

## SOME RESULTS ON THE VARIETY OF COMPLEXES

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### §1. Main theorems

Let  $R$  be a (commutative) Noetherian ring, and let  $\{n_0, n_1, \dots, n_m\}$   $\{k_1, k_2, \dots, k_m\}$  be two sequences of integers satisfying  $m > 0$ ,  $k_i \geq 0$  ( $i = 1, 2, \dots, m$ ) and  $n_i \geq k_i + k_{i+1}$  ( $i = 0, 1, 2, \dots, m$ ) with  $k_0 = k_{m+1} = 0$ . We consider the  $m$ -ple of matrices  $(X^{(1)}, X^{(2)}, \dots, X^{(m)})$ , where  $X^{(s)} = (x_{ij}^{(s)})$  is an  $n_{s-1} \times n_s$  matrix of indeterminates over  $R$  ( $s = 1, 2, \dots, m$ ). In this case we denote by  $R[X^{(1)}, X^{(2)}, \dots, X^{(m)}]$  a polynomial ring over  $R$  with the indeterminates  $\{x_{i_s j_s}^{(s)} \mid 1 \leq s \leq m, 1 \leq i_s \leq n_{s-1}, 1 \leq j_s \leq n_s\}$ . Now we define an algebra  $B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  as follows,

$$B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right) := R[X^{(1)}, X^{(2)}, \dots, X^{(m)}] / I$$

where  $I$  is the ideal generated by all the elements of matrices  $X^{(s-1)} X^{(s)}$  ( $s = 2, 3, \dots, m$ ) and the minors of  $X^{(s)}$  of size  $k_s + 1$  ( $s = 1, 2, \dots, m$ ).

These algebras first appeared in the following setting: In the affine space  $A_R^N = \bigoplus_{i=1}^m \text{Hom}_R(A_R^{n_i}, A_R^{n_{i-1}})$  if we define the subvariety  $W(k_1, k_2, \dots, k_m)$  of  $A_R^N$  to be the closed subset consisting of  $(f_1, f_2, \dots, f_m)$  with  $f_i \in \text{Hom}_R(A_R^{n_i}, A_R^{n_{i-1}})$ ,  $f_i f_{i-1} = 0$  ( $i = 2, 3, \dots, m$ ) and  $\text{rank}(f_i) = k_i$  ( $i = 1, 2, \dots, m$ ), then the algebra  $B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  is nothing but the coordinate ring of  $W(k_1, k_2, \dots, k_m)$ . (See [3; Theorem 1.7].)

These varieties are called the (Buchsbaum-Eisenbud) varieties of complexes. (Notice that for  $m = 1$ , we obtain the determinantal varieties, whose properties have been considerably studied.)

Recently in their paper [3] DeConcini and Strickland have proved that  $B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  is normal iff  $R$  is normal, and Cohen-Macaulay iff  $R$  is Cohen-Macaulay, and their work encourages us to make our way

into the deeper study of these algebras.

In relation with their results, one of the purposes of the present paper is to determine the divisor class group of  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  in normal case and another is to get a necessary and sufficient condition for these algebras to be Gorenstein.

Our main theorems are the following.

**THEOREM 1.1.** *Let  $R$  be a normal domain (and  $\{n_0, n_1, \dots, n_m\}, \{k_1, k_2, \dots, k_m\}$  as above). If we put  $t_i = n_i - k_i - k_{i+1}$  ( $i = 0, 1, \dots, m$ ) with  $k_0 = k_{m+1} = 0$ , then the divisor class group of  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is given as follows;*

$$\text{Cl}\left(B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)\right) = \text{Cl}(R) \oplus \mathcal{Z}^h$$

where  $h = \#\{i \mid 0 < k_i < n_i, t_{i-1} > 0\} + \#\{i \mid 0 < k_i < n_i, t_i = t_{i-1} = 0\}$ .

**THEOREM 1.2.** *Assume that  $k_i > 0$  ( $i = 1, 2, \dots, m$ ). If we denote  $t_i = n_i - k_i - k_{i+1}$  ( $i = 0, 1, \dots, m$ ), then  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is a Gorenstein ring if and only if  $R$  is Gorenstein and the integers  $t_0, t_1, \dots, t_m$  satisfy one of the following conditions;*

- (i)  $t_0 = t_1 = \dots = t_m$ ,
- (ii)  $t_0 = 0, t_1 = t_2 = \dots = t_m$ ,
- (iii)  $t_m = 0, t_0 = t_1 = \dots = t_{m-1}$ ,
- (iv)  $t_0 = t_m = 0, t_1 = t_2 = \dots = t_{m-1}$ .

*Remark 1.3.* The assumption that  $k_i > 0$  ( $i = 1, 2, \dots, m$ ) in Theorem 1.2 is not so serious. In fact if  $k_s = 0$  for some  $s$ , then by the definition of those algebras we have an isomorphism;

$$B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right) \simeq B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_{s-1} \\ k_1, \dots, k_{s-1} \end{smallmatrix}\right) \otimes_R B_R\left(\begin{smallmatrix} n_s, n_{s+1}, \dots, n_m \\ k_{s+1}, \dots, k_m \end{smallmatrix}\right).$$

So we can apply our theorem also in this case, because if  $A$  and  $B$  are finitely generated and faithfully flat  $R$ -algebras, then  $A \otimes_R B$  is Gorenstein if and only if  $A, B$  and  $R$  are all Gorenstein. (Cf. [10; Theorem 2])

*Remark 1.4.* In the case  $m = 1$ , our results are contained in Bruns [2] and Svanes [9] respectively.

Whole of this paper is devoted to the proof of these theorems. Since our method is mainly based on the one developed by DeConcini and Strickland, we will often refer the reader to their paper [3].

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## § 2. Preliminaries

Let  $B_R(n_0, n_1, \dots, n_m)$  be as in the previous section. (We always assume that  $k_i \geq 0$  and  $n_i \geq k_i + k_{i+1}$  for any  $i = 0, 1, \dots, m$  with  $k_0 = k_{m+1} = 0$  and denote  $t_i = n_i - k_i - k_{i+1}$ .)

The symbol  $[i_1, i_2, \dots, i_t | j_1, j_2, \dots, j_t]_s$  will denote the element of  $B_R(n_0, n_1, \dots, n_m)$  which is given by the minor of the matrix  $X^{(s)}$  whose rows are those of indices  $i_1, i_2, \dots, i_t$  and whose columns are those of indices  $j_1, j_2, \dots, j_t$ . If  $\{h_1, h_2, \dots, h_{n_s-t}, j_1, \dots, j_t\} = \{1, 2, \dots, n_s\}$ , then  $[i_1, i_2, \dots, i_t | j_1, j_2, \dots, j_t]_s$  can also be written as

$$\sigma[i_1, \dots, i_t | \hat{h}_1, \dots, \hat{h}_{n_s-t}]_s$$

where  $\sigma$  is the sign of permutation  $(h_1, \dots, h_{n_s-t}, j_1, \dots, j_s)$ . (For more detail see [3; § 1]).

The following lemma will be useful for us in proving inductively something about the property of these algebras.

