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POINCARÉ TYPE CONDITIONS OF THE REGULARITY FOR THE PARABOLIC OPERATOR OF ORDER α

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§ 1. Introduction

Let $R^{n+1} = R^n \times R$ denote the $(n+1)$-dimensional Euclidean space $(n \geqq 1)$. For $X \in R^{n+1}$, we write $X = (x, t)$ with $x \in R^n$ and $t \in R$. In this case, R^n is called the x-space of $R^{n+1} = R^n \times R$.

For an α with $0 < \alpha < 1$, we write

$$L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^{\alpha},$$

where $(-\Delta)^{\alpha}$ is the fractional power of the Laplacian $-\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ on the x-space. In the case of $\alpha = 1/2$, $L^{(1/2)}$ is called the Poisson operator on R^{n+1}.

First we shall examine some properties of the elementary solution $W^{(\alpha)}$ of $L^{(\alpha)}$. By using the reduced functions with respect to $W^{(\alpha)}$, we shall show the existence of swept-out measures with respect to $W^{(\alpha)}$. By using swept-out measures, we shall give the notion of the regularity for boundary points of an open set in R^{n+1}.

The purpose of this paper is to give a Poincaré type condition for the regularity of boundary points of an open set in R^{n+1}.

Our main theorem is the following

THEOREM. *Let Ω be an open set in R^{n+1} and X a boundary point of Ω. If there exists a non-empty open set ω in the x-space whose α-tusk $T_X^{(\alpha)}(\omega)$ at X is in $C\Omega$, then X is regular for the Dirichlet problem of $L^{(\alpha)}$ on Ω.*

For an $X = (x, t) \in R^{n+1}$ and an open set ω in the x-space, the α-tusk $T_X^{(\alpha)}(\omega)$ of ω at X is defined by

$$T_X^{(\alpha)}(\omega) = \{(x + py, t - p^{2\alpha});\, y \in \omega,\, 0 < p < p_0\}$$

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with some $p_0 > 0$.

For the heat equation, E. G. Effros and J. L. Kazdan [3] discussed a similar Poincaré type condition of the regularity.

§ 2. Superparabolic functions and the Riesz decomposition

Let $C_K^\infty(R^k)$ denote the usual topological vector space of all infinitely differentiable functions on R^k with compact support ($k \geqq 1$). For $0 < \alpha < 1$, we recall the fractional power $(-\Delta)^\alpha$ of $-\Delta$ on the x-space R^n; $(-\Delta)^\alpha$ is the convolution operator on R^n defined by the distribution $-C_{n,\alpha}\,\text{p.f.}|x|^{-n-2\alpha}$, where $|x|$ denotes the distance between x and the origin 0 in R^n and $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha)$, that is,

$$\text{p.f.}\,|x|^{-n-2\alpha}(\phi) = \lim_{\delta \downarrow 0} \int_{|x|>\delta} (\phi(x) - \phi(0))|x|^{-n-2\alpha}dx$$

for every $\phi \in C_K^\infty(R^n)$.

We denote by $(g_t)_{t\geqq 0}$ the Gaussian semi-group on R^n, namely $g_t(x) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ $(t > 0)$, and $g_0 = \varepsilon$. Here we denote by ε the Dirac measure at the origin of R^k for every $k \geqq 1$. Put

$$W^{(\alpha)}(X) = \begin{cases} (2\pi)^{-n} \int_{R^n} \exp(-t|\xi|^{2\alpha} + ix\cdot\xi)d\xi & t > 0 \\ 0 & t \leqq 0, \end{cases}$$

where $X = (x, t)$ and $x\cdot\xi$ denotes the inner product on R^n. By means of the Fourier transform, we see easily that $W^{(\alpha)}$ (resp. $\tilde{W}^{(\alpha)}$) is the elementary solution of $L^{(\alpha)}$ (resp. $\tilde{L}^{(\alpha)}$), where $\tilde{W}^{(\alpha)}(x, t) = W^{(\alpha)}(x, -t)$ and $\tilde{L}^{(\alpha)} = -\partial/\partial t + (-\Delta)^\alpha$ (see for example [4]). Let $(\sigma_t^\alpha)_{t\geqq 0}$ be the one-sided stable semi-group of order α on R^+, where R^+ denotes the semi-group of all non-negative numbers. Then for any $t > 0$ and $x \in R^n$,

$$W^{(\alpha)}(x, t) = \int_0^\infty g_s(x)d\sigma_t^\alpha(s) > 0 \tag{2.1}$$

(see [1], p. 74), $\int_{R^n} W^{(\alpha)}(x, t)dx = 1$ and $W^{(\alpha)}(x, t)$ is a decreasing function of $|x|$. Put

$$\psi_\alpha(t) = W^{(\alpha)}((1, 0, \cdots, 0), t);$$

then we have easily

$$W^{(\alpha)}(x, t) = |x|^{-n}\psi_\alpha(t|x|^{-2\alpha})$$

LEMMA 2.1. *$\psi_\alpha(t) = O(t)$ as $t \downarrow 0$.*

Proof. Let ν be the uniform measure on the unit sphere $\{x \in R^n; |x| = 1\}$ with $\int d\nu = 1$. Denoting by $\hat{\nu}$ the Fourier transform of ν, we have

$$\psi_\alpha(t) = (2\pi)^{-n/2} \int_{R^n} \exp(-t|\xi|^{2\alpha})\hat{\nu}(\xi)d\xi, \quad \lim_{t \downarrow 0} \psi_\alpha(t) = 0$$

