R_n \times [0, T])$ of the equation* (1) *satisfying the condition*

$$u(x, 0) + \sum_{i=1}^{\infty} \beta_i(x)u(x, T_i) = \Psi(x) \quad on\ R_n\,. \tag{9*}$$

The proof of Theorem 14 and 15 are similar to those of Theorems 5 and 8.

One can easily prove that under the assumptions of Theorems 15, the solution in $E_H(R_n \times (0, T])$ of the problem (1), (9*) is bounded on $R_n \times [0, T]$.

Remark. If 0 is an accumulation point of the sequence $\{T_i\}$ then the Lemma 3 remains true provided $\sum_{i=1}^{\infty} \beta_i(x) + 1 > 0$ and $\beta_i(x) \leq 0$ $(i = 1, 2, \cdots)$ on R_n.

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