LEMMA 2.1. (See also [3; Lemma 2.10].) *If we denote*

$$a = [i_1, \dots, i_t | j_1, \dots, j_t]_s \quad (1 \leq s \leq m, 1 \leq t \leq k_s, \\ 1 \leq i_1 < i_2 < \dots < i_t \leq n_{s-1}, 1 \leq j_1 < j_2 < \dots < j_t \leq n_s),$$

*then there exists an isomorphism;*

$$B_R(n_0, n_1, \dots, n_m) \left[ \frac{1}{a} \right] \simeq B_{\tilde{R}} \left( \begin{matrix} n_0, \dots, n_{s-1} - t, n_s - t, n_{s+1}, \dots, n_m \\ k_1, \dots, k_s - t, k_{s+1}, \dots, k_m \end{matrix} \right)$$

*as  $R$ -algebras, where*

$$\tilde{R} = R \left[ \{x_{ij}^{(s)} | i = i_1, \dots, i_t \text{ or } j = j_1, \dots, j_t\}, \frac{1}{a} \right].$$

*Proof.* To prove this lemma we have only to consider the case of  $a = [1, 2, \dots, t | 1, 2, \dots, t]_s$ . Then it is easy to see that there exist  $P \in GL(n_{s-1}; \tilde{R})$  and  $Q \in GL(n_s; \tilde{R})$  such that

$$PX^{(s)}Q = \left[ \begin{array}{ccc|c} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ \hline 0 & & & Y^{(s)} \end{array} \right]$$

where  $Y^{(s)}$  is an  $(n_{s-1} - t) \times (n_s - t)$  matrix. Furthermore if we put

$$Q^{-1}X^{(s+1)} = \left[ \begin{array}{c|c} * & \\ \hline Y^{(s+1)} & \end{array} \right]^t \quad \text{and} \quad X^{(s-1)}P^{-1} = \left[ \begin{array}{c|c} * & \\ \hline Y^{(s-1)} & \end{array} \right]$$

(we assume that  $X^{(0)} = X^{(m+1)} = 0$ ), and if  $Y^{(i)} = X^{(i)}$  ( $i \neq s-1, s, s+1$ ), then we can easily verify that  $B_{\tilde{R}}\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)\left[\frac{1}{a}\right]$  is generated by those elements of  $Y^{(i)}$ 's over  $\tilde{R}$  and all their relations must be induced by those of  $X^{(i)}$ 's. So we can consider this algebra to be the variety of complexes over  $\tilde{R}$  with the generic matrices  $(Y^{(1)}, \dots, Y^{(m)})$ . We leave to the reader the detail of this proof.

*Remark 2.2.* If  $a = [1, \dots, t | 1, \dots, t]_s$  is as in Lemma 2.1, then the matrices  $P$  and  $Q$  are given in the following form:

$$P = \left[ \begin{array}{ccc|c} P & & & 0 \\ & 1 & & \\ & & \ddots & \\ * & & & \\ \hline & & & 1 \end{array} \right] \in GL(n_{s-1}; \tilde{R}), \quad Q = \left[ \begin{array}{c|ccc} Q & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & \\ \hline & & & 1 \end{array} \right] \in GL(n_s; \tilde{R})$$

In this case let  $[i_1, \dots, i_p | j_1, \dots, j_p]_{\tilde{q}}$  denote the element of

$$B_{\tilde{R}}\left(\begin{smallmatrix} n_0, n_1, \dots, n_{s-1} - t, n_s - t, \dots, n_m \\ k_1, \dots, k_s - t, \dots, k_m \end{smallmatrix}\right)$$

which is given by the determinant of  $[y_{ij}^{(q)}]_{\substack{i=i_1, \dots, i_p \\ j=j_1, \dots, j_p}}$  in the notation of the above proof.

Then by this remark it is not so difficult to see the following

**LEMMA 2.3.** *Let  $a = [1, 2, \dots, t | 1, 2, \dots, t]_s$  be as in Lemma 2.1 and  $f$  be the natural map*

$$B_{\tilde{R}}\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right) \longrightarrow B_{\tilde{R}}\left(\begin{smallmatrix} n_0, \dots, n_{s-1} - t, n_s - t, \dots, n_m \\ k_1, \dots, k_s - t, \dots, k_m \end{smallmatrix}\right)$$

which is given in Lemma 2.1.

- (1) If  $\{i_1, \dots, i_p\} \subseteq \{t+1, t+2, \dots, n_{s-1}\}$  and  $\{j_1, j_2, \dots, j_p\} \subseteq \{t+1, t+2, \dots, n_s\}$ , then
- $$f([1, 2, \dots, t, i_1, \dots, i_p | 1, 2, \dots, t, j_1, \dots, j_p]_s) = u[i_1 - t, i_2 - t, \dots, i_p - t | j_1 - t, j_2 - t, j_p - t]_{\tilde{s}}$$

where  $u$  is a unit in  $\tilde{R}$ .

- (2) If  $\{i_1, \dots, i_p\} \subseteq \{t+1, t+2, \dots, n_s\}$  and  $\{j_1, \dots, j_p\} \subseteq \{1, 2, \dots, n_{s+1}\}$ , then

$$f([i_1, \dots, i_p | j_1, \dots, j_p]_{s+1}) = [i_1 - t, i_2 - t, \dots, i_p - t | j_1, \dots, j_p]_{\tilde{s+1}}.$$

- (3) If  $\{i_1, \dots, i_p\} \subseteq \{1, 2, \dots, n_{s-2}\}$  and  $\{j_1, \dots, j_p\} \subseteq \{t+1, t+2, \dots, n_{s-1}\}$ , then

$$f([i_1, \dots, i_p | j_1, \dots, j_p]_{s-1}) = [i_1, \dots, i_p | j_1 - t, \dots, j_p - t]_{\tilde{s-1}}$$

- (4) If  $q \neq s-1, s, s+1$ , then

$$f([i_1, \dots, i_p | j_1, \dots, j_p]_q) = [i_1, \dots, i_p | j_1, \dots, j_p]_{\tilde{q}}$$

In the ring  $B_R \begin{pmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{pmatrix}$  let  $I^{(r)}(i_1, \dots, i_p | j_1, \dots, j_q)$  denote the ideal generated by the maximal minors of the matrix  $[x_{ij}^{(r)}]_{\substack{i=1, \dots, i_p \\ j=1, \dots, j_q}}$ . And in the same way we define  $\tilde{I}^{(r)}(i_1, \dots, i_p | j_1, \dots, j_q)$  as the ideal of

$$B_{\tilde{R}} \begin{pmatrix} n_0, \dots, n_{s-1} - t, n_s - t, \dots, n_m \\ k_1, \dots, k_s - t, \dots, k_m \end{pmatrix}$$

generated by the maximal minors of the matrix  $[y_{ij}^{(r)}]_{\substack{i=1, \dots, i_p \\ j=1, \dots, j_q}}$ . Then as a corollary of Lemma 2.3 we get

**COROLLARY 2.4.** *Let  $a = [1, 2, \dots, t | 1, 2, \dots, t]_s$  be as in Lemma 2.3, then*

$$(1) \quad I^{(s)}(1, 2, \dots, t, i_1, \dots, i_p | 1, 2, \dots, t, j_1, \dots, j_q) \left[ \frac{1}{a} \right] = \tilde{I}^{(s)}(i_1 - t, i_2 - t, \dots, i_p - t | j_1 - t, j_2 - t, \dots, j_q - t)$$

if  $t < i_u \leq n_{s-1}$  ( $u = 1, 2, \dots, p$ ) and  $t < j_v \leq n_s$  ( $v = 1, 2, \dots, q$ ).

$$(2) \quad I^{(s+1)}(1, 2, \dots, n_s | j_1, \dots, j_q) \left[ \frac{1}{a} \right] = \tilde{I}^{(s+1)}(1, 2, \dots, n_s - t | j_1, \dots, j_q) \quad \text{if } q \leq n_s - t.$$

$$(3) \quad I^{(s-1)}(i_1, \dots, i_p | 1, 2, \dots, n_{s-1}) \left[ \frac{1}{a} \right] = \tilde{I}^{(s-1)}(i_1, \dots, i_p | 1, 2, \dots, n_{s-1} - t) \quad \text{if } p \leq n_{s-1} - t.$$

$$(4) \quad I^{(r)}(i_1, \dots, i_p | j_1, \dots, j_q) \left[ \frac{1}{a} \right] = \tilde{I}^{(r)}(i_1, \dots, i_p | j_1, \dots, j_q)$$

if  $r \neq s - 1, s, s + 1$ .