and

$$\begin{aligned} \frac{d}{dt}\psi_\alpha(t) &= (2\pi)^{-n/2} \int_{R^n} (-|\xi|^{2\alpha} \exp(-t|\xi|^{2\alpha})\hat{\nu}(\xi))d\xi \\ &= (2\pi)^{-n/2} \int_0^\infty \int_{R^n} (-|\xi|^{2\alpha} \exp(-s|\xi|^2)\hat{\nu}(\xi))d\xi\, d\sigma_t^\alpha(s) \end{aligned}$$

for $t > 0$ (see (2.1)). Let $\phi \in C_K^\infty(R^n)$ satisfying $0 \leqq \phi \leqq 1$, $\mathrm{supp}[\phi] \subset \{x \in R^n; |x| < 1\}$ and $\phi = 1$ on a neighborhood of 0, where $\mathrm{supp}[\phi]$ denotes the support of ϕ. For any $s > 0$, we have

$$\begin{aligned} &\int_{R^n} |\xi|^{2\alpha} \exp(-s|\xi|^2)\hat{\nu}(\xi)d\xi = (2\pi)^{n/2}(-\Delta)^\alpha(g_s * \nu)(0) \\ &\quad = (2\pi)^{n/2}C_{n,\alpha-1}(|x|^{-n-2\alpha+2}) * (\Delta g_s) * \nu(0) \\ &\quad = (2\pi)^{n/2}C_{n,\alpha-1}(\phi(x)|x|^{-n-2\alpha+2}) * (\Delta g_s) * \nu(0) \\ &\qquad + (2\pi)^{n/2}C_{n,\alpha-1}(\Delta((1-\phi(x))|x|^{-n-2\alpha+2})) * (g_s) * \nu(0)\,. \end{aligned}$$

Since $0 \notin \mathrm{supp}[(\phi(x)|x|^{-n-2\alpha}) * \nu]$ and Δg_s vanishes uniformly outside any neighborhood of 0,

$$\lim_{s \downarrow 0} (\phi(x)|x|^{-n-2\alpha+2}) * (\Delta g_s) * \nu(0) = 0\,.$$

Therefore the function $\int_{R^n} |\xi|^{2\alpha} \exp(-s|\xi|^2)\hat{\nu}(\xi)d\xi$ of s is bounded on $(0, \infty)$, so that $(d/dt)\psi_\alpha(t)$ is bounded on $(0, \infty)$, which shows Lemma 2.1.

Let $(P_t^{(\alpha)})_{t \geqq 0}$ be the convolution semi-group whose infinitesimal generator is equal to $-L^{(\alpha)}$ (see [7][1]); then

$$P_s^{(\alpha)} * u(x, t) = \int_{R^n} W^{(\alpha)}(x-y, s)u(y, t-s)dy$$

for every $u \in C_K(R^{n+1})$, where $C_K(R^{n+1})$ denotes the usual topological vector space of all finite continuous functions on R^{n+1} with compact support. For a non-negative continuous function $\phi(t)$ on $(0, \infty)$, we put

$$W_{(\phi)}^{(\alpha)}(x, t) = \phi(t)W^{(\alpha)}(x, t)\,.$$

1) Evidently $-L^{(\alpha)}$ is a generalized Laplacian, that is, for any $\phi \in C_K^\infty(R^{n+1})$ with $\phi \geqq 0$ and $\phi(0) = \max_{X \in R^{n+1}} \phi(X)$, $-(L^{(\alpha)}\phi)(0) \leqq 0$.

For a sequence $(\phi_m)_{m=1}^{\infty}$ in $C_K((0, \infty))$ with $\phi_m \geqq 0$, $\int \phi_m dt = 1$ and with $\mathrm{supp}[\phi_m] \subset ((m+1)^{-1}, m^{-1})$, we shall often use the sequence $(W_{(\phi_m)}^{(\alpha)})_{m=1}^{\infty}$. We say that such a $(\phi_m)_{m=1}^{\infty}$ is an approximate sequence of the Dirac measure.

Definition 1. A non-negative function u on R^{n+1} is said to be superparabolic of order α if the following two conditions are satisfied:

(1) u is lower semi-continuous on R^{n+1} and $u < \infty$ a.e..

(2) For any $s \geqq 0$, $u \geqq P_s^{(\alpha)} * u$ on R^{n+1}.

We denote by S_α (resp. $S_{\alpha,c}$) the set of all superparabolic (resp. all continuous superparabolic) functions of order α, and by $\tilde{S}_\alpha$ (resp. $\tilde{S}_{\alpha,c}$) the set of all functions u with $\tilde{u} \in S_\alpha$ (resp. $\tilde{u} \in S_{\alpha,c}$), where $\tilde{u}(x, t) = u(x, -t)$.

For a non-negative Borel measure μ on R^{n+1}, we denote by $W^{(\alpha)}\mu$ (resp. $\tilde{W}^{(\alpha)}\mu$) the function defined by the convolution $W^{(\alpha)} * \mu$ (resp. $\tilde{W}^{(\alpha)} * \mu$) and call it the $W^{(\alpha)}$-potential (resp. the $\tilde{W}^{(\alpha)}$-potential) of μ.

Remark 2.2. (1) $1 \in S_{\alpha,c}$ and for $u \in S_\alpha$, u is locally integrable.

(2) The condition (2) in Definition 1 is equivalent to $u \geqq W_{(\phi)}^{(\alpha)} * u$ for every $\phi \in C_K((0, \infty))$ with $\phi \geqq 0$ and $\int \phi dt = 1$.

(3) If $W^{(\alpha)}\mu < \infty$ (resp. $\tilde{W}^{(\alpha)}\mu < \infty$) a.e., then $W^{(\alpha)}\mu \in S_\alpha$ (resp. $\tilde{W}^{(\alpha)}\mu \in \tilde{S}_\alpha$).

