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EXISTENCE AND NON-EXISTENCE OF NULL-SOLUTIONS FOR SOME NON-FUCHSIAN PARTIAL DIFFERENTIAL OPERATORS WITH *T*-DEPENDENT COEFFICIENTS

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Dedicated to Professor S. Matsuura on his 60th birthday

§ 0. Introduction

Since M. S. Baouendi and C. Goulaouic ([2], [3]) defined partial differential operators *of Fuchs type* and proved theorems of Cauchy-Kowalevskaya type and Holmgren type, many authors have investigated operators of Fuchs type in various categories, that is, real-analytic, C^∞ and so on. (Cf. [1], [4], [6], [8], [9], [11], [12], [17], [18], [19], [20], [21] etc.)

DEFINITION 0.1. A partial differential operator P is called *of Fuchs type* (or *Fuchsian*) *with weight* $m-k$ $(0 \leq k \leq m)$, when P has the following form:

$$P = t^k\partial_t^m + a_1(x)t^{k-1}\partial_t^{m-1} + \cdots + a_k(x)\partial_t^{m-k} + \sum_{\substack{j+|\alpha|\leq m \\ j\leq m-1}} t^{\max(0,j+k-m+1)}a_{j,\alpha}(t, x)\partial_t^j\partial_x^\alpha, \tag{0.1}$$

where $a_j(x)$, $a_{j,\alpha}(t, x)$ are smooth, that is, real-analytic, C^∞ and so on. (Notations are given later.)

Remark 0.2. Note that the operator P is Fuchsian with weight $m-k$ if and only if $t^{m-k}P$ is Fuchsian with weight 0.

It has become known that Fuchsian operators have various "good" properties. Among them, we are concerned with the following uniqueness property. (See also [17].)

THEOREM 0.3 ([2]). *If P is Fuchsian with real-analytic coefficients, then there exists a positive integer N depending on P such that the following holds*:

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If a C^N *function* u *near* $(0, 0)$ *satisfies* $u=0$ *for* $t \leq 0$ *and* $Pu=0$ *near* $(0, 0)$, *then* $u=0$ *near* $(0, 0)$.

In this article, we study whether or not non-Fuchsian operators have such uniqueness property.

DEFINITION 0.4. A distribution u near $(0, 0)$ is called a *null-solution* for P at $(0, 0)$, if

(i) $Pu=0$ in a neighborhood of $(0, 0)$,

(ii) $(0, 0) \in \operatorname{supp} u \subset \{(t, x);\ t \geq 0\}$.

When u is of C^N class near $(0, 0)$ $(0 \leq N \leq \infty)$, it is called a C^N null-solution.

By this definition, our problem is whether P has the following property or not.

"For any positive integer N, there exists a C^N null-solution for P at $(0, 0)$."

Of course, this is weaker than that there exists a C^∞ null-solution.

Note that if P has this property, then $t^\rho P$ also has this property for any integer ρ. Hence, we give the following definition.

DEFINITION 0.5. A partial differential operator P is called *essentially Fuchsian* if $t^\rho P$ is Fuchsian with weight 0 for some integer ρ. If P is not essentially Fuchsian, then P is called *essentially non-Fuchsian.*

For many operators whose principal part is essentially non-Fuchsian, C^∞ null-solutions have been constructed by many authors. (Here, we refer only [15] and [16]. See the references of these papers.) As for the essentially non-Fuchsian operators whose principal part is essentially Fuchsian, there seems to be no reference. In this article, in order to obtain rough image of such operators, we investigate the existence and non-existence of null-solutions for operators including such operators, assuming that *the coefficients depend only on t.*

We use the following notations:

$\mathbf{R}$ (resp. $\mathbf{C}$, $\mathbf{Z}$) denotes the set of the real numbers (resp. the complex numbers, the integers);

$(t, x) = (t, x_1, \cdots, x_n)$ are the variables in $\mathbf{R}^{n+1}$;

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_x = (\partial_{x_1}, \cdots, \partial_{x_n}), \quad \partial_{x_j} = \frac{\partial}{\partial x_j};$$

$\max A$ (resp. $\min A$) denotes the maximum (resp. minimum) of the elements of A;

$\mathrm{ord}_\xi f(\xi)$ denotes the order of a polynomial $f(\xi)$ with respect to ξ;

$\mathrm{Re}\,\tau$ denotes the real part of $\tau \in \mathbf{C}$;

$\mathscr{D}'$ (resp. $\mathscr{E}'$) denotes the set of the distributions (resp. the distributions with compact support);

$\mathrm{supp}\, u$ denotes the support of $u \in \mathscr{D}'$.

In Section 1, we divide the essentially non-Fuchsian operators with t-dependent coefficients into four types and show the existence of null-solutions for three of the types. (Theorem A) As for first order operators, we can give another result. This is given in Section 2 as Theorem B. We also give sufficient conditions for the non-existence of C^1 null-solutions to first order operators as Theorem C. In Section 3, we consider some second order operators, which are excluded from Theorem A and have an essentially Fuchsian principal part. Theorem D gives sufficient conditions for the non-existence of C^N null-solutions. In order to prove Theorem A, we need some results on ordinary differential operators. We review them in Section 4. In Sections 5 and 6, we prove Theorems A, B and C. In the proof of Theorem D, an energy estimate plays an important role. This estimate is given in Section 7. In Section 8, we prove Theorem D. In Appendix, we show the C^∞ well-posedness of the flat Cauchy problem for the operators treated in Theorems C and D.

§ 1. Existence of null-solutions

We consider the following operator on $[0, T] \times \mathbf{R}^n$ $(T > 0)$:

$$P = p(t; \partial_t, \partial_x) = t^\kappa \partial_t^m + \sum_{j=0}^{m-1} a_j(t; \partial_x)\partial_t^j , \tag{1.1}$$

where κ is a positive integer and $a_j(t; \xi) = \sum_{|\alpha| \leq m-j} a_{j,\alpha}(t)\xi^\alpha$, $a_{j,\alpha} \in C^\infty[0, T]$. Put $a_m(t; \xi) \equiv t^\kappa$.

For $j = 0, 1, \cdots, m$, we put

$$r(j) = \max\{r \in \mathbf{Z};\ 0 \leq r \leq \kappa,\ \partial_t^i a_j\ (0; \xi) \equiv 0 \text{ for } 0 \leq i \leq r-1\}.$$

If we put $\mathring{a}_j(t; \xi) = t^{-r(j)} a_j(t; \xi)$, then the coefficients of $\mathring{a}_j$ also belong to $C^\infty[0, T]$. Put $d(j) = \mathrm{ord}_\xi\, \mathring{a}_j(0; \xi)$.

We draw a Newton polygon using the points $(j, r(j) - j)$ $(j = 0, 1, \cdots, m)$ as follows:

DEFINITION 1.1. The Newton polygon $\Delta(P)$ is the convex hull of the set $\bigcup_{j=0}^m \{(u, v);\ 0 \leq u \leq j,\ r(j) - j \leq v\}$. (Fig. 0) Let $0 \leq \mu_1 < \mu_2 < \cdots < \mu_r$

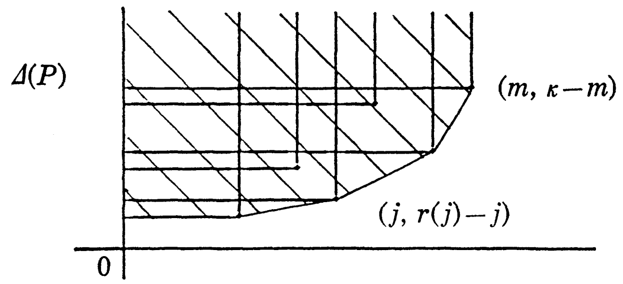

Figure 0.

be the slopes of the non-vertical sides of $\Delta(P)$ and put $S=\{\mu_1, \cdots, \mu_r\}$, $S_+=\{\mu \in S;\ \mu>0\}$. For $\mu \in S$, let $L(\mu)$ be the side with the slope μ including the both terminals and put $V(\mu)=\{j \in \mathbf{Z};\ 0 \leq j \leq m,\ (j, r(j)-j) \in L(\mu)\}$.

Remarks 1.2. (1) Let (j_k^+, ρ_k^+) (resp. (j_k^-, ρ_k^-)) be the right (resp. left) terminal of $L(\mu_k)$ $(k=1, \cdots, r)$. If $\mu_k>0$, then $j_k^\pm \in V(\mu_k)$ and if $\mu_1=0$, then $j_1^+ \in V(0)$. (Note that there hold $j_k^+=j_{k+1}^-$ $(k=1, \cdots, r-1)$.)