**FACT 2.5.** Let  $H$  be the set consisting of all the determinants. We partially order  $H$  by the following; When  $x = [i_1, \dots, i_t | j_1, \dots, j_t]_s$  and  $x' = [i'_1, \dots, i'_{t'} | j'_1, \dots, j'_{t'}]_{s'}$ ,  $x$  and  $x'$  are incomparable if  $s \neq s'$ , while if  $s = s'$  then  $x \leq x'$  if  $t \geq t'$  and  $i_u \leq i'_u, j_u \leq j'_u$  for  $u = 1, 2, \dots, t'$ . The product  $xx'$  ( $s' \geq s$ ) is said to be *standard* if one of the following conditions holds;

$$(1) \quad s' > s + 1$$

$$(2) \quad s' = s \text{ and } x \text{ and } x' \text{ are comparable in the partial order on } H.$$

$$(3) \quad s' = s + 1, n_s - t \geq t' \text{ and writing } u_1 < u_2 < \dots < u_{n_s-t}$$

for the complement of  $\{j_1, \dots, j_t\}$  in  $\{1, 2, \dots, n_s\}$ , we have

$$u_{n_s-t-v+1} \geq j'_{v-1} \quad \text{for } v = 1, 2, \dots, t'.$$

(If we symbolically write  $x = [I|J]_s$ ,  $x' = [I'|J']_{s+1}$  and if we define a permutation  $\pi_s$  on  $\{1, 2, \dots, n_s\}$  as

$$\pi_s(u) = n_s - u + 1 \quad (1 \leq u \leq n_s),$$

then this condition could be re-written as  $\pi_s(\hat{J}) \leq \pi_s(I')$ .)

Now we define an arbitrary product  $x_1 x_2 \dots x_n$  of minors to be standard if each product  $x_i x_j$  is standard in a suitable order.

Then the theorem of DeConcini and Strickland says that the algebra  $B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  is a Hodge algebra over  $R$  generated by  $H$ . (Cf. [3] and [5; § 16])

*Remark.* Our notion of standardness does not coincide with that of [3]. In fact the condition (3) above corresponds to the standardness given in [3] by considering the product matrices

$$X^{(s)} \begin{bmatrix} & & 1 \\ & \cdot & \\ & \cdot & \\ 1 & & \end{bmatrix}, \quad \begin{bmatrix} & & 1 \\ & \cdot & \\ & \cdot & \\ 1 & & \end{bmatrix} X^{(s+1)}$$

instead of  $X^{(s)}$  and  $X^{(s+1)}$ . This is necessary if one wishes to make  $B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  a Hodge algebra defined in [5].

**LEMMA 2.6.** *Suppose that  $R$  is a reduced ring and  $k_i \geq 1$ . If we denote*

$d = [1, 2, \dots, k_1 | 1, 2, \dots, k_1]_1$  and  $\mathfrak{p} = I^{(1)}(1, 2, \dots, k_1 | 1, 2, \dots, n_1)$ , then the ideals  $(d)$  and  $\mathfrak{p}$  are radical ideals.

Moreover if  $m \geq 2$ ,  $k \geq 1$  and if we denote  $w = [1, 2, \dots, k_m | 1, 2, \dots, k_m]_m$  then  $w$  is a non zero divisor on  $B/\mathfrak{p}$  and  $B/(d)$ , where

$$B = B_R \binom{n_0, n_1, \dots, n_m}{k_1, \dots, k_m}.$$

*Proof.* Recall that a subset  $I$  of  $H$  is called a poset ideal if  $I \ni a \geq b \in H$  implies  $b \in I$ . In our case  $\{d\}$  and the system of generators

$$\{[1, 2, \dots, k_1 | i_1, \dots, i_{k_1}]_1 | 1 \leq i_1 < i_2 < \dots < i_{k_1} \leq n_1\}$$

of  $\mathfrak{p}$  are in fact poset ideals of  $H$ . (Notice that  $d$  is a minimal element in  $H$ .) Thus by Proposition 1.2 of [5] we have that both  $B/(d)$  and  $B/\mathfrak{p}$  are also Hodge algebras over  $R$ . In particular they are reduced.

Next assume that  $w(\sum_i a_i x_i) \in \mathfrak{p}$  (resp.  $\in (d)$ ) where each  $x_i$  is a standard monomial and  $a_i \in R$ . By the straightening law we can write  $w x_i = \sum_j b_{ij} y_{ij}$  ( $y_{ij}$  is a standard monomial and  $b_{ij} \in R$ ). In this expression each monomial  $y_{ij}$  contains  $w$ , for  $w$  is minimal in the order on  $H$ . Since  $\sum_{i,j} a_i b_{ij} y_{ij} \in \mathfrak{p}$  (resp.  $\in (d)$ ) and since this expression is unique, each  $y_{ij}$  also contains one of  $[1, 2, \dots, k_1 | i_1, \dots, i_{k_1}]_1$  ( $1 \leq i_1 < \dots < i_{k_1} \leq n_1$ ) (resp.  $d$ ).

Thus we have  $w(\sum a_i x_i) \in w\mathfrak{p}$ . (resp.  $\in (wd)$ ). Because  $w$  is a non zero divisor on  $B$ , we conclude that  $\sum a_i x_i \in \mathfrak{p}$  (resp.  $\in (d)$ ). Hence  $w$  is a non zero divisor on  $B/\mathfrak{p}$  (resp.  $B/(d)$ ).

Now we shall prove the main proposition of this section.

**PROPOSITION 2.7.** *Assume that  $R$  is an integral domain and  $k_1 \geq 1$ . If we put  $\mathfrak{p} = I^{(1)}(1, 2, \dots, k_1 | 1, 2, \dots, n_1)$  as in Lemma 2.6, then  $\mathfrak{p}$  is a prime ideal and*

$$\text{ht}(\mathfrak{p}) = \begin{cases} 1 & (k_1 < n_0) \\ n_1 - k_1 - k_2 + 1 & (k_1 = n_0). \end{cases}$$

*Proof.* We prove the proposition by the induction on  $m$ . If  $m = 1$ , then this algebra  $B := B_R \binom{n_0, n_1}{k_1}$  is the determinantal variety and so the result has been known by [6; Theorem 1].

So assume that  $m \geq 2$ . If we denote  $w = [1, 2, \dots, k_m | 1, 2, \dots, k_m]_m$ , then by Lemma 2.6 we have only to prove that  $\mathfrak{p}B_w$  is prime and its height is given as in the proposition.

On the other hand  $B_w$  is isomorphic to  $\tilde{B} := B_{\tilde{R}} \left( \begin{smallmatrix} n_0, \dots, n_{m-2}, n_{m-1} - k_m \\ k_1, \dots, k_{m-1} \end{smallmatrix} \right)$  with a certain integral domain  $\tilde{R}$  and by this isomorphism  $\mathfrak{p}B_w$  corresponds to the ideal  $\tilde{\mathfrak{p}}$  which is defined in  $\tilde{B}$  as in the same way as  $\mathfrak{p}$ . (See Corollary 2.4.)

By the induction hypothesis  $\tilde{\mathfrak{p}}$  is in fact a prime ideal and

$$\text{ht}(\tilde{\mathfrak{p}}) = \begin{cases} 1 & (k_1 < n_0) \\ n_1 - k_1 - k_2 + 1 & (k_1 = n_0, m \geq 3) \\ (n_1 - k_2) - k_1 + 1 & (k_1 = n_0, m = 2). \end{cases}$$

This is what we wanted.

*Remark 2.8.*  $I := I^{(1)}(1, 2, \dots, n_0 | 1, 2, \dots, k_1)$  may not be a prime ideal. In fact if we consider

$$B := B_R \left( \begin{smallmatrix} 1, 2, 2 \\ 1, 1 \end{smallmatrix} \right) = R[u, v, x, y, z, w]/(ux + vz, uy + vw, xw - yz),$$

then  $I = (u)$ ,  $\mathfrak{p} = (u, v)$  and so

$$B/I \simeq R[v, x, y, z, w]/(vz, vw, xw - yz)$$

which is not an integral domain, while

$$B/\mathfrak{p} \simeq R[x, y, z, w]/(xw - yz)$$

is an integral domain whenever  $R$  is a domain.

### § 3. Proof of Theorem 1.1.

In order to prove Theorem 1.1 we need some lemmas on the decomposition of a certain principal ideal, which will be stated in (3.2).

In the following we always denote  $B := B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$ .

**LEMMA 3.1.** *Assume that  $R$  is an integral domain and  $1 \leq k_1 < n_1$ . If we denote*

$$d = [1, 2, \dots, k_1 | 1, 2, \dots, k_1]_1$$

*and*

$$e = [1, 2, \dots, k_1 | 1, 2, \dots, k_1 - 1, k_1 + 1]_1,$$

*then we have*  $(d) = \mathfrak{p} \cap (dB_e \cap B)$ .

*Proof.* It is enough to see that for an element  $x$  of  $\mathfrak{p}$ ,  $ex \in (d)$  implies  $x \in (d)$ .