We denote by M_α (resp. $M_{\alpha,c}$, $\tilde{M}_\alpha$ and $\tilde{M}_{\alpha,c}$) the set of all positive Borel measures μ with $W^{(\alpha)}\mu \in S_\alpha$ (resp. $W^{(\alpha)}\mu \in S_{\alpha,c}$, $\tilde{W}^{(\alpha)}\mu \in \tilde{S}_\alpha$ and $\tilde{W}^{(\alpha)}\mu \in \tilde{S}_{\alpha,c}$). For a Borel measure μ and a Borel set A, we denote by $\mu|_A$ the Borel measure defined by $\mu|_A(E) = \mu(A \cap E)$ for every Borel set E.

Lemma 2.3. *For $u \in S_\alpha$, we have*

$$\int_a^b \int_{|x| \geqq 1} u(x, t)|x|^{-n-2\alpha} dx\, dt < \infty$$

for every finite interval $[a, b]$.

Proof. Let $\phi \in C_K^\infty(R^{n+1})$ with $0 \leqq \phi \leqq 1$, $\phi(X) = 1$ on $\{X = (x, t); |x| \leqq 1/2, a \leqq t \leqq b\}$ and with $\phi(X) = 0$ on $\{X = (x, t); |x| \geqq 3/4\}$. Since for any $X = (x, t) \in C\,\mathrm{supp}[\phi]$,

$$\tilde{L}^{(\alpha)}\phi(x, t) = -C_{n,\alpha} \int_{R^n} \phi(y, t)|x - y|^{-n-2\alpha} dy \leqq 0,$$

$\mathrm{supp}[(\tilde{L}^{(\alpha)}\phi)^+] \subset \mathrm{supp}[\phi]$. On the other hand for any open ball B containing $\mathrm{supp}[\phi]$,

$$\int_B u(\tilde{L}^{(\alpha)}\phi)dX = \lim_{s\downarrow 0}\int_B u\frac{\phi - \tilde{P}_s^{(\alpha)}*\phi}{s}dX$$
$$= \lim_{s\downarrow 0}\left(\int_{R^{n+1}}\frac{u - P_s^{(\alpha)}*u}{s}\phi\, dX + \int_{CB} u\frac{\tilde{P}_s^{(\alpha)}*\phi}{s}dX\right) \geqq 0\,,$$

where $\tilde{P}_s^{(\alpha)}$ is defined by $\int f d\tilde{P}_s^{(\alpha)} = \int f(-X)dP_s^{(\alpha)}(X)$ for every $f \in C_K(R^{n+1})$. Hence

$$\infty > \int_{R^{n+1}} u(\tilde{L}^{(\alpha)}\phi)^+ dX \geqq \int_{R^{n+1}} u(L^{(\alpha)}\phi)^- dX$$
$$\geqq \int_a^b\int_{|x|\geqq 1} u(x, t)\left(C_{n,\alpha}\int_{R^n}\phi(y, t)|x-y|^{-n-2\alpha}dy\right)dx\, dt$$
$$\geqq 2^{-n-2\alpha}C_{n,\alpha}\int_{|y|\leqq 1/2} dy\int_a^b\int_{|x|\geqq 1} u(x, t)|x|^{-n-2\alpha}dx\, dt\,,$$

which shows Lemma 2.3.

LEMMA 2.4. (1) *S_α and $S_{\alpha,c}$ are convex semi-lattices by $u\wedge v(X) = \min(u(X), v(X))$.*

(2) *Let $u \in S_\alpha$ and let $(\phi_m)_{m=1}^\infty$ be an approximate sequence of the Dirac measure. Then $W_{(\phi_m)}^{(\alpha)} * u \in S_{\alpha,c}$ and $W_{(\phi_m)}^{(\alpha)} * u \uparrow u$ with $m\uparrow\infty$.*

(3) *Let $u, v \in S_\alpha$ and ω be an open set in R^{n+1}. If $u \leqq v$ a.e. on ω, then $u \leqq v$ on ω.*

Proof. The assertion (1) is evident (see Definition 1). Since $W_{(\phi_m)}^{(\alpha)}$ is finite continuous, Lemmas 2.1, 2.3 give $W_{(\phi_m)}^{(\alpha)} * u \in S_{\alpha,c}$. Since $(W_{(\phi_m)}^{(\alpha)}(X)dX)_{m=1}^\infty$ converges vaguely to ε as $m\to\infty$, we have the second part of (2) (see Definition 1). The assertion (3) follows from (2).

LEMMA 2.5. *For $u \in S_\alpha$, the family $\left(\frac{u - P_s^{(\alpha)}*u}{s}dX\right)_{s>0}$ of positive measures converges vaguely as $s\downarrow 0$, where dX denotes the Lebesgue measure on R^{n+1}. Denote by μ its vague limit. Then*

$$\int_{R^{n+1}} u\tilde{L}^{(\alpha)}\phi\, dX = \int_{R^{n+1}}\phi\, d\mu$$

for every $\phi \in C_K^\infty(R^{n+1})$.