(2) The operator P is essentially Fuchsian if and only if $S=\{0\}$ and $d(j)=0$ for any $j \in V(0)$.

In order to clarify the meaning of our results, we first give a conjecture.

CONJECTURE. *If the principal part P_m of P is essentially non-Fuchsian, then there exists a C^∞ null-solution for P at* $(0, 0)$.

If the coefficients of P are real-analytic, then this conjecture is valid. ([16, Theorem 1.8]) In Section 2, it is shown that if P is of first order, then this conjecture is also valid. (Theorem B) Though it is important to prove this conjecture in full generality, our present interest mainly lies in the complementary case when P_m is essentially Fuchsian but P is not. Theorem A in the following has less meaning when P_m is essentially non-Fuchsian, though we do not assume that P_m is essentially Fuchsian.

Now, we divide the essentially non-Fuchsian operators into two types.

Type (I); There exist $\mu \in S_+$ and $j \in V(\mu)$ such that $d(j) \geq 1$.

Type (II); If $\mu \in S_+$, then $d(j)=0$ for any $j \in V(\mu)$.

(Note that if $S=\{0\}$, then P is of type (II).)

Let P be of type (II). If $\mu \in S_+$, then $\mathring{a}_j(0; \xi)=\mathring{a}_j$ are constants for $j \in V(\mu)$. We put $f_\mu(\tau) = \sum_{j \in V(\mu)} \mathring{a}_j \tau^j$ for $\mu \in S_+$ and divide the operators of type (II) into three types:

Type (II-a); There exist $\mu \in S_+$ and a root τ_0 of $f_\mu(\tau) = 0$ such that $\operatorname{Re} \tau_0 > 0$.

Type (II-b); $S \ni 0$ and there exists $j \in V(0)$ such that $d(j) \geq 1$.

Type (II-c); Otherwise. That is, (1) if $\mu \in S_+$ and $f_\mu(\tau) = 0$, then $\operatorname{Re} \tau \leq 0$, (2) if $j \in V(0)$, then $d(j) = 0$.

Note that Type (II-a) and Type (II-b) may have an intersection.

Now, the following is the main theorem.

THEOREM A. *Let P be an essentially non-Fuchsian operator given by* (1.1).

(1) *If P is of type* (I) *or* (II-a), *then there exists a C^∞ null-solution for P at* $(0, 0)$.

(2) *If P is of type* (II-b), *then there exists a C^N null-solution for P at* $(0, 0)$ *for any positive integer N.*

COROLLARY 1.3. *If $P = p(t; \partial_t, \partial_x)$ is essentially non-Fuchsian and $p(t; \partial_t, 0)$ is essentially Fuchsian, then there exists a C^N null-solution for P at* $(0, 0)$ *for any positive integer N.*

Remarks 1.4. (1) It is expected that there exists a C^∞ null-solution also in the case of Type (II-b). The author, however, could not prove it. The C^N null-solution constructed in the proof of the theorem is not C^∞. It is an interesting question whether or not there exists an operator which has a C^N null-solution for any positive integer N but has no C^∞ null-solutions.

(2) In the case of Type (II-c), there are both possibilities that P has a C^N null-solution and that P has no C^N null-solution. As is already stated, it is expected that if the principal part of P is essentially non-Fuchsian, then there exists a C^∞ null-solution. In Sections 2 and 3, we shall show the non-existence of C^N null-solutions for some operators of type (II-c) whose principal part is essentially Fuchsian.

EXAMPLE 1.5. The simplest example of an operator of type (II-c) is

$$P = t^2\partial_t - t^r\partial_x + b\,,$$

where r is a positive integer and b is a non-zero constant such that

$\operatorname{Re} b \geq 0$. If $r=1$ (, that is, the principal part is essentially non-Fuchsian,) then P has a C^∞ null-solution, while if $r>1$ and $\operatorname{Re} b>0$, then P has no C^1 null-solutions. (See the next section.)

§ 2. First order operators

In this section, we consider the following first order operator:

$$P = t^\kappa \partial_t + \sum_{j=1}^{n} a_j(t)\partial_{x_j} + b(t)\,, \tag{2.1}$$

where a_j $(j=1,\cdots,n)$, $b \in C^\infty[0, T]$, $\kappa \in \mathbf{Z}$ and $\kappa \geq 1$. Put

$$s(1) = \max\{s \in \mathbf{Z};\ 0 \leq s \leq \kappa,\ \partial_t^i a_j(0) = 0 \text{ for } 1 \leq j \leq n,\ 0 \leq i \leq s-1\},$$
$$s(0) = \max\{s \in \mathbf{Z};\ 0 \leq s \leq \kappa,\ \partial_t^i b(0) = 0 \text{ for } 0 \leq i \leq s-1\}.$$

Note that P is essentially Fuchsian if and only if $\kappa - 1 \leq s(0)$ and $\kappa - 1 < s(1)$. The four types given in Section 1 is as follows. (Put $\mathring{b}(t) = t^{-s(0)} b(t)$.)

Type (I) $s(1) \leq s(0)$ and $s(1) < \kappa - 1$,

Type (II-a) $s(0) < s(1)$, $s(0) < \kappa - 1$ and $\operatorname{Re} \mathring{b}(0) < 0$,

Type (II-b) $s(1) = \kappa - 1 \leq s(0)$,

Type (II-c) $s(0) < s(1)$, $s(0) < \kappa - 1$ and $\operatorname{Re} \mathring{b}(0) \geq 0$.

The next theorem shows that the conjecture given in Section 1 is valid if P is of first order.

THEOREM B. *If $s(1) \leq \kappa - 1$, then there exists a C^∞ null-solution for P at* $(0, 0)$.

By Theorems A and B, the only possibility that a first order operator P given by (2.1) is essentially non-Fuchsian but has no C^∞ null-solutions is the case when $s(1) > \kappa - 1$, $s(0) < \kappa - 1$ and $\operatorname{Re} \mathring{b}(0) \geq 0$. If $s(1) > \kappa - 1$, $s(0) < \kappa - 1$, a_j are real-valued and $\operatorname{Re} \mathring{b}(0) > 0$, then we can really show the non-existence of C^1 null-solutions. More strongly, we have the following theorem.

THEOREM C. *Consider the operator*

$$P = t^\kappa \partial_t + t^\kappa \sum_{j=1}^{n} \mathring{a}_j(t, x)\partial_{x_j} + t^\sigma \mathring{b}(t, x)\,, \tag{2.2}$$

where κ, $\sigma \in \mathbf{Z}$ and $\mathring{a}_j$ $(1 \leq j \leq n)$, $\mathring{b} \in C^\infty([0, T] \times \mathbf{R}^n)$.

Assume that

(2.3) $\sigma < \kappa - 1$,

(2.4) $\mathring{a}_j(t, x)$ $(1 \leq j \leq n)$ *are real-valued*,

(2.5) $\operatorname{Re} \mathring{b}(0, 0) > 0$.

If $u \in C^1((0, T]; \mathscr{D}'(\mathbf{R}^n))$ *satisfies*

(2.6) $t^M u(t, \cdot) \longrightarrow 0$ $(t \longrightarrow +0)$ *in* $\mathscr{D}'(\mathbf{R}^n)$ *for some* M,

(2.7) $Pu = 0$ *in* $(0, T_0] \times \Omega_0$, *where* $T_0 > 0$ *and* Ω_0 *is an open neighborhood of* $x = 0$,

then $u = 0$ *in* $(0, T_1] \times \Omega_1$ *for some* $T_1 > 0$ *and some open neighborhood* Ω_1 *of* $x = 0$.

Remark 2.1. Assume the same assumptions (2.3), (2.4) and (2.5). Then, the flat Cauchy problem for P is C^∞ well-posed near (0, 0). (See Proposition A.1 in Appendix.)

§ 3. Non-existence of null-solutions for a class of second order operators

In this section, we show the non-existence of C^N null-solutions for a class of second order operators of type (II-c) with essentially Fuchsian principal part. We consider the following operator in $[0, T] \times \mathbf{R}^n$.

$$P = t^\alpha \partial_t^2 - t^p Q(t; \partial_x) + t^\beta b(t)\partial_t + t^q \sum_{j=1}^n c_j(t)\partial_{x_j} + t^\gamma d(t), \tag{3.1}$$

where $\alpha, \beta, \gamma, p, q \in \mathbf{Z}$ and b, c_j $(j = 1, \cdots, n)$, $d \in C^\infty[0, T]$. Since we can increase β (resp. γ) if $b(0) = 0$ (resp. $d(0) = 0$), we may assume that (i) $b(0) \neq 0$ or $\beta \geq \alpha$, (ii) $d(0) \neq 0$ or $\gamma \geq \alpha$.