Let us describe  $x$  as a linear combination of standard monomials as follows;

$$x = \sum_i a_i x_i \quad (a_i \in R)$$

where each standard monomial  $x_i$  contains one of  $[1, 2, \dots, k_1 | i_1, \dots, i_{k_1}]_1$  ( $1 \leq i_1 < \dots < i_{k_1} \leq n_1$ ), since  $x \in \mathfrak{p}$ .

Assume that  $x_i$  does not contain  $d$ . Then by the definition of standardness we can easily check that  $ex_i$  is also a standard monomial. Since  $ex \in (d)$  and since  $d$  is a minimal element of  $H$ , the straightening law shows that the monomial  $ex_i$  contains  $d$ . It also means that  $x_i$  contains  $d$ , which is a contradiction. Thus  $d$  is included in each monomial  $x_i$ , so  $x \in (d)$ .

**PROPOSITION 3.2.** *Assume that  $R$  is an integral domain,  $k_1 > 0$  and  $d := [1, 2, \dots, k_1 | 1, 2, \dots, k_1]_1$ .*

(1) *If  $k_1 < \min\{n_0, n_1\}$  or  $k_1 = n_0 = n_1 - k_2 < n_1$  then we have*

$$(d) = \mathfrak{p} \cap \mathfrak{q}$$

where  $\mathfrak{p} = I^{(1)}(1, 2, \dots, k_1 | 1, 2, \dots, n_1)$ , which is a prime ideal of height one as shown in Proposition 2.7 and  $\mathfrak{q}$  denotes another prime ideal of height one.

(2) *If  $k_1 = n_1$  or  $k_1 = n_0 < n_1 - k_2$ , then the ideal  $(d)$  is prime itself.*

*Proof.* If  $k_1 = n_1$ , then  $(d) = \mathfrak{p}$ , so it is prime by Proposition 2.7.

Assume that  $k_1 < n_1$ . In this case let us denote

$$e = [1, 2, \dots, k_1 | 1, 2, \dots, k_1 - 1, k_1 + 1]_1$$

as in Lemma 3.1. First we claim that  $dB_e$  is a proper prime ideal.

In fact we have already known that  $B_e \simeq B_{\tilde{R}} \left( \begin{smallmatrix} n_1 - k_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{smallmatrix} \right)$  with  $\tilde{R} = R[\{x_{ij}^{(1)} | i = 1, 2, \dots, k_1 \text{ or } j = 1, 2, \dots, k_1 - 1, k_1 + 1\}, 1/e]$  and by this isomorphism  $d$  corresponds to the element  $\tilde{d} := \det [x_{ij}^{(1)}]_{\substack{i=1,2,\dots,k_1 \\ j=1,2,\dots,k_1}}$  of  $R$ . It is well known that  $\tilde{d}$  is a prime element of  $\tilde{R}$ . (See [6; Theorem 1].) Hence  $B_e/dB_e \simeq B_{\tilde{R}/d\tilde{R}} \left( \begin{smallmatrix} n_1 - k_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{smallmatrix} \right)$  which is in fact an integral domain, and so we conclude that  $dB_e$  is prime as we wanted.

Thus if we denote  $\mathfrak{q} = dB_e \cap B$ , then by Lemma 3.1 we get the decomposition of  $(d)$  into prime ideals;

$$(d) = \mathfrak{p} \cap \mathfrak{q} \quad (*)$$

(Remark that clearly  $\mathfrak{p} \not\subset \mathfrak{q}$ .)

In the first case of proposition, the height of  $\mathfrak{p}$  is one as shown in Proposition 2.7. So  $(*)$  gives the irredundant decomposition of  $(d)$ .

On the other hand if  $k_1 = n_0 < n_1 - k_2$ , then  $\text{ht}(\mathfrak{p}) \geq 2$  by Proposition 2.7, hence  $\mathfrak{q}$  must be contained in  $\mathfrak{p}$  since  $(d)$  is a radical ideal. Q.E.D.

*Proof of Theorem 1.1.* Assume that  $R$  is a normal domain as in the assumption of Theorem 1.1.

To prove the theorem it will be sufficient to see the following claim;

$$(3.3) \quad \text{Cl}\left(B_R\left(\begin{matrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{matrix}\right)\right) \simeq Z^e \oplus \text{Cl}\left(B_{\tilde{R}}\left(\begin{matrix} n_1 - k_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{matrix}\right)\right)$$

where

$$e = \begin{cases} 1 & \text{(if } 0 < k_1 < \min\{n_0, n_1\} \text{ or } 0 < k_1 = n_0 = n_1 - k_2 < n_1), \\ 0 & \text{(otherwise),} \end{cases}$$

where  $\tilde{R}$  is a certain normal domain which is a faithfully flat  $R$ -algebra and has the same divisor class group as  $R$ .

If the claim (3.3) is verified, then we can prove Theorem 1.1 by the induction on  $m$ . In fact if  $m = 1$ , then by the claim we have

$$\text{Cl}\left(B_R\left(\begin{matrix} n_0, n_1 \\ k_1 \end{matrix}\right)\right) \simeq Z^e \oplus \text{Cl}(R)$$

where

$$e = \begin{cases} 1 & \text{if } 0 < k_1 < \min\{n_0, n_1\}, \\ 0 & \text{otherwise,} \end{cases}$$

that is nothing but Theorem 1.1 in case  $m = 1$ .

If  $m > 1$  and if we put

$$\begin{aligned} h\left(\begin{matrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{matrix}\right) &= \#\{i \mid 0 < k_i < n_i, t_{i-1} > 0\} \\ &\quad + \#\{i \mid 0 < k_i < n_i, t_{i-1} = t_i = 0\} \end{aligned}$$

(recall that  $t_i = n_i - k_i - k_{i+1}$ ), then by (3.3) and by the induction hypothesis we get

$$\text{Cl}\left(B_R\left(\begin{matrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{matrix}\right)\right) \simeq Z^h \oplus \text{Cl}(R)$$

where  $h = e + h\left(\begin{smallmatrix} n_1 - k_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{smallmatrix}\right)$ . Here it is not so difficult to see that  $h = h\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$ .

Now we proceed to the proof of (3.3).

Our proof will be divided into the following three cases;

- (1)  $k_1 = 0$ ,
- (2)  $k_1 = n_1$  or  $k_1 = n_0 < n_1 - k_2$ ,
- (3)  $0 < k_1 < \min\{n_0, n_1\}$  or  $0 < k_1 = n_0 = n_1 - k_2 < n_1$ .

(Observe that they exhaust all the possible cases.)

In case (1) it is clear that  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right) = B_R\left(\begin{smallmatrix} n_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{smallmatrix}\right)$ .

Thus there is nothing to prove.

In case (2) the element  $d = [1, 2, \dots, k_1 | 1, 2, \dots, k_1]_1$  is a prime element by Proposition 3.2. Therefore the divisor class group does not change after localizing by  $d$ . On the other hand there is an isomorphism

$$B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)\left[\frac{1}{d}\right] \simeq B_{\tilde{R}}\left(\begin{smallmatrix} n_1 - k_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{smallmatrix}\right)$$

as we have shown in Lemma 2.1. Since

$$\tilde{R} = R[[i|j]_1 | 1 \leq i \leq k_1, 1 \leq j \leq k_1], 1/d]$$

and since  $d$  is a prime element of  $R[[i|j]_1 | 1 \leq i \leq k_1, 1 \leq j \leq k_1]$  (see [6; Theorem 1]), we have an isomorphism  $\text{Cl}(R) \simeq \text{Cl}(\tilde{R})$ . Hence we get the claim in this case.