Proof. For any $\phi \in C_K^\infty(R^{n+1})$ with $\phi \geqq 0$, we take $r > 0$ with $\mathrm{supp}[\phi] \subset \{X; |X| \leqq r\}$. For $(x, t) \in R^{n+1}$ with $|x| \geqq 2r$ and any $s > 0$, Lemma 2.1 shows

$$\left|\frac{\phi(x,t)-\tilde{P}_s^{(\alpha)}*\phi(x,t)}{s}\right| \leqq \frac{1}{s}\int W^{(\alpha)}(x-y,s)\phi(y,t+s)dy$$
$$\leqq \frac{1}{s}\int W^{(\alpha)}\left(\frac{x}{2},s\right)\phi(y,t+s)dy$$
$$\leqq C|x|^{-n-2\alpha}$$

for some constant C. The Lebesgue theorem and Lemma 2.3 give

$$\int u\tilde{L}^{(\alpha)}\phi\, dX = \lim_{s\downarrow 0}\int \frac{u-P_s^{(\alpha)}*u}{s}\phi\, dX. \tag{2.2}$$

Hence $\left(\frac{u-P_s^{(\alpha)}*u}{s}dX\right)_{s>0}$ converges vaguely as $s\downarrow 0$ and we get

$$\int u\tilde{L}^{(\alpha)}\phi\, dX = \int \phi\, d\mu$$

for every $\phi\in C_K^\infty(R^{n+1})$.

The above positive Borel measure μ is called the associated measure of u.

Remark 2.6. Let $\mu\in M_\alpha$. Then the associated measure of $W^{(\alpha)}\mu$ is equal to μ, because $\frac{W^{(\alpha)}\mu-P_s^{(\alpha)}*W^{(\alpha)}\mu}{s}=\frac{1}{s}\int_0^s P_t^{(\alpha)}*\mu dt$[2] (see (2.2)).

LEMMA 2.7. *Let $u\in S_\alpha$, $(u_m)_{m=1}^\infty$ a sequence in S_α, μ the associated measure of u and μ_m the associated measure of $u_m (m\geqq 1)$. If $\lim_{m\to\infty}u_m=u$ a.e. and if there exixts $v\in S_\alpha$ such that for any $m\geqq 1$, $u_m\leqq v$, then $(\mu)_{m=1}^\infty$ converges vaguely to μ as $m\to\infty$.*

Proof. For any $\phi\in C_K^\infty(R^{n+1})$ with $\phi\geqq 0$, Lemmas 2.3, 2.5 and $\int v|\tilde{L}^{(\alpha)}\phi|dX<\infty$ give

$$\int\phi\, d\mu=\int u\tilde{L}^{(\alpha)}\phi\, dX=\lim_{m\to\infty}u_m\tilde{L}^{(\alpha)}\phi\, dX=\lim_{m\to\infty}\int\phi\, d\mu_m,$$

which shows Lemma 2.7.

LEMMA 2.8. *Let u be a non-negative continuous function on R^{n+1}. If $u=P_s^{(\alpha)}*u$ for every $s>0$, u is constant.*

For the proof, we use the following

LEMMA 2.9 (Choquet-Deny [2]). *Let σ be a positive Borel measure on*

2) $\int_0^s P_t^{(\alpha)}*\mu dt$ is a positive measure defined by $\int_0^s\int\phi d(P_t^{(\alpha)}*\mu)dt$ for every $\phi\in C_K(R^{n+1})$.

R^k $(k \geqq 1)$ *with* $\int d\sigma = 1$ *and* h *a non-negative Borel function on* R^k. *Assume that* R^k *is generated by* $\operatorname{supp}[\sigma]$ *as a group and that* $h * \sigma = h$ *on* R^k. *Then* h *has the following representation*:

$$h(x) = \int \exp(a \cdot x) d\nu(a) \text{ a.e.}$$

with some positive Borel measure ν *on* R^k.

Proof of Lemma 2.8. Let ϕ be a non-negative continuous function on $(0, \infty)$ with compact support and with $\int \phi(t) dt = 1$. Then $u = P_s^{(\alpha)} * u$ give $u = W_{(\phi)}^{(\alpha)} * u$. Applying Lemma 2.9 with $\sigma = W_{(\phi)}^{(\alpha)}$, we see that there exists a positive measure ν on R^{n+1} such that

$$u(x, t) = \int_{R^{n+1}} \exp(a \cdot x + bt) d\nu(a, b) \text{ a.e.}$$

By Lemma 2.3, we have

$$\int_{R^{n+1}} \int_{|x| \geqq 1} \exp(a \cdot x + bt) |x|^{-n-2\alpha} dx\, d\nu(a, b) < \infty ,$$

so that $\operatorname{supp}[\nu] \subset \{0\} \times R$. By using $u = P_s^{(\alpha)} * u$ for every $s > 0$ again, we conclude that u is constant.

PROPOSITION 2.10. *Let* $u \in S_\alpha$ *and the associated measure of* u. *Then*

$$u = W^{(\alpha)}\mu + c \qquad \text{on } R^{n+1}$$

with some constant $c \geqq 0$. *Furthermore if for any positive Borel measure* ν *on* R^{n+1}, $u - W^{(\alpha)}\nu = a$ a.e. *with some constant* a, *then* $\nu = \mu$ *and* $a = c$.

Proof. For a positive integer m, we put $\mu_m = \mu|_{B(0, m)}$, where $B(\mathrm{C}, m)$ denotes the open ball in R^{n+1} with center 0 and with radius m. For $\phi \in C_K^\infty(R^{n+1})$ with $\phi \geqq 0$ and for any $s > 0$, Lemma 2.5 gives

$$\left(\int_0^s P_\tau^{(\alpha)} d\tau\right) * (u - W^{(\alpha)}\mu_m) * (\tilde{L}^{(\alpha)}\phi)^\sim(X) \geqq 0 ,$$

so that

$$\int u \phi\, dX - \int u \cdot (\tilde{P}_s^{(\alpha)} * \phi) dX \geqq \int \left(\int_0^s \tilde{P}_\tau^{(\alpha)} d\tau\right) * \phi\, d\mu_m .$$