We assume the following five conditions:

(A-1) $p > \alpha - 2$.

(A-2) $Q(t; \xi) = \sum_{j,k=1}^n a_{j,k}(t)\xi_j\xi_k$, where $a_{j,k} \in C^\infty[0, T]$ and $a_{j,k}$ are real-valued. Further, there exists $\varepsilon > 0$ such that $Q(t; \xi) \geq \varepsilon|\xi|^2$ for any $(t; \xi) \in [0, T] \times \mathbf{R}^n$.

(A-3) P is essentially non-Fuchsian and of type (II).

(A-4) If $\mu \in S_+$ and $f_\mu(\tau) = 0$, then $\operatorname{Re} \tau < 0$. If $j \in V(0)$, then $d(j) = 0$.

(A-5) $2q \geq p - 1 + \min(\beta, (\alpha + \gamma)/2)$.

Remarks 3.1. (1) The condition (A-1) means that the principal part of P is essentially Fuchsian. The condition (A-2) implies that P is strictly hyperbolic in $\{t > 0\}$.

(2) The conditions (A-3) and (A-4) imply that P is of type (II-c). In fact, the difference between the conditions (A-3), (A-4) and the conditions to be of type (II-c) is whether "$\mathrm{Re}\,\tau < 0$" or "$\mathrm{Re}\,\tau \leq 0$".

THEOREM D. *If the above five conditions are satisfied, then there exists a positive integer N for which the following holds:*

If $u \in C^2((0, T]; \mathscr{D}'(\mathbf{R}^n))$ satisfies that

(i) *$Pu = 0$ in $(0, T_0] \times \Omega_0$, where $T_0 > 0$ and Ω_0 is an open neighborhood of $x = 0$,*

(ii) *$t^{-N}u(t, \cdot)$, $t^{-N+1}\partial_t u(t, \cdot) \to 0$ in $\mathscr{D}'(\mathbf{R}^n)$ $(t \downarrow 0)$,*
then $u = 0$ in $(0, T_1] \times \Omega_1$ for some $T_1 > 0$ and some open neighborhood Ω_1 of $x = 0$.

Remarks 3.2. (1) If $p \leq \alpha - 2$ and (A-2) is satisfied, then P has a C^∞ null-solution for arbitrary lower order terms. ([15])

(2) Under the same assumptions as in the theorem, the flat Cauchy problem $Pu = f$ is C^∞ well-posed in $[0, T] \times \mathbf{R}^n$. (See Proposition A.2 in Appendix.)

In the rest of this section, we shall clarify the meaning of the conditions (A-3) and (A-4). Put $P_0 = t^\alpha\partial_t^2 + t^\beta b(t)\partial_t + t^\gamma d(t)$. Assume the conditions (A-3) and (A-4). Then, there holds $\Delta(P) = \Delta(P_0)$. There are three possibilities about the shape of $\Delta(P_0)$.

Case (1) $\alpha - 1 > \beta$ and $\gamma + 1 \geq \beta$. (Fig. 1)

Case (2) $\beta - 1 > \gamma$ and $\alpha + \gamma > 2\beta$. (Fig. 2)

Case (3) $\alpha - 2 > \gamma$ and $\alpha + \gamma \leq 2\beta$. (Fig. 3)

(Note that P_0 is essentially Fuchsian if and only if $\beta - 1 \geq \alpha - 2$ and $\gamma \geq \alpha - 2$.)

We can easily show the following lemma. (Cf. [13], see also the next section.)

LEMMA 3.3. *P_0 can be factored as*

$$P_0 = t^{\alpha-2}(t\partial_t - \Lambda(t))(t\partial_t - \Theta(t)), \tag{3.2}$$

where Λ and Θ have the following properties in each of the above three cases.

Case (1) $\Lambda(t) = t^{-\kappa}\lambda(t)$, *where* $\lambda \in C^\infty_{\text{frac}}$, $\kappa = \alpha - \beta - 1 > 0$ *and* $\mathrm{Re}\,\lambda(0) < 0$. $\Theta \in C^\infty_{\text{frac}}$.

Here, $C^\infty_{\text{frac}} = \{f(t) \in C^\infty(0, T];\ f(t^M) \in C^\infty[0, T]$ *for some positive integer* $M\}$.

Case (2) $\Lambda(t) = t^{-\kappa}\lambda(t)$, $\Theta(t) = t^{-\rho}\theta(t)$, *where* $\lambda, \theta \in C^\infty_{\text{frac}}$, $\kappa = \alpha - \beta - 1 > \rho = \beta - \gamma - 1 > 0$, $\mathrm{Re}\,\lambda(0) < 0$ *and* $\mathrm{Re}\,\theta(0) < 0$.

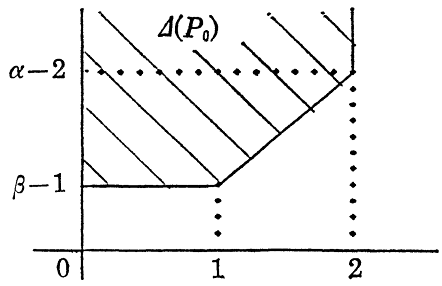

Figure 1.

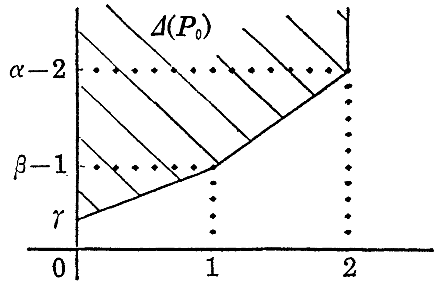

Figure 2.

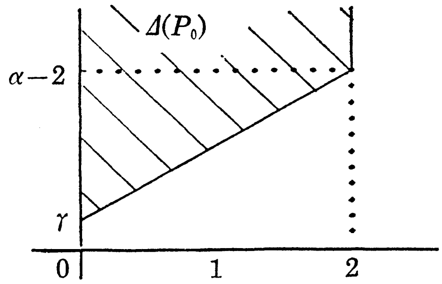

Figure 3.

Case (3) $\Lambda(t) = t^{-\kappa}\lambda(t)$, $\Theta(t) = t^{-\kappa}\theta(t)$, *where* $\lambda, \theta \in C^{\infty}_{\mathrm{frac}}$, $\kappa = (\alpha - \gamma - 2)/2 > 0$ *and* $\mathrm{Re}\,\lambda(0) \leq \mathrm{Re}\,\theta(0) < 0$.

We shall use these factorization in Section 7.

§ 4. Review on ordinary differential operators with C^∞ coefficients

In this section, we consider an ordinary differential operator

$$Q = \sum_{j=0}^{m} b_j(t)\partial_t^j, \tag{4.1}$$

where $b_j \in C^\infty[0, T]$ and $b_m \equiv t^\kappa$ (κ is a positive integer). We can define

$r(j)$, $\mathring{b}_j(t)$, $\Delta(Q)$, S, S_+, $L(\mu)$ and $V(\mu)$ in the same way as in Section 1. We put

$$f_\mu(\tau) = \sum_{j \in V(\mu)} \mathring{b}_j(0)\tau^j \qquad \text{for} \quad \mu \in S_+ \quad \text{and}$$
$$f_0(\tau) = \sum_{j \in V(0)} \mathring{b}_j(0)\tau(\tau-1)\cdots(\tau-j+1) \quad (\text{if } 0 \in S).$$

THEOREM 4.1. (1) *If there exists $\mu \in S_+$ and a root τ_0 of $f_\mu(\tau) = 0$ such that* $\operatorname{Re}\tau_0 > 0$, *then there exists $v \in C^\infty[0, T]$ such that $Qv = 0$ and $\partial_t^j v(0) = 0$ for any non-negative integer j.*

(2) *Let N be an arbitrary positive integer. If $0 \in S$ and there exists a root τ_0 of $f_0(\tau) = 0$ such that* $\operatorname{Re}\tau_0 > N$, *then there exists $v \in C^N[0, T]$ such that $Qv = 0$ and $\partial_t^j v(0) = 0$ for any $j \leq N$.*

Proof. First, we consider formal solutions. As is well-known, the equation $Qv = 0$ has formal solutions of the form $v^\wedge = e^{R(t)}t^\rho w(t)$, where

(i) $R(t)$ is a polynomial of $t^{-1/M}$, with no constant term, for a positive integer M,

(ii) $\rho \in \mathbf{C}$,

(iii) $w(t)$ is a formal power series of $t^{1/M}$ such that $w(0) \neq 0$.