In case (3) the ideal  $(d)$  is decomposed into the intersection of two prime ideals;

$$(d) = \mathfrak{p} \cap \mathfrak{q}$$

as shown in Proposition 3.2. If  $D(A)$  denotes the free abelian group generated by height-one prime ideals in a normal domain  $A$ , and  $F(A)$  the subgroup of  $D(A)$  consisting of principal divisors, then we have the following commutative diagram induced by the natural map  $B \rightarrow B_d$ , where

$$B = B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right).$$

$$\begin{array}{ccc} F(B) & \xrightarrow{\xi} & F(B_d) \\ \downarrow & & \downarrow \\ D(B) & \xrightarrow{\eta} & D(B_d) \\ \downarrow & & \downarrow \\ \text{Cl}(B) & \xrightarrow{\zeta} & \text{Cl}(B_d) \end{array}$$

Immediately it can be known that

$$\begin{aligned}\text{Ker}(\eta) &= \mathbf{Z} \text{div}(p) + \mathbf{Z} \text{div}(q) \\ \text{Ker}(\xi) &= \mathbf{Z} \text{div}(d)\end{aligned}$$

where “div” expresses the class in  $D(B)$ . (Notice that  $F(B) \simeq Q(B) - \{0\}/U(B)$  where  $Q(B)$  is the quotient field of  $B$  and  $U(B)$  the group of units in  $B$ . See [1; Chapter 7].) Furthermore by the above decomposition of  $(d)$  it is clear that  $\text{div}(d) = \text{div}(p) + \text{div}(q)$  in  $D(B)$ . Hence we conclude that the kernel of  $\zeta$  is generated by  $\text{cl}(p)$  and it is  $\mathbf{Z}$ -free. Anyway we get the following exact sequence of abelian groups;

$$\begin{aligned} (*) \quad 0 &\longrightarrow \mathbf{Z} \longrightarrow \text{Cl}(B) \xrightarrow{\zeta} \text{Cl}(\tilde{B}) \longrightarrow 0 \\ &1 \longmapsto \text{cl}(p)\end{aligned}$$

where  $B := B_d \simeq B_{\tilde{R}} \left( \begin{smallmatrix} n_1 - k_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{smallmatrix} \right)$  as in Lemma 2.1.

It remains only to prove that this sequence  $(*)$  is split. We shall prove this by the induction on  $m$ .

Since we may assume that  $\tilde{B}$  also satisfies our claim (3.3),  $\text{Cl}(\tilde{B})$  can be expressed by the direct sum of  $\text{Cl}(R)$  and  $\mathbf{Z}^{\tilde{h}}$  for some  $\tilde{h} \geq 0$ , as we have shown above. On the other hand by the diagram of algebras;

$$\begin{array}{ccc} B & \longrightarrow & \tilde{B} \\ \uparrow & & \uparrow \\ R & \longrightarrow & \tilde{R} \end{array}$$

we get the following commutative diagram of abelian groups;

$$\begin{array}{ccc} \text{Cl}(B) & \xrightarrow{\zeta} & \text{Cl}(\tilde{B}) \\ \alpha \uparrow & & \uparrow \tilde{\alpha} \\ \text{Cl}(R) & \xrightarrow{\beta} & \text{Cl}(\tilde{R}) \end{array}$$

where  $\beta$  is an isomorphism as in the proof of case (2).

If  $m = 1$ ,  $\tilde{\alpha}$  is also an isomorphism and so  $\zeta$  has the right inverse  $\alpha\beta^{-1}\tilde{\alpha}^{-1}$  as wanted.

Assume that  $m > 1$ . Then by the induction hypothesis  $\tilde{\alpha}$  has the left inverse  $\delta$  and the cokernel of  $\tilde{\alpha}$  is isomorphic to  $\mathbf{Z}^{\tilde{h}}$ . So there exists a map  $\gamma: \mathbf{Z}^{\tilde{h}} \rightarrow \text{Cl}(\tilde{B})$  such that  $\pi\zeta\gamma$  gives the identity map on  $\mathbf{Z}^{\tilde{h}}$  where  $\pi$  denotes the natural projection from  $\text{Cl}(\tilde{B})$  to  $\text{Coker}(\tilde{\alpha}) \simeq \mathbf{Z}^{\tilde{h}}$ . Thus we can conclude that  $\alpha\beta^{-1}\delta \oplus \gamma$  is the right inverse of  $\zeta$ .

This completes the proof of (3.3) and so Theorem 1.1.

*Remark 3.4.* There exists an automorphism of  $B := B_R \begin{pmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{pmatrix}$  which permutes some columns of the matrix  $X^{(1)}$  and leaves the ideal  $\mathfrak{p}$  stable. From this fact we see that even if  $\mathfrak{d}$  denotes  $[1, 2, \dots, k_1 | i_1, \dots, i_{k_1}]_1$  for arbitrary  $i_1 < \dots < i_{k_1} \leq n_1$  instead of  $[1, 2, \dots, k_1 | 1, 2, \dots, k_1]_1$ , then the natural map from  $\text{Cl}(B)$  to  $\text{Cl}(B_{\mathfrak{d}})$  is an isomorphism in case (2) and the sequence (\*) is also split exact in case (3).

Carefully pursuing the proof of the theorem, we shall have the following corollary which will be used in Sections 4 and 5 to prove Theorem 1.2.

**COROLLARY 3.5.** *Let us denote  $B := B_R \begin{pmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{pmatrix}$  with a normal domain  $R$  and  $x := [1 | 1]_s$  for some  $s$ . If one of the following conditions holds;*

- (i)  $k_s \geq 2$ ,
- (ii)  $s = 1, k_1 = k_2 = n_0 = 1$  and  $n_1 > 2$ ,
- (iii)  $s = m, k_m = k_{m-1} = n_m = 1$  and  $n_{m-1} > 2$ ,

*then the group homomorphism  $\text{Cl}(B) \rightarrow \text{Cl}(B_x)$  induced by the natural map of  $B$  to  $B_x$  is an isomorphism.*

*Proof.* In case (ii) or (iii) the element  $x$  is prime by Proposition 3.2 and so the result will be obtained by [1; Chapter 7 § 1 Proposition 17].

In case (i) we prove this by the induction on  $m$ . It is also divided into the following three cases as in the proof of (3.3):

- (1)  $k_1 = 0$ ,
- (2)  $k_1 = n_1 > 0$  or  $0 < k_1 = n_0 < n_1 - k_2$ ,
- (3)  $0 < k_1 < \min\{n_0, n_1\}$  or  $0 < k_1 = n_0 = n_1 - k_2 < n_1$ .

In case (1) this obviously reduces to the case with less  $m$  since  $B \simeq B_R \begin{pmatrix} n_1, n_2, \dots, n_m \\ k_2, \dots, k_m \end{pmatrix}$ .

In case (2) let us denote

$$\mathfrak{d} = \begin{cases} [1, 2, \dots, k_1 | 1, 2, \dots, k_1]_1 & \text{if } s \neq 2, \\ [1, 2, \dots, k_1 | n_1 - k_1 + 1, \dots, n_1]_1 & \text{if } s = 2, \end{cases}$$

and

$$\tilde{d} = \begin{cases} [1, 2, \dots, k_1 - 1 | 1, 2, \dots, k_1 - 1]_{\tilde{1}} & \text{if } s = 1, \\ [1, 2, \dots, k_1 | n_1 - k_1, \dots, n_1 - 1]_{\tilde{1}} & \text{if } s = 2, \\ [1, 2, \dots, k_1 | 1, 2, \dots, k_1]_{\tilde{1}} & \text{if } s \geq 3. \end{cases}$$

Then by Lemma 2.3 we know that  $d$  corresponds to  $\tilde{d}$  under the natural map from  $B$  to  $B_x$ . (Notice that in case  $s = 2$ ,  $n_1 - k_1 \geq k_2 \geq 2$ .) Applying Remark 3.4 to  $B$  and  $B_x$ , we get the commutative diagram;

$$\begin{array}{ccc} \text{Cl}(B_x) & \longrightarrow & \text{Cl}((B_x)_d) \simeq \text{Cl}((B_d)_x) \\ \uparrow \alpha & & \uparrow \beta \\ \text{Cl}(B) & \longrightarrow & \text{Cl}(B_d) \end{array}$$

where the horizontal arrows are isomorphisms. On the other hand the induction hypothesis shows that  $\beta$  is also an isomorphism. Thus  $\alpha$  is an isomorphism as we wanted.