Hence

$$\int u \phi\, dX \geqq \int W^{(\alpha)}\mu_m \phi\, dX .$$

Thus $u \geqq W^{(\alpha)}\mu_m$ a.e. By Lemma 2.4, $u \geqq W^{(\alpha)}\mu_m$. Letting $m \to \infty$, we obtain $u \geqq W^{(\alpha)}\mu$. Put

$$h = u - W^{(\alpha)}\mu \qquad \text{on } \{X \in R^{n+1};\ W^{(\alpha)}\mu(X) < \infty\}.$$

Then Remark 2.6 gives

$$\int (h - \tilde{P}_s^{(\alpha)} * h)\phi dX = \left(\int_0^s P_\tau^{(\alpha)} d\tau\right) * h * (\tilde{L}^{(\alpha)}\phi)^{\check{}}(0) = 0$$

for every $s > 0$ and $\phi \in C_K(R^{n+1})$. Hence $h = P_s^{(\alpha)} * h$ a.e. For any $\psi \in C_K((0, \infty))$ with $\psi \geqq 0$ and with $\int \psi dt = 1$, $h = W_{(\psi)}^{(\alpha)} * h$ a.e. and $(W_{(\psi)}^{(\alpha)} * h) = P_s^{(\alpha)} * (W_{(\psi)}^{(\alpha)} * h)$ on R^{n+1}, so that Lemma 2.8 gives $W_{(\psi)}^{(\alpha)} * h = c$ with some constant $c \geqq 0$, that is, $h = c$ a.e., which gives $u = W^{(\alpha)}\mu + c$ a.e. Lemma 2.4 leads to $u = W^{(\alpha)}\mu + c$, which shows the first equality. By Remark 2.6, we obtain the second part of this proposition. Thus Proposition 2.10 is shown.

COROLLARY 2.11. *Let $u \in S_\alpha$ and $\mu \in M_\alpha$. If $u \leqq W^{(\alpha)}\mu$, then u is the $W^{(\alpha)}$-potential of the associated measure of u.*

§ 3. Reduced functions and swept-out measures

For $u \in S_{\alpha, c}$ and a compact set K in R^{n+1}, we put

$$Q_K^{(\alpha)}u(X) = \inf\{v(X);\ v \in S_\alpha,\ v \geqq u \text{ on } K\}$$

and

$$R_K^{(\alpha)}u(X) = \underset{\smile}{Q_K^{(\alpha)}u}(X),$$

where $\underset{\smile}{Q_K^{(\alpha)}u}$ is the lower regularization of $Q_K^{(\alpha)}u$, namely for a function v on R^{n+1}, $\underset{\smile}{v}(X) = \liminf_{Y \to X} v(Y)$. Furthermore, for $u \in S_\alpha$ and a set A in R^{n+1}, we put

$$R_A^{(\alpha)}u(X) = \sup\{R_K^{(\alpha)}v(X);\ v \in S_{\alpha, c},\ v \leqq u \text{ and } A \supset K\text{: compact set}\},$$

$$\bar{Q}_A^{(\alpha)}u(X) = \inf\{R_\omega^{(\alpha)}u(X);\ A \subset \omega\text{: open set}\}$$

and

$$\bar{R}_A^{(\alpha)}u(X) = \underset{\smile}{\bar{Q}_A^{(\alpha)}u}(X).$$

We say that $R_A^{(\alpha)}u$ and $\bar{R}_A^{(\alpha)}u$ are the reduced function of u to A and the outer reduced function of u to A with respect to $L^{(\alpha)}$, respectively.

For a set A in R^{n+1} and for $u \in \tilde{S}_\alpha$, the reduced function $\tilde{R}_A^{(\alpha)}u$ of u to A and the outer reduced function $\tilde{\bar{R}}_A^{(\alpha)}u$ of u to A with respect to $\tilde{L}^{(\alpha)}$

can be defined analogously. For all the results for $L^{(\alpha)}$ in this paragraph, the analogies for $\tilde{L}^{(\alpha)}$ hold.

Remark 3.1. Let ω be an open set in R^{n+1} and $u \in S_\alpha$. Then we have:
(1) ω is redusable, that is, $\bar{R}_\omega^{(\alpha)} u = R_\omega^{(\alpha)} u$.
(2) $R_\omega^{(\alpha)} u = u$ on ω.

LEMMA 3.2 (G. Choquet, [6] p. 34). *Let $(f_i)_{i \in I}$ be an arbitrary family of functions on R^{n+1}. Then there exists a countable subset I_0 of I such that for any lower semi-continuous function g, $g \leqq f_{I_0}$ implies $g \leqq f_I$. Here for a subset J of I, we write $f_J(X) = \inf_{i \in J} f_i(X)$.*

LEMMA 3.3. *Let u be a positive and locally integrable Borel function on R^{n+1} and assume $u \geqq P_s^{(\alpha)} * u$ for $s > 0$. Then $\underset{\sim}{u} \in S_\alpha$, $\underset{\sim}{u} = u$ a.e. and for any approximate sequence $(\phi_m)_{m=1}^\infty$ of the Dirac measure, $(W_{(\phi_m)}^{(\alpha)} * u(X))_{m=1}^\infty$ converges increasingly to $\underset{\sim}{u}(X)$ with $m \to \infty$.*

Proof. Take an approximate sequence $(\phi_m)_{m=1}^\infty$ of the Dirac measure. The semi-group property of $(P_s^{(\alpha)})_{s \geqq 0}$ shows that $P_{s_1}^{(\alpha)} * u \geqq P_{s_2}^{(\alpha)} * u$ on R^{n+1} if $0 < s_1 < s_2$, so that $(W_{(\phi_m)}^{(\alpha)} * u(X))_{m=1}^\infty$ is increasing. For $X \in R^{n+1}$, we choose a sequence $(X_k)_{k=1}^\infty \subset R^{n+1}$ convergent to X satisfying $\underset{\sim}{u}(X) = \lim_{k \to \infty} u(X_k)$. Then for any $m \geqq 1$,

$$\underset{\sim}{u}(X) \geqq \liminf_{k \to \infty} (W_{(\phi_m)}^{(\alpha)} * u(X_k)) \geqq W_{(\phi_m)}^{(\alpha)} * u(X) \geqq W_{(\phi_m)}^{(\alpha)} * \underset{\sim}{u}(X) .$$