(The equation may also have formal solutions with logarithmic terms, but we do not need such solutions.)

The leading term of $R(t)$ and the value of ρ when $R(t) \equiv 0$ are calculated easily from $\Delta(Q)$ as follows. (See [5], [7], [10] etc.)

(I) For any $\mu \in S_+$ and any non-zero root τ_0 of $f_\mu(\tau) = 0$, there exists a formal solution $v^\wedge$ such that $R(t) = -(\tau_0/\mu)t^{-\mu} +$ (higher order terms).

(II) Assume that $0 \in S$. If τ_1 satisfies $f_0(\tau_1) = 0$ and $f_0(\tau_1 + k) \neq 0$ for $k = 1, 2, 3, \cdots$, then there exists a formal solution $v^\wedge$ such that $R(t) \equiv 0$ and $\rho = \tau_1$. (As for this solution, we can take $M = 1$.) Thus, if $0 \in S$, $f_0(\tau_0) = 0$ and $\operatorname{Re}\tau_0 > N$, then there exists a formal solution $v^\wedge$ such that $R(t) \equiv 0$ and $\operatorname{Re}\rho > N$.

By A.N. Kuznetsov [13] or B. Malgrange [14], if there exists a formal solution $v^\wedge$, then there exists an actual solution v which has the formal expansion $v^\wedge$. Thus, the theorem follows.

§ 5. Proof of Theorem A

In this section, we prove Theorem A. Let P be an operator given by (1.1). We shall construct null-solutions in the form $u = v(t)\exp\langle\xi, x\rangle$, where $\xi \in \mathbf{C}^n$ and $\langle\xi, x\rangle = \sum_{j=1}^n \xi_j x_j$. Thus, the equation $Pu = 0$ is reduced to the equation $Q_\xi v = 0$, where $Q_\xi = \sum_{j=0}^{m} a_j(t;\xi)\partial_t^j$ is an ordinary differential operator with a parameter ξ. Put

$$f_\mu(\xi;\tau) = \sum_{j\in V(\mu)} \mathring{a}_j(0;\xi)\tau^j \qquad \text{for} \quad \mu \in S_+ \quad \text{and}$$
$$f_0(\xi;\tau) = \sum_{j\in V(0)} \mathring{a}_j(0;\xi)\tau(\tau-1)\cdots(\tau-j+1) \quad (\text{if } 0 \in S).$$

In the case of Type (II-a), by Theorem 4.1-(1), there exists $v \in C^\infty[0, T]$ such that $Q_\xi v = 0$, $\partial_t^j v(0) = 0$ for any j. (In this case, ξ can be taken arbitrarily.) Thus, we have a C^∞ null-solution $u = v(t)\exp\langle\xi, x\rangle$.

To prove Theorem A in the cases of Type (I) and (II-b), we need the following lemma.

LEMMA 5.1. *Consider a polynomial* $F(\xi;\tau) = \sum_{j=0}^{h} c_j(\xi)\tau^j$ *of* τ, *where* $c_j(\xi)$ $(j = 0, 1, \cdots, h)$ *are polynomials of* $\xi \in \mathbf{C}^n$ *and* $c_h(\xi) \not\equiv 0$. *If there exists* j *such that* $\mathrm{ord}_\xi c_j > \mathrm{ord}_\xi c_h$, *then for any real number* M, *there exist* $\xi_0 \in \mathbf{C}^n$ *and* $\tau_0 \in \mathbf{C}$ *which satisfy* $F(\xi_0;\tau_0) = 0$ *and* $\mathrm{Re}\,\tau_0 > M$.

Proof. We may assume that $n = 1$ without loss of generality. Put $s_j = \mathrm{ord}_\xi c_j$ $(j = 0, 1, \cdots, h)$ and $\omega = \max_{0\le j\le h-1}(s_j - s_h)/(h-j)$. By the assumption, we have $\omega > 0$. Put $J = \{j;\ 0 \le j \le h,\ s_j = s_h + \omega(h-j)\}$. Note that $h \in J$ and $J\backslash\{h\} \ne \emptyset$. Putting $\tau = \rho^\omega\sigma$ and $\xi = \rho e^{i\theta}$ $(\rho > 0, \theta \in \mathbf{R})$, we have $F(\xi;\tau) = \rho^{s_h+\omega h}F_\rho(\theta;\sigma)$, where

$$F_\rho(\theta;\sigma) = \sum_{j=0}^{h} c_j(\rho e^{i\theta})\rho^{-s_h-\omega(h-j)}\sigma^j .$$

Since $s_j = \mathrm{ord}_\xi c_j$, we can write $c_j(\xi) = c_j^0\xi^{s_j} + (\text{lower order terms})$, where $c_j^0 \ne 0$, unless $c_j(\xi) \equiv 0$. Since $s_j \le s_h + \omega(h-j)$ and since the equality holds if and only if $j \in J$, we have

$$F_\rho(\theta;\sigma) \longrightarrow F_\infty(\theta;\sigma) = e^{i\theta(s_h+\omega h)} \sum_{j\in J} c_j^0(e^{-i\omega\theta}\sigma)^j. \qquad (\rho \to \infty) \tag{5.1}$$

Since $h \in J$ and $J\backslash\{h\} \ne \emptyset$, the equation $\sum_{j\in J} c_j^0\nu^j = 0$ has a non-zero root ν. Hence, by choosing a suitable $\theta_0 \in \mathbf{R}$, we can see that there exists $\sigma_0 \in \mathbf{C}$ which satisfies $F_\infty(\theta_0;\sigma_0) = 0$ and $\mathrm{Re}\,\sigma_0 > 0$. From (5.1) it follows that for a sufficiently large ρ, there exists $\sigma_\rho \in \mathbf{C}$ which satisfies $F_\rho(\theta_0;\sigma_\rho) = 0$ and $\mathrm{Re}\,\sigma_\rho \ge \frac{1}{2}\mathrm{Re}\,\sigma_0$ (> 0). Thus, for a sufficiently large ρ, the equation $F(\rho e^{i\theta_0};\tau) = 0$ has a root $\tau_\rho \in \mathbf{C}$ which satisfies $\mathrm{Re}\,\tau_\rho \ge \frac{1}{2}(\mathrm{Re}\,\sigma_0)\rho^\omega$. This implies the lemma.

Now, we return to the proof of Theorem A. First, we consider the case of Type (I). By the condition of Type (I), we can define

$$\mu^{\sim} = \max\{\mu \in S_+ ;\ d(j) \geq 1 \text{ for some } j \in V(\mu)\}.$$

Let $f_{\mu^{\sim}}(\xi; \tau) = \sum_{j=0}^{h} c_j(\xi)\tau^j$, where $c_h(\xi) \not\equiv 0$. By the following lemma, we see that $c_h(\xi)$ is a constant.

LEMMA 5.2. $\operatorname{ord}_\xi c_h = 0$.

Proof. Note that $h = \max V(\mu^{\sim})$. If $h = m$, then $\operatorname{ord}_\xi c_h = d(m) = 0$. If $h < m$, then, by Remark 1.2-(1), there exists $\mu \in S_+$ such that $\mu > \mu^{\sim}$ and $h \in V(\mu)$. Hence, by the definition of $\mu^{\sim}$, we have $\operatorname{ord}_\xi c_h = 0$.

Since there exists j such that $\operatorname{ord}_\xi c_j \geq 1$ by the definition of $\mu^{\sim}$, we can use Lemma 5.1, and hence there exist $\xi_0 \in \mathbf{C}^n$ and $\tau_0 \in \mathbf{C}$ which satisfy $f_{\mu^{\sim}}(\xi_0; \tau_0) = 0$ and $\operatorname{Re}\tau_0 > 0$. Hence, by Theorem 4.1-(1), there exists $v \in C^\infty[0, T]$ such that $Q_{\xi_0}v = 0$ and $\partial_t^j v(0) = 0$ for any j. Thus, we have a C^∞ null-solution $u = v(t)\exp\langle\xi_0, x\rangle$.