In case (3) let  $d$  and  $\tilde{d}$  be as above and let  $\tilde{\mathfrak{p}}$  denote the ideal;

$$\begin{aligned} & \tilde{I}^{(1)}(1, 2, \dots, k_1 - 1 | 1, 2, \dots, n_1 - 1) & \text{if } s = 1, \\ & \tilde{I}^{(1)}(1, 2, \dots, k_1 | 1, 2, \dots, n_1 - 1) & \text{if } s = 2, \\ \text{or} & \tilde{I}^{(1)}(1, 2, \dots, k_1 | 1, 2, \dots, n_1) & \text{if } s \geq 3. \end{aligned}$$

Then  $d$  (resp.  $\mathfrak{p}$ ) corresponds to  $\tilde{d}$  (resp.  $\tilde{\mathfrak{p}}$ ) as we have shown in Lemma 2.3 and Corollary 2.4. Thus we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z \text{cl}(\mathfrak{p}) & \longrightarrow & \text{Cl}(B_x) & \longrightarrow & \text{Cl}(B_{x_d}) \longrightarrow 0 \\ & & \uparrow \alpha & & \uparrow \beta & & \uparrow \gamma \\ 0 & \longrightarrow & Z \text{cl}(\tilde{\mathfrak{p}}) & \longrightarrow & \text{Cl}(B) & \longrightarrow & \text{Cl}(B_d) \longrightarrow 0 \end{array}$$

(Notice that  $\alpha$  is an isomorphism.) On the other hand by the hypothesis of induction we can easily see that  $\gamma$  is an isomorphism. So it is concluded that  $\beta$  is also an isomorphism. Q.E.D.

#### § 4. Reduction

In this section we shall show that it is sufficient to prove the following proposition in order to get Theorem 1.2.

**PROPOSITION 4.1.** *Assume that  $R$  is a field. Then  $B_R \binom{n_0, n_1, \dots, n_m}{1, \dots, 1}$  is Gorenstein if and only if one of the following four conditions holds:*

$$(1) \quad n_0 + 1 = n_1 = n_2 = \dots = n_{m-1} = n_m + 1,$$

- (2)  $n_0 = 1, n_1 = n_2 = \dots = n_{m-1} = n_m + 1,$
- (3)  $n_m = 1, n_0 + 1 = n_1 = n_2 = \dots = n_{m-1},$
- (4)  $n_0 = n_m = 1, n_1 = n_2 = \dots = n_{m-1}.$

The proof of this proposition will be done in the next section, and for a while we assume that this holds actually.

For the purpose of reducing Theorem 1.2 to this proposition we need some lemmas, the following one of which is rather well known.

LEMMA 4.2. (1)  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is Gorenstein if and only if  $R$  and  $B_{k(\mathfrak{p})}\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  are Gorenstein for any prime ideal  $\mathfrak{p}$  of  $R$  where  $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ .

(2) Let  $R \rightarrow S$  be a faithfully flat extension of Noetherian rings with Gorenstein fibres. Then  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is Gorenstein if and only if  $B_S\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is Gorenstein.

*Proof.* (1) Since the natural map  $R \rightarrow B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is faithfully flat, the result is obtained from [10; Theorem 1].

(2) By the assumption we see that the natural map from

$$B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right) \text{ to } B_S\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$$

is faithfully flat. Thus if  $B_S\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is Gorenstein, then by [10; Theorem 1] we see that  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is also Gorenstein. Conversely assume that  $B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  is Gorenstein. Then  $R$  is Gorenstein and so is  $S$ . (Use (1) and [10; Theorem 1].) Hence the Gorensteinness of

$$B_S\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right) \simeq B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right) \otimes_R S.$$

follows from [10; Theorem 2].

LEMMA 4.3. Let us denote  $B = B_R\left(\begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix}\right)$  and  $x = [1|1]_s$  for some  $s$ . If one of the following conditions holds;

- (i)  $k_s \geq 2,$

(ii)  $s = 1$ ,  $k_1 = k_2 = n_0 = 1$  and  $n_1 > 2$ ,

(iii)  $s = m$ ,  $k_m = k_{m-1} = n_m = 1$  and  $n_{m-1} > 2$ ,

then  $B$  is Gorenstein if and only if  $B_x$  is Gorenstein.

*Proof.* By the above lemma we may assume that  $R$  is a field, and so we can consider the canonical module  $K_B$  of the graded ring  $B$ . (For the definition of  $K_B$  see [7; Chapter 2].) Remark that  $K_B$  is isomorphic to a certain divisorial ideal of  $B$  since  $B$  is a Cohen-Macaulay normal domain. (Cf. [8; Korollar 6.7]) Therefore  $K_B$  defines the divisor class  $\text{cl}(K_B)$  in  $\text{Cl}(B)$  and thus  $B$  is Gorenstein if and only if  $\text{cl}(K_B) = 0$ . (See [8; Satz 5.9].)

If we denote  $B' = B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_{s-1} - 1, n_s - 1, \dots, n_m \\ k_1, \dots, k_s - 1, \dots, k_m \end{smallmatrix} \right)$  and  $\tilde{R} = R[\{[i|j]_s \mid i = 1 \text{ or } j = 1\}, 1/x]$ , then  $B_x \simeq B' \otimes_R \tilde{R}$  as we have shown in Lemma 2.1, and so we have the natural maps among the divisor class groups;

$$\text{Cl}(B) \xrightarrow{f} \text{Cl}(B_x) \simeq \text{Cl}(B' \otimes_R \tilde{R}) \xleftarrow{g} \text{Cl}(B')$$

which are all isomorphisms by Theorem 1.1. and Corollary 3.5. Here it can be easily checked that  $f(\text{cl}(K_B))$  and  $g(\text{cl}(K_{B'}))$  give the same element in  $\text{Cl}(B_x)$ . (Use [7; Proposition (2.2.9)].)

Thus  $\text{cl}(K_B) = 0$  if and only if  $\text{cl}(K_{B'}) = 0$  and this establishes the lemma.

Now let  $k_i > 0$  ( $i = 1, 2, \dots, m$ ) be as in the assumption of Theorem 1.2. (Recall that we always assume that  $n_i \geq k_i + k_{i+1}$  with  $k_0 = k_{m+1} = 0$ .)

Then the Gorensteinness of  $B_R \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  is equivalent to the Gorensteinness of  $R$  and  $R_{k(p)} \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ k_1, \dots, k_m \end{smallmatrix} \right)$  for all  $p \in \text{Spec}(R)$ . Lemma 4.3 and Lemma 2.1 show that if  $k_s > 1$  then it is also equivalent to the Gorensteinness of  $R$  and  $B_{\tilde{k}(p)} \left( \begin{smallmatrix} n_0, \dots, n_{s-1} - 1, n_s - 1, \dots, n_m \\ k_1, \dots, k_s - 1, \dots, k_m \end{smallmatrix} \right)$  for any  $p \in \text{Spec}(R)$  where  $\tilde{k}(p) := k(p)[\{[i|j]_s \mid i = 1 \text{ or } j = 1\}, 1/[1|1]_s]$ . Then by Lemma 4.2 (2) we see that it holds if and only if  $R$  and

$$B_{k(p)} \left( \begin{smallmatrix} n_0, \dots, n_{s-1} - 1, n_s - 1, \dots, n_m \\ k_1, \dots, k_s - 1, k_m \end{smallmatrix} \right)$$

are both Gorenstein. If we continue this process using Lemma 4.3 until

each integer of the lower row of the parenthesis becomes 1, then it will be concluded that it is equivalent to the Gorensteinness of  $R$  and

$$B_{k(\mathfrak{p})} \begin{pmatrix} t_0 + 1, t_1 + 2, t_2 + 2, t_{m-1} + 2, t_m + 1 \\ 1, 1, 1, \dots, 1 \end{pmatrix}$$

for all  $\mathfrak{p} \in \text{Spec}(R)$ . (Recall that  $t_i = n_i - k_i - k_{i+1}$ .)

*Thus if we assume the validity of Proposition 4.1, then we shall get Theorem 1.2.*

**§ 5. Final step of the proof**

Now we shall prove Proposition 4.1. Our main tool is the Poincaré series of graded rings which is defined as follows; If  $A = \bigoplus_{n \geq 0} A_n$  is an  $N$ -graded ring with Artinian  $A_0$  and  $A_n$  of finite length over  $A_0$ , then the Poincaré series  $P_A(\lambda)$  of  $A$  is the formal power series in  $\lambda$  given by

$$\sum_{n \geq 0} \text{length}_{A_0}(A_n) \lambda^n.$$

(We remark that this is always a rational function in  $\lambda$  if  $A$  is Noetherian.)