For any $\phi \in C_K(R^{n+1})$ with $\phi \geqq 0$, the Fatou lemma gives

$$\int u \phi \, dX \leqq \liminf_{m \to \infty} \int u \cdot (\tilde{W}_{(\phi_m)}^{(\alpha)} * \phi) dX = \liminf_{m \to \infty} \int (W_{(\phi_m)}^{(\alpha)} * u) \phi \, dX \leqq \int \underset{\sim}{u} \phi \, dX ,$$

so that $u \leqq \underset{\sim}{u}$ a.e., that is, $u = \underset{\sim}{u}$ a.e. Since w*-$\lim_{m \to \infty} (W_{(\phi_m)}^{(\alpha)} dX) = \varepsilon$[3] and $\underset{\sim}{u}$ is lower semi-continuous, we have

$$\liminf_{m \to \infty} (W_{(\phi_m)}^{(\alpha)} * \underset{\sim}{u}(X)) \geqq \underset{\sim}{u}(X) \text{ on } R^{n+1} .$$

Thus we have

$$\underset{\sim}{u}(X) = \lim_{m \to \infty} (W_{(\phi_m)}^{(\alpha)} * \underset{\sim}{u}(X)) = \lim_{m \to \infty} (W_{(\phi_m)}^{(\alpha)} * u(X)) \text{ on } R^{n+1} .$$

This gives $\underset{\sim}{u} \in S_\alpha$, which shows Lemma 3.3.

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3) For a sequence $(\mu_m)_{m=1}^\infty$ of Borel measures and a Borel measure μ, we write $\mu = \text{w*-}\lim_{m \to \infty} \mu_m$ if $(\mu_m)_{m=1}^\infty$ converges vaguely to μ as $m \to \infty$.

Lemmas 3.2 and 3.3 give the following

Remark 3.4. For $u \in S_\alpha$ and any set A in R^{n+1}, we have:

(1) $R_A^{(\alpha)}u = \lim_{m\to\infty} R_A^{(\alpha)}(W_{(\phi_m)}^{(\alpha)} * u)$, $R_A^{(\alpha)}u \in S_\alpha$, $\bar{R}_A^{(\alpha)}u \in S_\alpha$,

(2) $R_A^{(\alpha)}u$ is a $W^{(\alpha)}$-potential if A is relatively compact (see Corollary 2.11) and $R_A^{(\alpha)}R_A^{(\alpha)}u = R_A^{(\alpha)}u$ if A is open (see Remark 3.1).

In general, a closed set F is not always reducible, that is, $\bar{R}_F^{(\alpha)}u \neq R_F^{(\alpha)}u$ for some $u \in S_\alpha$. But we have the following

LEMMA 3.5. *Let F be a closed set in R^{n+1} and $u \in S_\alpha$. If u is continuous on a neighborhood of F and if $\lim_{X\in F,\, X\to\infty} u(X) = 0$, then $\bar{R}_F^{(\alpha)}u = R_F^{(\alpha)}u$.*

Proof. For any $\delta > 0$, we choose a compact set $K \subset F$ such that $u \leqq \delta$ on $F\backslash K$. Then we have

$$R_K^{(\alpha)}u \leqq R_F^{(\alpha)}u \leqq \bar{R}_F^{(\alpha)}u \leqq \bar{R}_K^{(\alpha)}u + \delta \text{ on } R^{n+1},$$

so that it suffices to show that $\bar{R}_K^{(\alpha)}u = R_K^{(\alpha)}u$ for every compact set $K \subset F$. Let $v \in S_\alpha$ with $v \geqq u$ on K. Then for any $\delta > 0$, continuity of u on some neighborhood of K shows that $v + \delta \geqq \bar{R}_K^{(\alpha)}u$ on R^{n+1}. Letting $\delta \to 0$ and taking the lower regularizations, we obtain $R_K^{(\alpha)}u \geqq \bar{R}_K^{(\alpha)}u$ on R^{n+1}, that is, $R_K^{(\alpha)}u \geqq \bar{R}_K^{(\alpha)}u$, which shows Lemma 3.5.

For $\mu \in M_\alpha$ (resp. $\mu \in \tilde{M}_\alpha$) and for a set A in R^{n+1}, Corollary 2.11 shows that $R_A^{(\alpha)}W^{(\alpha)}\mu$ (resp. $\tilde{R}_A^{(\alpha)}\tilde{W}^{(\alpha)}\mu$) is a $W^{(\alpha)}$-potential (resp. $\tilde{W}^{(\alpha)}$-potential). We denote by μ'_A (resp. μ''_A) the associated measure of $R_A^{(\alpha)}W^{(\alpha)}\mu$ (resp. $\tilde{R}_A^{(\alpha)}\tilde{W}^{(\alpha)}\mu$). We say that μ'_A (resp. μ''_A) is the inner $W^{(\alpha)}$-swept-out (resp. $\tilde{W}^{(\alpha)}$-swept-out) measure of μ to A.