Next, we consider the case of Type (II-b). Let $f_0(\xi; \tau) = \sum_{j=0}^{h} c_j(\xi)\tau^j$, where $c_h(\xi) \not\equiv 0$. By a similar argument to the proof of Lemma 5.2, we can easily show that $\operatorname{ord}_\xi c_h = 0$, using the condition of Type (II). Further, by the condition of Type (II-b), there exists j such that $\operatorname{ord}_\xi c_j \geq 1$. By Lemma 5.1, for an arbitrary positive integer N, there exists $\xi_N \in \mathbf{C}^n$ and $\tau_N \in \mathbf{C}$ which satisfy $f_0(\xi_N; \tau_N) = 0$ and $\operatorname{Re}\tau_N > N$. Hence, by Theorem 4.1-(2), there exists $v_N \in C^N[0, T]$ such that $Q_{\xi_N}v_N = 0$ and $\partial_t^j v_N(0) = 0$ for any $j \leq N$. Thus, we have a C^N null-solution $u = v_N(t)\exp\langle\xi_N, x\rangle$.

§ 6. Proof of Theorems B and C

In this section, we prove Theorems B and C. First we give an easy lemma, without proof, which gives solutions to first order equations on $(0, T] \times \mathbf{R}$.

LEMMA 6.1. *Let $n = 1$ and consider $P = \partial_t + A(t)\partial_x + B(t)$, where $A, B \in C^\infty(0, T]$. Let $\mathscr{A}(t), \mathscr{B}(t) \in C^\infty(0, T]$ satisfy $(d/dt)\mathscr{A} = A$ and $(d/dt)\mathscr{B} = B$. Assume that Ω is a neighborhood of $x = 0$ and that a domain W of $\mathbf{C}$ satisfies $W \supset \{x - \mathscr{A}(t);\ t \in (0, T],\ x \in \Omega\}$. If $F(z)$ is holomorphic on W, then $u(t, x) = \exp\{-\mathscr{B}(t)\}F(x - \mathscr{A}(t))$ is a solution of $Pu = 0$ on $(0, T] \times \Omega$.*

Now, we prove Theorem B.

Proof of Theorem B. By considering solutions of the form $u(t, x) = u^{\sim}(t, x_j)$ for a suitable j, we may assume that $n = 1$. First, we consider the case when $s(1) = \kappa - 1$. In this case,

$$P = t^{\kappa}\{\partial_t + t^{-1}a(t)\partial_x + t^{-h}b(t)\},$$

where $a, b \in C^{\infty}[0, T]$, $h \in \mathbf{Z}$ and $a(0) \neq 0$. We can take $\mathscr{A}(t), \mathscr{B}(t) \in C^{\infty}(0, T]$ which satisfy $(d/dt)\mathscr{A}(t) = t^{-1}a(t)$ and $(d/dt)\mathscr{B}(t) = t^{-h}b(t)$, in the form

$$\mathscr{A}(t) = a(0)\log t + A_1(t), \qquad \mathscr{B}(t) = t^{-h+1}B_1(t) + B_2 \log t,$$

where $A_1, B_1 \in C^{\infty}[0, T]$, $A_1(0) = 0$ and B_2 is a constant. Note that

$$x - \mathscr{A}(t) = -a(0)\log\left\{t\cdot\exp\left(\frac{-x + A_1(t)}{a(0)}\right)\right\}.$$

If Ω is a sufficiently small neighborhood of $x = 0$ and T_0 is a sufficiently small positive number, then

$$\left\{z = t\cdot\exp\left(\frac{-x + A_1(t)}{a(0)}\right);\ (t, x) \in (0, T_0] \times \Omega\right\} \subset W_0$$
$$= \left\{z \in \mathbf{C};\ |z| < 1,\ |\arg z| < \frac{\pi}{4}\right\}.$$

Hence, by Lemma 6.1, if $G(z)$ is holomorphic on W_0, then

$$u = t^{-B_2}\exp(-t^{-h+1}B_1(t))G\left(t\cdot\exp\left(\frac{-x + A_1(t)}{a(0)}\right)\right)$$

is a solution of $Pu = 0$ on $(0, T_0] \times \Omega$. If we choose a suitable $G(z)$, then $\partial_t^j u(t, \cdot) \to 0$ $(t \to +0)$ in $C^{\infty}(\Omega)$ for any j and $(0, 0) \in \operatorname{supp} u$, hence u is a C^{∞} null-solution at $(0, 0)$.

Next, we consider the case when $s(1) < \kappa - 1$, that is,

$$P = t^{\kappa}\{\partial_t + t^{-1-\varepsilon}a(t)\partial_x + t^{-h}b(t)\},$$

where $a, b \in C^{\infty}[0, T]$, $a(0) \neq 0$, ε, $h \in \mathbf{Z}$ and $\varepsilon > 0$. By an argument similar to the above case, we have a solution of $Pu = 0$ in the form

$$u = t^{-B_2}\exp\{-t^{-h+1}B_1(t)\}G(\exp\{-t^{-\varepsilon} + A_1(t)\log t + xA_2(t)\}),$$

where $A_1, A_2, B_1 \in C^{\infty}[0, T]$ and B_2 is a constant. By choosing a suitable $G(z)$, we also obtain a C^{∞} null-solution for P at $(0, 0)$.

Next, we prove Theorem C.

Proof of Theorem C. Let P be an operator given by (2.2) and assume (2.3), (2.4) and (2.5). Dividing P by t^{κ}, we may assume that $\kappa = 0$ and hence $\sigma = -1 - \varepsilon$ for some positive integer ε.

We can solve the system of ordinary differential equations

$$\frac{dx}{dt} = \mathring{a}(t, x), \ x(0) = y \qquad \text{near } (0, 0).$$

Let the solution be $x = X(t, y)$. Note that if t is sufficiently small positive number, then $y \to X(t, y) = x$ is a diffeomorphism between a neighborhood of $y = 0$ and a neighborhood of $x = 0$. By the coordinate transformation $t = s$, $x = X(s, y)$, we have $\partial_t + \sum_{j=0}^{n} \mathring{a}_j(t, x)\partial_{x_j} = \partial_s$. Since the conditions for P and u are invariant under this coordinate transformation, we may assume that $\mathring{a}_j(t, x) \equiv 0$ $(j = 1, \cdots, n)$, without loss of generality. We may also assume that $\mathrm{Re}\, \mathring{b}(0, x) > 0$ on $\mathbf{R}^n$.

Now, assume that $u \in C^1((0, T]; \mathscr{D}'(\mathbf{R}^n))$ satisfies (2.6) and (2.7). Put

$$\mathscr{B}(t, x) = -\int_t^T \tau^{-1-\varepsilon} \mathring{b}(\tau, x) d\tau .$$

By the equation

$$(\partial_t + t^{-1-\varepsilon} \mathring{b}(t, x))u = 0 ,$$

we have

$$\partial_t\{u(t, x) \exp (\mathscr{B}(t, x))\} = 0 ,$$

hence there exists $F(x) \in \mathscr{D}'(\mathbf{R}^n)$ such that $u(t, x) \exp (\mathscr{B}(t, x)) = F(x)$ near $x = 0$ for any $t \in (0, T_0]$. Note that

$$\mathscr{B}(t, x) = -\frac{\mathring{b}(0, x)}{\varepsilon} t^{-\varepsilon} + B_1(t, x)t^{-\varepsilon+1} + B_2(x) \log t ,$$

where $B_1 \in C^\infty([0, T] \times \mathbf{R}^n)$ and $B_2 \in C^\infty(\mathbf{R}^n)$. Since $\mathrm{Re}\, \mathring{b}(0, x) > 0$ on $\mathbf{R}^n$, we have

$$t^{-M} \exp (\mathscr{B}(t, \cdot)) \longrightarrow 0 \ (t \to +0) \text{ in } C^\infty(\mathbf{R}^n) \text{ for any } M.$$

Hence, by (2.6), we obtain

$$u(t, \cdot) \exp (\mathscr{B}(t, \cdot)) \longrightarrow 0 \ (t \to +0) \text{ in } \mathscr{D}'(\mathbf{R}^n) ,$$

which implies that $F(x) \equiv 0$ near $x = 0$. Thus, there exist $T_1 > 0$ and a neighborhood Ω_1 of $x = 0$ such that $u = 0$ in $(0, T_1] \times \Omega_1$.

§ 7. Basic estimate

In this section, we shall show a basic energy estimate for the operator P given by (3.1). We use the same notations as in Section 3. Assume the conditions (A-2)–(A-5).