The reason why we are going to consider the Poincaré series is the following lemma due to R.P. Stanley.

LEMMA 5.1 [11: Theorem 4.4]. *Suppose that  $A$  is a Cohen-Macaulay domain with  $A_0$  a field and  $d = \dim A$ . Then  $A$  is Gorenstein if and only if the equality;*

$$P_A(1/\lambda) = (-1)^d \lambda^a P_A(\lambda)$$

holds for some  $a \in \mathbb{Z}$ .

First of all we shall compute the Poincaré series of  $B_R \begin{pmatrix} r, s \\ 1 \end{pmatrix}$  where  $r, s \geq 1$ . (In the rest of this paper we assume that  $R$  is a field.) If we denote it by  $F_{r,s}(\lambda)$  then it is given by

$$(5.2) \quad F_{r,s}(\lambda) = \sum_{n \geq 0} \binom{n+r-1}{r-1} \binom{n+s-1}{s-1} \lambda^n$$

where  $\binom{u}{0} = 1$  for any  $u \geq 0$ .

In fact each standard monomial of  $B_R \begin{pmatrix} r, s \\ 1 \end{pmatrix}$  of degree  $n$  is written as

$$\left( \begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{array} \right)$$

for some  $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq r$  and  $1 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq s$ . (See [5; Theorem 12.1].) Thus the number of standard monomials of degree  $n$  is given by  $\binom{n+r-1}{r-1} \binom{n+s-1}{s-1}$ . This establishes the equality (5.2).

On the other hand  $F_{rs}(\lambda)$  should be written as a rational function in  $\lambda$ . In fact we get the equality;

$$(5.3) \quad F_{rs}(\lambda) = \frac{f_{rs}(\lambda)}{(1-\lambda)^{r+s-1}}$$

where  $f_{rs}(\lambda) = \sum_{j \geq 0} \binom{r-1}{j} \binom{s-1}{j} \lambda^j \in \mathbb{Z}[\lambda]$ .

Remark that  $\binom{u}{v} = 0$  if  $v < 0$  or  $v > u$ .

To prove this equality we have only to notice that both functions in (5.2) and (5.3) are uniquely determined by the following condition.

$$(*) \quad \begin{aligned} F_{1,s}(\lambda) &= \frac{1}{(1-\lambda)^s} \\ rF_{r+1,s}(\lambda) - sF_{r,s+1}(\lambda) &= (r-s)F_{rs}(\lambda). \end{aligned}$$

In the same way we define another formal power series  $G_{rs}(\lambda)$  as follows:

$$(5.4) \quad \begin{aligned} &\text{If } r \geq 1 \text{ and } s \geq 2, \text{ then} \\ G_{rs}(\lambda) &:= \sum_{n \geq 1} \binom{n+r-1}{r-1} \binom{n+s-2}{s-1} \lambda^n. \end{aligned}$$

Then the same method as above shows that it is also written as a rational function in  $\lambda$ :

$$(5.5) \quad G_{rs}(\lambda) = \frac{g_{rs}(\lambda)}{(1-\lambda)^{r+s-1}}$$

where  $g_{rs}(\lambda) = \sum_{j \geq 1} \binom{r}{j} \binom{s-2}{j-1} \lambda^j \in \mathbb{Z}[\lambda]$ .

It is easy to check the following facts about those functions.

$$(5.6) \quad \begin{aligned} \deg f_{rs}(\lambda) &= \min\{r, s\} - 1, \\ \deg g_{rs}(\lambda) &= \min\{r, s-1\}. \end{aligned}$$

(5.7) (i) If  $r, s \geq 1$ , then  $f_{rs}(\lambda)$  is a monic polynomial if and only if either  $r = s$  or  $(r - 1)(s - 1) = 0$ .

(ii) If  $r \geq 1$  and  $s \geq 2$ , then  $g_{rs}(\lambda)$  is monic if and only if either  $r = s - 1$  or  $r = 1$ .

If  $r \geq 1$ , then

$$(5.8) \quad \begin{aligned} F_{rr}(1/\lambda) &= -\lambda^r F_{rr}(\lambda), \\ F_{\tau+1, r}(1/\lambda) &= \lambda^r G_{\tau, \tau+1}(\lambda), \\ G_{\tau+1, \tau+1}(1/\lambda) &= -\lambda^r G_{\tau+1, \tau+1}(\lambda), \\ G_{\tau, \tau+1}(1/\lambda) &= \lambda^r F_{\tau+1, r}(\lambda). \end{aligned}$$

*Remark 5.9.* If  $B_R\left(\begin{smallmatrix} r, & s \\ 1 & \end{smallmatrix}\right)$  is Gorenstein, then  $F_{rs}(\lambda)$  satisfies the equality in Lemma 5.1. In particular the coefficient of highest degree term in  $f_{rs}(\lambda)$  should be one. Thus in this case it must be  $r = s$  or  $(r - 1)(s - 1) = 0$ . Conversely if  $r = s$  or  $(r - 1)(s - 1) = 0$ , then  $F_{rs}(\lambda)$  certainly satisfies the condition in Lemma 5.1.

In such a way we can get the necessary and sufficient condition for  $B_R\left(\begin{smallmatrix} r, & s \\ 1 & \end{smallmatrix}\right)$  to be Gorenstein.

We shall generalize this method to prove Proposition 4.1.

Now we denote the Poincaré series of

$$B_R\left(\begin{smallmatrix} n_0, & n_1, & \dots, & n_m \\ 1, & \dots, & 1 & \end{smallmatrix}\right) \text{ by } P_{n_0, n_1, \dots, n_m}(\lambda).$$

(Recall that  $n_0 \geq 1$ ,  $n_m \geq 1$  and  $n_i \geq 2$  if  $1 \leq i < m$ .)

LEMMA 5.10. (1)  $P_{n_0, n_1}(\lambda) = F_{n_0, n_1}(\lambda)$ ,

(2)  $P_{n_0, n_1, \dots, n_m}(\lambda) = F_{n_0, n_1-1}(\lambda)P_{n_1, \dots, n_m}(\lambda) + G_{n_0, n_1}(\lambda)P_{n_1-1, n_2, \dots, n_m}(\lambda)$

whenever  $m \geq 2$ .

*Proof.* We have already known (1). To prove (2) we must notice that each standard monomial of  $B_R\left(\begin{smallmatrix} n_0, & n_1, & \dots, & n_m \\ 1, & \dots, & 1 & \end{smallmatrix}\right)$  is given by

$$\left( \begin{array}{c|c|c|c|c} a_1^1 & b_1^1 & b_1^2 & & b_1^{m-1} & b_1^m \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{v_1}^1 & b_{v_1}^1 & b_{v_2}^2 & \dots & b_{v_{m-1}}^{m-1} & b_{v_m}^m \\ \hline & a_1^2 & a_1^3 & & a_1^m & \\ & \vdots & \vdots & & \vdots & \\ & a_{v_2}^2 & a_{v_3}^3 & & a_{v_m}^m & \end{array} \right)$$

where  $1 \leq a_1^i \leq a_2^i \leq \dots \leq a_{v_i}^i \leq n_{i-1}$ ,  $1 \leq b_1^i \leq b_2^i \leq \dots \leq b_{v_i}^i \leq n_i$  for any  $i = 1, 2, \dots, m$  and  $a_{v_i}^i < n_{i-1}$  whenever  $b_{v_i}^i = n_i$ . (See (2.5).) Remark that the degree of this standard monomial is  $v_1 + v_2 + \dots + v_m$ . Thus it is concluded that the number of standard monomials of degree  $n$  in  $B_R(n_0, n_1, \dots, n_m)$  is

$$\begin{aligned} c_{n_0 n_1 \dots n_m}^{(n)} &= \sum_{v_1 + v_2 + \dots + v_m = n} \binom{v_1 + n_0 - 1}{n_0 - 1} \left\{ \binom{v_1 + n_1 - 2}{n_1 - 2} \binom{v_2 + n_1 - 1}{n_1 - 1} \right. \\ &\quad \left. + \binom{v_1 + n_1 - 2}{n_1 - 1} \binom{v_2 + n_1 - 2}{n_1 - 2} \right\} \dots \left\{ \binom{v_{m-1} + n_{m-1} - 2}{n_{m-1} - 2} \binom{v_m + n_{m-1} - 1}{n_m - 1} \right. \\ &\quad \left. + \binom{v_{m-1} + n_{m-1} - 2}{n_{m-1} - 1} \binom{v_m + n_{m-1} - 2}{n_{m-1} - 2} \right\} \binom{v_m + n_m - 1}{n_m - 1}. \end{aligned}$$