PROPOSITION 3.6. *Let A be a set in R^{n+1} and $\mu \in M_\alpha$. Then*

$$\int d\mu'_A \leqq \int d\mu\,.$$

Proof. Let $(\omega_m)_{m=1}^\infty$ be an exhaustion of R^{n+1}. By Remark 3.1, there exists a positive measure ν_m with $\tilde{R}_{\omega_m}^{(\alpha)}1 = \tilde{W}^{(\alpha)}\nu_m$. Then we have

$$\begin{aligned}\int d\mu'_A &= \lim_{m\to\infty}\int \tilde{W}^{(\alpha)}\nu_m d\mu'_A = \lim_{m\to\infty}\int W^{(\alpha)}\mu'_A d\nu_m \\ &\leqq \lim_{m\to\infty}\int W^{(\alpha)}\mu\, d\nu_m = \lim_{m\to\infty}\int \tilde{W}^{(\alpha)}\nu_m d\mu = \int d\mu\,.\end{aligned}$$

PROPOSITION 3.7. *Let $u \in S_\alpha$ and let A be a set in R^{n+1}. Then the support of associated measure of $R_A^{(\alpha)}u$ is in $\bar{A}$.*

Proof. By the definition of $R_A^{(\alpha)}u$, Lemma 2.7 and by Remarks 3.5, 3.4, we may assume that A is compact and that u is a continuous $W^{(\alpha)}$-potential. Put $u = W^{(\alpha)}\mu$ with $\mu \in M_\alpha$ and let $(\omega_m)_{m=1}^\infty$ be a sequence of relatively compact open sets with $\overline{\omega_{m+1}} \subset \omega_m$ and with $\bigcap_{m=1}^\infty \omega_m = A$. Since $W^{(\alpha)}\mu'_{\omega_m} \leqq W^{(\alpha)}\mu$ for all m and since Lemmas 3.3 and 3.5 give $\lim_{m\to\infty}(W^{(\alpha)}\mu'_{\omega_m}) = W^{(\alpha)}\mu'_A$ a.e., we obtain $\mu'_A = \text{w*-}\lim_{m\to\infty}\mu'_{\omega_m}$ (see Lemma 2.7). Hence it suffices to show $\operatorname{supp}[\mu'_\omega] \subset \bar{\omega}$ for every open set ω in R^{n+1}. Suppose that there exists a point $X_0 \in C\bar{\omega} \cap \operatorname{supp}[\mu'_\omega]$. Let $(V_m)_{m=1}^\infty$ be a sequence of open sets in R^{n+1} with $\overline{V}_1 \subset C\bar{\omega}$, $\overline{V_{m+1}} \subset V_m$ and with $\bigcap_{m=1}^\infty V_m = \{X_0\}$. We put $\mu_m = \mu'_\omega|_{V_m}$. Then

$$W^{(\alpha)}\mu'_\omega \geqq W^{(\alpha)}(\mu'_\omega - \mu_m) + W^{(\alpha)}(\mu_m)'_\omega \qquad \text{on } R^{n+1}$$

and

$$W^{(\alpha)}(\mu'_\omega - \mu_m) + W^{(\alpha)}(\mu_m)'_\omega = W^{(\alpha)}\mu \qquad \text{on } \omega\,.$$

Hence

$$W^{(\alpha)}\mu_m = W^{(\alpha)}(\mu_m)'_\omega \qquad \text{on } R^{n+1}\,,$$

so that

$$\lim_{m\to\infty} W^{(\alpha)}\Big((\mu_m)'_\omega \Big/ \int d\mu_m\Big) = \lim_{m\to\infty} W^{(\alpha)}\Big(\mu_m \Big/ \int d\mu_m\Big) = W^{(\alpha)}\varepsilon_{X_0} \qquad \text{on } C\{X_0\}\,,$$

which contradicts the unboundedness of $W^{(\alpha)}\varepsilon_{X_0}$ on a neighborhood of X_0. Thus Proposition 3.7 is shown.

PROPOSITION 3.8. *Let $\mu \in M_\alpha$ and $\nu \in \tilde{M}_\alpha$. For a set A in R^{n+1}, we have*

$$\int W^{(\alpha)}\mu'_A\, d\nu = \int W^{(\alpha)}\mu\, d\nu''_A \quad \textit{and} \quad W^{(\alpha)}\mu'_A(X) = \int W^{(\alpha)}\varepsilon'_{Y,A}(X)\, d\mu(Y)\,,$$

where we denote by ε_Y and by $\varepsilon'_{Y,A}$ the Dirac measure at Y and its inner $W^{(\alpha)}$-swept-out measure to A. In particular if A is open,

$$\int W^{(\alpha)}\mu'_A\, d\nu = \int W^{(\alpha)}\mu'_A\, d\nu''_A\,.$$

Proof. First we assume that A is open. Let $(\omega_m)_{m=1}^\infty$ be an exhaustion of A. Then Proposition 3.7 and Remark 3.1 show that

$$\begin{aligned}\int W^{(\alpha)}\mu'_A\, d\nu &= \lim_{m\to\infty}\int W^{(\alpha)}\mu'_{\omega_m}\, d\nu = \lim_{m\to\infty}\int \tilde{W}^{(\alpha)}\nu\, d\mu'_{\omega_m} = \lim_{m\to\infty}\int \tilde{W}^{(\alpha)}\nu''_A\, d\mu'_{\omega_m} \\ &= \lim_{m\to\infty}\int W^{(\alpha)}\mu'_{\omega_m}\, d\nu''_A = \int W^{(\alpha)}\mu'_A\, d\nu''_A\,.\end{aligned}$$