By Lemma 3.3, we have the factorization (3.2) of P_0. Put

$$G(t)=\int_t^T \frac{1}{\sigma}\Theta(\sigma)d\sigma,\quad \alpha'=\kappa+1,\quad p'=p-\alpha+\kappa+1$$

and

$$\begin{aligned}(7.1)\qquad E_s^*(u;t)=\{&t^{\alpha'}\|u_t(t,\cdot)+G'(t)u(t,\cdot)\|_s^2\\ &-t^{p'}(Q(t;\partial_x)u(t,\cdot),u(t,\cdot))_s+\|u(t,\cdot)\|_s^2\}^{1/2},\end{aligned}$$

for $u\in C^2((0,T];H^{s+2})$. Here, $u_t=\partial_t u$ and $(\cdot,\cdot)_s$ (resp. $\|\cdot\|_s$) denotes the inner product (resp. the norm) of the Sobolev space H^s of order s on $\mathbf{R}^n$.

Remark 7.1. By the condition (A-2), we have $(Qu,u)_s\le 0$ and hence E_s^* is well-defined.

The following energy estimate is vital to the proof of Theorem D.

PROPOSITION 7.2. *Assume the conditions* (A-2)–(A-5). *Then, there exist positive constants* $\tilde{N}$, T_0 *and* C *for which the following inequality holds*:

$$(7.2)\qquad E_s^*(u;t)\le C\left\{\int_{t_1}^t \tau^{-\tilde{N}}\|Pu(\tau,\cdot)\|_s d\tau+t_1^{-\tilde{N}}E_s^*(u;t_1)\right\}\quad (0<t_1\le t\le T_0)$$

for any $u\in C^2((0,T_0];H^{s+2})$.

To prove this proposition, we transform P using $G(t)$.

If we put $u=e^{-G(t)}v$, then the equation $Pu=f$ is transformed to $P_1v=e^{G(t)}t^{-\alpha+\kappa+1}f$, where

$$(7.3)\qquad P_1=t^{\alpha'}\partial_t^2+\tilde{b}(t)\partial_t-t^{p'}Q(t;\partial_x)+t^{q'}\sum_{j=1}^n c_j(t)\partial_{x_j}.$$

Here, $q'=q-\alpha+\kappa+1$ and $\tilde{b}(t)=t^{\kappa}(1-\Lambda(t)+\Theta(t))\in C^{\infty}_{\mathrm{frac}}$.

LEMMA 7.3. *The condition* (A-5) *implies* $2q'\ge p'-1$.

Proof. Consider the three cases given in Section 3. In Cases (1) and (2), there holds $2\beta<\alpha+\gamma$, hence the condition (A-5) is "$2q\ge p-1+\beta$". Since $\kappa=\alpha-\beta-1$, we have $2q'\ge p'-1$. In Case (3), there holds $2\beta\ge\alpha+\gamma$, hence the condition (A-5) is "$2q\ge p-1+(\alpha+\gamma)/2$". Since $\kappa=(\alpha-\gamma-2)/2$, we also have $2q'\ge p'-1$.

Now, we shall give an energy estimate for P_1. Put

$$(7.4)\qquad E_s(v;t)=\{t^{\alpha'}\|v_t(t,\cdot)\|_s^2-t^{p'}(Q(t;\partial_x)v(t,\cdot),v(t,\cdot))_s+\|v(t,\cdot)\|_s^2\}^{1/2}$$

for $v \in C^2((0, T]; H^{s+2})$.

Remarks 7.4. (1) By the condition (A-2), we have

$$t^{p'}\|v_x(t, \cdot)\|_s^2 \leq C\{E_s(v; t)\}^2 \tag{7.5}$$

for some constant C.

(2) Note that $E_s^*(u; t) = |e^{-G(t)}|E_s(e^{G(t)}u; t)$.

LEMMA 7.5. *Consider the operator P_1 given by (7.3). Assume that $\alpha' > 1$, $2q' \geq p' - 1$ and the condition (A-2). If $\operatorname{Re}\tilde{b}(0) > -\varepsilon_0$ ($\varepsilon_0 \geq 0$), then we have the following estimate for some positive constants N', T_0 and C:*

$$e^{\psi(t)}E_s(v; t) \leq C\left\{\int_{t_1}^{t} \tau^{-N'}e^{\psi(\tau)}\|P_1 v(\tau, \cdot)\|_s d\tau + t_1^{-N'}e^{\psi(t_1)}E_s(v; t_1)\right\} \tag{7.6}$$
$$(0 < t_1 \leq t \leq T_0)$$

for any $v \in C^2((0, T_0]; H^{s+2})$, where $\psi(t) = \varepsilon_0/(\alpha' - 1)t^{-\alpha'+1}$.

Proof. We can take $T_0 > 0$ and $b_1 > -\varepsilon_0$ such that

$$\operatorname{Re}\tilde{b}(t) \geq b_1 \quad \text{on} \quad [0, T_0]. \tag{7.7}$$

In this proof, C denotes an unspecified constant whose value may be different each time it appears.

We shall estimate

$$2\operatorname{Re}(P_1 v(t, \cdot), v_t(t, \cdot))_s = 2t^{\alpha'}\operatorname{Re}(v_{tt}, v_t)_s - 2t^{p'}\operatorname{Re}(Qv, v_t)_s + 2\operatorname{Re}(\tilde{b}v_t, v_t)_s + 2t^{q'}\operatorname{Re}\left(\sum_{j=1}^{n} c_j v_{x_j}, v_t\right)_s,$$

by means of $E_s(v; t)$ as follows.

(a) $\partial_t\{t^{\alpha'}\|v_t\|_s^2\} = \alpha' t^{\alpha'-1}\|v_t\|_s^2 + t^{\alpha'}2\operatorname{Re}(v_{tt}, v_t)_s$.

Hence, we have

$$2t^{\alpha'}\operatorname{Re}(v_{tt}, v_t)_s \geq \partial_t\{t^{\alpha'}\|v_t\|_s^2\} - \frac{\alpha'}{t}\{E_s\}^2.$$

(b) $\partial_t\{-t^{p'}Qv, v)_s\}$

$$= -p't^{p'-1}(Qv, v)_s - t^{p'}(Q_t v, v)_s - t^{p'}(Qv_t, v)_s - t^{p'}(Qv, v_t)_s$$
$$= -p't^{p'-1}(Qv, v)_s - t^{p'}(Q_t v, v)_s - t^{p'}2\operatorname{Re}(Qv, v_t)_s,$$

where $Q_t = (\partial_t Q)(t; \partial_x)$. By Remark 7.4, there holds

$$|(Q_t v, v)_s| \leq C\|v_x\|_s^2 \leq Ct^{-p'}\{E_s\}^2.$$

Hence, we obtain

$$
\begin{aligned}
-2t^{p'}\operatorname{Re}(Qv, v_t)_s &\geq \partial_t\{-t^{p'}(Qv, v)_s\} + \frac{p'}{t}t^{p'}(Qv, v)_s - C\{E_s\}^2 \\
&\geq \partial_t\{-t^{p'}(Qv, v)_s\} - \frac{C}{t}\{E_s\}^2 .
\end{aligned}
$$

(c) $2\operatorname{Re}(\tilde{b}v_t, v_t)_s = 2\operatorname{Re}\tilde{b}(t)\|v_t\|_s^2 \geq 2b_1\|v_t\|_s^2 .$

(d)
$$
\begin{aligned}
2t^{q'}\operatorname{Re}\left(\sum_{j=1}^{n} c_j v_{x_j}, v_t\right)_s &\geq -Ct^{q'}\|v_x\|_s\|v_t\|_s \\
&\geq -Ct^{2q'}\|v_x\|_s^2 - \varepsilon_1\|v_t\|_s^2 \geq -\frac{C}{t}\{E_s\}^2 - \varepsilon_1\|v_t\|_s^2
\end{aligned}
$$

for an arbitrary $\varepsilon_1 > 0$, by $2q' \geq p' - 1$ and Remark 7.4.

(e)
$$
\begin{aligned}
\partial_t\{\|v\|_s^2\} &= 2\operatorname{Re}(v, v_t)_s \leq 2\|v\|_s\|v_t\|_s \\
&\leq C\|v\|_s^2 + \varepsilon_2\|v_t\|_s^2 \leq C\{E_s\}^2 + \varepsilon_2\|v_t\|_s^2
\end{aligned}
$$

for an arbitrary $\varepsilon_2 > 0$.