In particular we have that

$$\begin{aligned} c_{n_0 n_1 \dots n_m}^{(n)} &= \sum_{v_1=0}^n \left\{ \binom{v_1 + n_0 - 1}{n_0 - 1} \binom{v_1 + n_1 - 2}{n_1 - 2} c_{n_1 n_2 \dots n_m}^{(n-v_1)} \right. \\ &\quad \left. + \binom{v_1 + n_0 - 1}{n_0 - 1} \binom{v_1 + n_1 - 2}{n_1 - 1} c_{n_1-1, n_2, \dots, n_m}^{(n-v_1)} \right\}. \end{aligned}$$

Since  $P_{n_0 n_1 \dots n_m}(\lambda) = \sum_{n_0 \geq 0} c_{n_0 n_1 \dots n_m}^{(n)} \lambda^n$ , this establishes the lemma.

As the corollary of this lemma we can write  $P_{n_0 n_1 \dots n_m}(\lambda)$  as a rational function in  $\lambda$ .

COROLLARY 5.11.

$$P_{n_0 n_1 \dots n_m}(\lambda) = \frac{p_{n_0 n_1 \dots n_m}(\lambda)}{(1 - \lambda)^d}$$

where  $p_{n_0 n_1}(\lambda) = f_{n_0 n_1}(\lambda)$ ,

$$p_{n_0 n_1 \dots n_m}(\lambda) = f_{n_0 n_1-1}(\lambda) p_{n_1 n_2 \dots n_m}(\lambda) + g_{n_0 n_1}(\lambda) p_{n_1-1, n_2, \dots, n_m}(\lambda) \in Z[\lambda] \quad \text{if } m \geq 2$$

and

$$d = n_0 + n_m + 2 \sum_{i=1}^{m-1} n_i - 2m + 1.$$

(Remark that the integer  $d$  in this lemma is the Krull dimension of the ring  $B_R(n_0, n_1, \dots, n_m)$ . See [3; Lemma 2.3]. And also remark that all the coefficients of the polynomial  $p_{n_0 n_1 \dots n_m}(\lambda)$  are non-negative integers.)

Here by using (5.6) we know inductively that

$$\begin{aligned}
 \deg f_{n_0 n_1 - 1} p_{n_1 n_2} &= \min \{n_0, n_1 - 1\} + \min \{n_1, n_2\} - 2, \\
 \deg f_{n_0 n_1 - 1} p_{n_1 n_2 \dots n_m} &= \min \{n_0, n_1 - 1\} + \min \{n_1, n_2 - 1\} \\
 &\quad + \min \{n_{m-1} - 1, n_m\} + \sum_{i=2}^{m-2} \min \{n_i, n_{i+1}\} - m + 1 \\
 &\quad \text{if } m \geq 3, \\
 \deg p_{n_0 n_1 \dots n_m} &= \deg g_{n_0 n_1} p_{n_1 - 1, n_2, \dots, n_m} \\
 &= \min \{n_0, n_1 - 1\} + \min \{n_{m-1} - 1, n_m\} \\
 &\quad + \sum_{i=1}^{m-2} \min \{n_i, n_{i+1}\} - m + 1 \quad \text{if } m \geq 2.
 \end{aligned}$$

In particular we have the following lemma.

**LEMMA 5.12.**  $\deg f_{n_0 n_1 - 1} p_{n_1 \dots n_m} \leq \deg g_{n_0 n_1} p_{n_1 - 1, n_2 \dots n_m}$  and the equality holds if and only if  $n_1 < n_2$ .

Now let us prove Proposition 4.1. If  $m = 1$ , then we have already proved it in Remark 5.9. So in the following we assume that  $m \geq 2$ .

Suppose that  $B_{\mathbb{R}} \left( \begin{smallmatrix} n_0, n_1, \dots, n_m \\ 1, \dots, 1 \end{smallmatrix} \right)$  is a Gorenstein ring. Then  $P_{n_0 n_1 \dots n_m}(\lambda)$  satisfies the condition in Lemma 5.1, so in particular the polynomial  $p_{n_0 n_1 \dots n_m}(\lambda)$  is monic. Thus Lemma 5.12 gives that  $n_1 \geq n_2$  and  $g_{n_0 n_1} p_{n_1 - 1, n_2 \dots n_m}$  is also monic. By (5.7) this is equivalent to say that  $p_{n_1 - 1, n_2 \dots n_m}$  is monic and either  $n_0 + 1 = n_1 \geq n_2$  or  $n_0 = 1, n_1 \geq n_2$ . If we continue this process, we can easily get one of the four conditions in Proposition 4.1.

Conversely assume that  $n_0 + 1 = n_1 = \dots = n_{m-1} = n_m + 1$ . For the convenience let us denote  $r = n_0 = n_m$  and

$$\begin{aligned}
 S_m(\lambda) &:= \underbrace{P_{r, r+1, \dots, r+1, r}}_{m+1}(\lambda) \\
 T_m(\lambda) &:= \underbrace{P_{r+1, r+1, \dots, r+1, r}}_{m+1}(\lambda)
 \end{aligned}$$

with  $S_1 = F_{rr}$  and  $T_1 = F_{r+1, r}$ .

Then by Lemma 5.10 we see that

$$\begin{aligned}
 \begin{bmatrix} S_m(\lambda) \\ T_m(\lambda) \end{bmatrix} &= \begin{bmatrix} G_{r, r+1}(\lambda), F_{rr}(\lambda) \\ G_{r+1, r+1}(\lambda), F_{r+1, r}(\lambda) \end{bmatrix} \begin{bmatrix} S_{m-1}(\lambda) \\ T_{m-1}(\lambda) \end{bmatrix} \\
 &= \dots = \begin{bmatrix} G_{r, r+1}(\lambda), F_{rr}(\lambda) \\ G_{r+1, r+1}(\lambda), F_{r+1, r}(\lambda) \end{bmatrix}^m \begin{bmatrix} 0 \\ 1 \end{bmatrix}
 \end{aligned}$$

Thus by (5.8) we have an equality;

$$\begin{bmatrix} S_m(1/\lambda) \\ T_m(1/\lambda) \end{bmatrix} = \lambda^{mr} \begin{bmatrix} F_{r+1, r}(\lambda), -F_{rr}(\lambda) \\ -G_{r+1, r+1}(\lambda), G_{r, r+1}(\lambda) \end{bmatrix}^m \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Comparing these equalities we easily conclude that

$$S_m(1/\lambda) = -\lambda^{mr} S_m(\lambda).$$

Hence  $B_R\left(\begin{smallmatrix} r, r+1, \dots, r+1, r \\ 1, \dots, 1 \end{smallmatrix}\right)$  is Gorenstein by virtue of Lemma 5.1.

Next we assume that  $n_0 = 1$  and  $n_1 = n_2 = \dots = n_{m-1} = n_m + 1 (= : r)$ . If  $r = 2$ , this case is included in the previous one, and there is nothing to say. So we may take  $r > 2$ . Then using Lemma 4.3 (ii), we have that  $B_R\left(\begin{smallmatrix} 1, r, \dots, r, r-1 \\ 1, \dots, 1 \end{smallmatrix}\right)$  is Gorenstein if and only if  $B_R\left(\begin{smallmatrix} r-1, r, \dots, r, r-1 \\ \underbrace{1, \dots, 1}_{m-1} \end{smallmatrix}\right)$  is Gorenstein. Thus by the previous case it is concluded that  $B_R\left(\begin{smallmatrix} 1, r, \dots, r, r-1 \\ 1, \dots, 1 \end{smallmatrix}\right)$  is always Gorenstein whenever  $r \geq 2$ .

The other cases in Proposition 4.1 can be treated as in the same method using Lemma 4.3, and so we omit the detail.

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