From these estimates (a)–(e), we obtain

$$
\begin{aligned}
2\operatorname{Re}(P_1 v, v_t)_s &\geq \partial_t\{t^{\alpha'}\|v_t\|_s^2\} - \frac{\alpha'}{t}\{E_s\}^2 + \partial_t\{-t^{p'}(Qv, v)_s\} \\
&\quad - \frac{C}{t}\{E_s\}^2 + 2b_1\|v_t\|_s^2 - \frac{C}{t}\{E_s\}^2 - \varepsilon_1\|v_t\|_s^2 \\
&\geq \partial_t\{E_s\}^2 - \frac{C}{t}\{E_s\}^2 + (2b_1 - \varepsilon_1 - \varepsilon_2)\|v_t\|_s^2 .
\end{aligned}
$$

If we take $\varepsilon_1, \varepsilon_2$ as $\varepsilon_1 + \varepsilon_2 \leq 2b_1 + 2\varepsilon_0$, then we have

$$
\partial_t\{E_s\}^2 - \frac{C}{t}\{E_s\}^2 - 2\varepsilon_0\|v_t\|_s^2 \leq 2\operatorname{Re}(P_1 v, v_t)_s \leq 2t^{-\alpha'/2}\|P_1 v\|_s E_s ,
$$

and hence

$$
\partial_t\{E_s\}^2 - \frac{C}{t}\{E_s\}^2 - 2\varepsilon_0 t^{-\alpha'}\{E_s\}^2 \leq 2t^{-\alpha'/2}\|P_1 v\|_s E_s .
$$

Dividing by $2E_s$, we have

$$
\partial_t E_s - \frac{C}{t}E_s - \varepsilon_0 t^{-\alpha'}E_s \leq t^{-\alpha'/2}\|P_1 v\|_s .
$$

From this, we obtain

$$
\partial_t\{t^{-C}e^{\psi(t)}E_s(v; t)\} \leq t^{-C-\alpha'/2}e^{\psi(t)}\|P_1 v(t, \cdot)\|_s ,
$$

and hence, we obtain

$$t^{-C}e^{\psi(t)}E_s(v;t) \le \int_{t_1}^{t} \tau^{-C-\alpha'/2}e^{\psi(\tau)}\|P_1v(\tau,\cdot)\|_s d\tau + t_1^{-C}e^{\psi(t_1)}E_s(v;t_1)$$
$$\text{for } 0<t_1\le t\le T_0 .$$

(Note that $\psi'(t) = -\varepsilon_0 t^{-\alpha'}$.)

Thus, we obtain the estimate (7.6).

Now, we prove Proposition 7.2.

Proof of Proposition 7.2. Note that if we put $v = e^{G(t)}u$, then we have $P_1 v = e^{G(t)}t^{-\alpha+\kappa+1}Pu$ and $E_s(v;t) = |e^{G(t)}|E_s^*(u;t)$. From (7.6), we obtain the following estimate for some constant N''.

$$E_s^*(u;t) \le C|e^{-\psi(t)-G(t)}|\left\{\int_{t_1}^{t} \tau^{-N''}|e^{\psi(\tau)+G(\tau)}|\,\|Pu(\tau,\cdot)\|_s d\tau \right.$$
$$\left. + t_1^{-N''}|e^{\psi(t_1)+G(t_1)}|E_s^*(u;t_1)\right\} \quad (0<t_1\le t\le T_0).$$

Comparing to (7.2), we have only to show that

$$(7.8) \qquad |e^{-\psi(t)-G(t)}e^{\psi(\tau)+G(\tau)}| \le C\tau^{-M} \qquad (0<\tau\le t\le T_0)$$

for some constant M.

By Lemma 3.3, we can prove (7.8) as follows.

In Case (1), we have $\operatorname{Re}\tilde{b}(0) = -\operatorname{Re}\lambda(0) > 0$, hence we can take $\varepsilon_0 = 0$, that is, $\psi(t) \equiv 0$. Further, we have $e^{G(t)} = t^{-\theta(0)}\varphi(t)$ for some $\varphi \in C^\infty_{\text{frac}}$. Thus, we obtain (7.8).

In Case (2), we also have $\operatorname{Re}\tilde{b}(0) = -\operatorname{Re}\lambda(0) > 0$, hence we can take $\varepsilon_0 = 0$ and $\psi(t) \equiv 0$. Further, $\operatorname{Re} G(t)$ is increasing near $t=0$, since $\operatorname{Re} G'(t) = -(1/t)\operatorname{Re}\Theta(t) > 0$ near $t=0$. Hence, we obtain (7.8) with $M=0$.

In Case (3), we have $\operatorname{Re}\tilde{b}(0) = -\operatorname{Re}\lambda(0) + \operatorname{Re}\theta(0) \ge 0$. Further, there holds

$$-\psi(t) - G(t) + \psi(\tau) + G(\tau) = \int_{\tau}^{t} \{\theta(\sigma)+\varepsilon_0\}\sigma^{-\kappa-1}d\sigma .$$

Since we can take ε_0 as $0<\varepsilon_0< -\operatorname{Re}\theta(t)$ near $t=0$, we obtain (7.8) with $M=0$.

§ 8. Proof of Theorem D

In this section, we shall prove Theorem D. Consider the operator P given by (3.1) and assume the conditions (A-1)–(A-5).

Before the proof, we give two lemmas. The first lemma shall make it possible to use the Sobolev norm.

LEMMA 8.1. *Assume that* $u \in C^0((0, T]; \mathscr{D}'(\mathbf{R}^n))$ $(T > 0)$ *satisfies* $\operatorname{supp} u(t, \cdot) \subset K$ *for any* $t \in (0, T]$, *where* K *is a compact set in* $\mathbf{R}^n$. *If* $u(t, \cdot)$ *is bounded in* $\mathscr{D}'(\mathbf{R}^n)$ (*resp.* $u(t, \cdot) \to 0$, *in* $\mathscr{D}'(\mathbf{R}^n)$) *as* $t \to +0$, *then there exists an integer* s *such that* $u \in C^0((0, T]; H^s)$ *and* $\|u(t, \cdot)\|_s$ *is bounded* (*resp.* $\|u(t, \cdot)\|_s \to 0$) *as* $t \to +0$.

Proof. If we put $\mathscr{L} = \{u(t, \cdot) \in \mathscr{D}'(\mathbf{R}^n);\ 0 < t \le T\}$, then its closure $\bar{\mathscr{L}}$ is bounded in $\mathscr{E}'(\mathbf{R}^n)$, and hence compact. By the structure theorem of $\mathscr{E}'(\mathbf{R}^n)$, there exists an integer s such that $\bar{\mathscr{L}} \subset H^s$ and that the topologies on $\bar{\mathscr{L}}$ induced from $\mathscr{D}'(\mathbf{R}^n)$ and from H^s coincide. This implies the results.

The second lemma shows the existence of "good" dependence domains for the Cauchy problem in $\{t > 0\}$.

LEMMA 8.2. *Assume the conditions* (A-1) *and* (A-2). *For* $t_1 \in (0, T]$, *consider the Cauchy problem with the initial surface* $t = t_1$:

$$(CP)_{t_1} \qquad \begin{cases} Pu = f \in C^0([t_1, T];\ \mathscr{D}'(\mathbf{R}^n)), \\ u|_{t=t_1} = \varphi_1 \in \mathscr{D}'(\mathbf{R}^n), \\ \partial_t u|_{t=t_1} = \varphi_2 \in \mathscr{D}'(\mathbf{R}^n). \end{cases}$$

For any $T_0 > 0$ *and any neighborhood* Ω_0 *of* $x = 0$ *in* $\mathbf{R}^n$, *there exists a compact set* $D \subset [0, T_0] \times \Omega_0$ *which satisfies the following*:

(i) *There exist* $T_0' > 0$ *and a neighborhood* Ω_0' *of* $x = 0$ *such that* $D \supset [0, T_0'] \times \Omega_0'$.

(ii) *For any* $t_1 \in (0, T_0)$, *if* $f = 0$ *in* $D \cap \{t \ge t_1\}$ *and* $\varphi_1 = \varphi_2 = 0$ *on* $D \cap \{t = t_1\}$, *then the solution* u *of* $(CP)_{t_1}$ *satisfies* $u = 0$ *in* $D \cap \{t \ge t_1\}$.

It is the point of this lemma that D is independent of t_1.

Proof. Let ν be a positive integer such that $\nu \ge 2/(p - \alpha + 2)$ (> 0). Let $P^\sim$ be the operator transformed from P by $t^{1/\nu} = s$. By $\partial_t = (1/\nu)s^{1-\nu}\partial_s$, we have

$$\begin{aligned} P^\sim &= (1/\nu^2)s^{\alpha\nu+2-2\nu}\partial_s^2 - s^{p\nu}Q(s^\nu; \partial_x) + \text{l.o.t. [lower order terms]} \\ &= (1/\nu^2)s^{\alpha\nu+2-2\nu}(\partial_s^2 - \nu^2 s^{(p-\alpha+2)\nu-2}Q(s^\nu; \partial_x)) + \text{l.o.t.} \end{aligned}$$

Since $(p - \alpha + 2)\nu - 2 \ge 0$, the operator $P^\sim$ satisfies that

(i) $P^{\sim}$ is strictly hyperbolic in $(0, T^{1/\nu}] \times \mathbf{R}^n$,

(ii) the characteristic roots $\sigma = \pm\lambda(s; \xi)$ of $P^{\sim}$ is bounded as $s \to +0$. That is, there exists a constant M such that $|\lambda(s; \xi)| \leq M|\xi|$ for any $(s; \xi) \in (0, T^{1/\nu}] \times \mathbf{R}^n$.

By the well-known result for strictly hyperbolic operators, the Cauchy problem for $P^{\sim}$ with the initial surface $s = s_1$ (> 0) has dependence domains $D^{\sim}$ of the form

$$D^{\sim} = \{(s, x);\ s_1 \leq s \leq s_0,\ |x - x_0| \leq M(s_0 - s)\}.$$

Note that M does not depend on s_1.

Thus, taking $x_0 = 0$ and a sufficiently small $t_0 > 0$, we see that the compact set

$$D = \{(t, x);\ 0 \leq t \leq t_0,\ |x| \leq M(t_0^{1/\nu} - t^{1/\nu})\}$$

satisfies the required properties.

Now, we shall prove Theorem D.

Proof of Theorem D. Assume that $u \in C^2((0, T]; \mathscr{D}'(\mathbf{R}^n))$ satisfies the conditions (i) and (ii) in Theorem D for sufficiently large N. By cutting off and by Lemma 8.1, we may assume that

(8.1) $\operatorname{supp} u(t, \cdot) \subset K$ for $t \in (0, T]$, where K is a compact set in $\mathbf{R}^n$,

(8.2) $u \in C^2((0, T]; H^{s+4})$ for some integer s,

(8.3) $t^{-N}u(t, \cdot),\ t^{-N+1}u_t(t, \cdot) \to 0$ $(t \downarrow 0)$ in H^{s+2}.

Fix an arbitrary $t_1 \in (0, T)$. Since P is strictly hyperbolic on $[t_1, T] \times \mathbf{R}^n$, we can take the solution $w[t_1] \in C^2([t_1, T]; H^{s+2})$ of the Cauchy problem

$$\begin{cases} Pw[t_1] = 0 & \text{on } [t_1, T] \times \mathbf{R}^n, \\ w[t_1]|_{t=t_1} = u|_{t=t_1} & (\in H^{s+4}), \\ \partial_t w[t_1]|_{t=t_1} = \partial_t u|_{t=t_1} & (\in H^{s+4}). \end{cases} \tag{8.4}$$

Applying the estimate (7.2) to $w[t_1]$, we obtain

$$\begin{aligned} E_s^*(w[t_1]; t) &\leq C t_1^{-\tilde{N}} E_s^*(w[t_1]; t_1) \\ &= C t_1^{-\tilde{N}} E_s^*(u; t_1) \qquad \text{for } t_1 \leq t \leq T_0. \end{aligned}$$

Note that this constant C does not depend on t_1. By (8.3), we have $t_1^{-\tilde{N}} E_s^*(u; t_1) \to 0$ $(t_1 \to +0)$, if N is sufficiently large. Hence, we obtain

$$w[t_1](t, \cdot) \longrightarrow 0 \ (t_1 \downarrow 0) \text{ in } H^s \quad \text{for any } t \in (0, T_0]. \tag{8.5}$$

Now, take D in Lemma 8.2. Since $P(w[t_1]-u)=0$ in $D\cap\{t\geq t_1\}$, and $(w[t_1]-u)|_{t=t_1}=\partial_t(w[t_1]-u)|_{t=t_1}=0$ on $D\cap\{t=t_1\}$, we have $w[t_1]=u$ in $D\cap\{t\geq t_1\}$. Hence, by (8.5), we obtain $u=0$ in D.

Appendix. C^∞ well-posedness of the flat Cauchy problem for some non-Fuchsian operators

In this appendix, we show C^∞ well-posedness of some flat Cauchy problems as is stated in Remark 2.1 and Remarks 3.2-(2). First, we shall give the precise statements.

For an open set U in $\mathbf{R}^{n+1}$, put $U_+=U\cap\{t\geq 0\}$ and $\mathscr{F}^\infty(U_+)=\{u\in C^\infty(U_+);\ \partial_t^j u|_{t=0}=0 \text{ for any } j\geq 0\}$.

PROPOSITION A.1. *Consider the operator P given by* (2.2). *Assume that conditions* (2.3), (2.4) *and* (2.5). *Then, there exists an open neighborhood Ω of* $(0,0)$ *such that for any $f\in\mathscr{F}^\infty(\Omega_+)$, there exists a unique $u\in\mathscr{F}^\infty(\Omega_+)$ which satisfies $Pu=f$ on Ω_+.*

PROPOSITION A.2. *Consider the operator P given by* (3.1) *and assume that conditions* (A-1)–(A-5) *in Section* 3. *Then, for any $f\in\mathscr{F}^\infty([0,T]\times\mathbf{R}^n)$, there exists a unique $u\in\mathscr{F}^\infty([0,T]\times\mathbf{R}^n)$ which satisfies $Pu=f$ on* $[0,T]\times\mathbf{R}^n$.

Since the proof of Proposition A.1 is similar to that of Theorem C, we shall prove only Proposition A.2.

Proof of Proposition A.2. Put $H^\infty=\bigcap_{s\in\mathbf{R}}H^s$ and $\mathscr{F}^\infty([0,T];H^\infty)=\{f\in C^\infty([0,T]);H^\infty);\ \partial_t^j f|_{t=0}=0 \text{ for any } j\geq 0\}$. Since there exist bounded dependence domains (Lemma 8.2), we have only to show the following:

For any $f\in\mathscr{F}^\infty([0,T];H^\infty)$, there exists $u\in\mathscr{F}^\infty([0,T];H^\infty)$ such that $Pu=f$ on $[0,T]\times\mathbf{R}^n$.

Let $f\in\mathscr{F}^\infty([0,T];H^\infty)$. For any $j\geq 1$, put

$$f_j(t,x)=\begin{cases} f(t-1/j,x) & (t\geq 1/j)\\ 0 & (t\leq 1/j)\end{cases}\in\mathscr{F}^\infty([0,T];H^\infty).$$

Since P is strictly hyperbolic in $\{t>0\}$, we can solve the Cauchy problem;

$$(\mathrm{CP})_j\qquad \begin{cases} Pv=f_j & \text{in } [1/j,T]\times\mathbf{R}^n,\\ v|_{t=1/j}=0,\\ \partial_t v|_{t=1/j}=0.\end{cases}$$

Let the solution be $v_j(t, x)$ and put

$$u_j(t, x) = \begin{Bmatrix} v_j(t, x) & (t \geq 1/j) \\ 0 & (t \leq 1/j) \end{Bmatrix} \in \mathscr{F}^\infty([0, T]; H^\infty).$$

Applying the energy estimate (7.2) to $u_j - u_k$ for any s, we can show that $u = \lim_{j\to\infty} u_j$ exists in $C^1((0, T]; H^\infty)$. This u satisfies $Pu = f$ in $(0, T] \times \mathbf{R}^n$ and $E_s^*(u, t) \leq C\int_0^t \tau^{-\tilde{N}}\|f(\tau, \cdot)\|_s d\tau$ for any s. It is easy to show that $t^{-M}u(t, \cdot)$, $t^{-M}u_t(t, \cdot) \to 0$ $(t \downarrow 0)$ in H^∞ for any M. Hence, by $Pu = f$, we obtain $u \in \mathscr{F}^\infty([0, T]; H^\infty)$.

Remark A.3. Since the condition (A-1) is used only to show the existence of bounded dependence domains (Lemma 8.2), we can show the H^∞ well-posedness of the flat Cauchy problem for P under the assumptions (A-2)–(A.5) without assuming (A-1).

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