Let A be an ar bitrary set. By the definition of inner $W^{(\alpha)}$-swept-out measures and Lemma 3.3, we may assume that A is compact, $\mu \in M_{\alpha, c}$ and that $\nu \in \tilde{M}_{\alpha, c}$. Take a sequence $(\Omega_m)_{m=1}^{\infty}$ of relatively compact open sets with $\overline{\Omega_{m+1}} \subset \Omega_m$ and with $\bigcap_{m=1}^{\infty} \Omega_m = A$. By Lemma 3.5 and the above result, we have

$$\int W^{(\alpha)} \mu'_A \, d\nu = \lim_{m\to\infty} \int W^{(\alpha)} \mu'_{\Omega_m} \, d\nu = \lim_{m\to\infty} \int \tilde{W}^{(\alpha)} \nu''_{\Omega_m} \, d\mu = \int \tilde{W}^{(\alpha)} \nu''_A \, d\mu .$$

In particular, we have $W^{(\alpha)} \varepsilon'_{Y,A}(X) = \tilde{W}^{(\alpha)} \varepsilon''_{X,A}(Y)$. Hence

$$W^{(\alpha)} \mu'_A(X) = \int W^{(\alpha)} \mu'_A \, d\varepsilon_X = \int \tilde{W}^{(\alpha)} \varepsilon''_{X,A}(Y) d\mu(Y) = \int W^{(\alpha)} \varepsilon'_{Y,A}(X) d\mu(Y) .$$

This completes the proof.

By Remark 3.4, (2) and Proposition 3.8, we have the following

COROLLARY 3.9. *Let ω be an open set in R^{n+1}. Then the mapping $M_\alpha \ni \mu \to \mu'_\omega$ is positively linear, and for any $\mu \in M_\alpha$ and any positive measure ν with $\nu \leqq \mu'_\omega$, we have $\nu'_\omega = \nu$.*

Proof. It follows immediately from Proposition 3.8 that the mapping $\mu \to \mu'_\omega$ is positively linear. By Remark 3.4, (2) we have $(\mu'_\omega)'_\omega = \mu'_\omega$, so that by Proposition 3.8, for any $X \in R^{n+1}$,

$$\int (W^{(\alpha)} \varepsilon_Y(X) - W^{(\alpha)} \varepsilon'_{Y,\omega}(X)) d\mu'_\omega(Y) = 0 .$$

Since $W^{(\alpha)} \varepsilon_Y \geqq W^{(\alpha)} \varepsilon'_{Y,\omega}$, we have $W^{(\alpha)} \varepsilon_Y = W^{(\alpha)} \varepsilon'_{Y,\omega}$ μ'_ω-a.e. as functions of Y, so that

$$\int (W^{(\alpha)} \varepsilon_Y(X) - W^{(\alpha)} \varepsilon'_{Y,\omega}(X)) d\nu(Y) = 0 ,$$

that is,

$$W^{(\alpha)} \nu = W^{(\alpha)} \nu'_\omega ,$$

which gives $\nu = \nu'_\omega$.

PROPOSITION 3.10. *Let $\mu \in M_\alpha$. Then we have:*

(1) *For two sets A_1 and A_2 in R^{n+1} with $A_1 \subset A_2$, we have $\mu'_{A_1} \geqq \mu'_{A_2}$ on* Int(A_1), *where* Int(A_1) *denotes the interior of A_1.*

(2) *For a set A in R^{n+1} with $\int_{\overline{CA}} d\mu = 0$, we have $\mu'_A = \mu$.*

Proof. (1): Choose $\phi \in C_K^\infty(R^{n+1})$ with $\phi \geqq 0$ and supp$[\phi] \subset$ Int(A_1). Let λ be the real Borel measure such that $\phi = \tilde{W}^{(\alpha)} \lambda$. Then we have

$$\int \tilde{W}^{(\alpha)}\lambda\, d\mu'_{A_2} = \int W^{(\alpha)}\mu'_{A_2}\, d\lambda^+ - \int W^{(\alpha)}\mu'_{A_2}\, d\lambda^-$$
$$\leqq \int W^{(\alpha)}\mu'_{A_1}\, d\lambda^+ - \int W^{(\alpha)}\mu'_{A_1}\, d\lambda^-$$
$$= \int \tilde{W}^{(\alpha)}\lambda\, d\mu'_{A_1},$$

because $\mathrm{supp}[\lambda^+] \subset \mathrm{Int}(A_1)$ and $W^{(\alpha)}\mu'_{A_1} = W^{(\alpha)}\mu'_{A_2}$ on $\mathrm{Int}(A_1)$.

By using Proposition 3.8 and Remark 3.1, (2), we show (2) in the same manner as in (1). This completes the proof.

PROPOSITION 3.11 (the domination principle). *Let Ω be an open set in R^{n+1}, $u \in S_\alpha$ and $\mu \in M_\alpha$ with* $\mathrm{supp}[\mu] \subset \Omega$. *Put*

$$E = \{X \in \Omega;\, u(X) - R^{(\alpha)}_{C\Omega}u(X) \geqq W^{(\alpha)}\mu(X) - W^{(\alpha)}\mu'_{C\Omega}(X)\}.$$

If $\mu'_E = \mu$, then $u - R^{(\alpha)}_{C\Omega}u \geqq W^{(\alpha)}\mu - W^{(\alpha)}\mu'_{C\Omega}$ on R^{n+1}.

Proof. Since μ is a sum of positive measures with compact support, we may assume that $\mathrm{supp}[\mu]$ is compact. Let ω be an open set with $\omega \supset C\Omega$. Then

$$u + R^{(\alpha)}_{\omega}W^{(\alpha)}\mu \geqq R^{(\alpha)}_{C\Omega}u + W^{(\alpha)}\mu \text{ on } E \cup \omega.$$

Let ν be the associated measure of $R^{(\alpha)}_{C\Omega}u$ and put $R^{(\alpha)}_{C\Omega}u = W^{(\alpha)}\nu + c$ with $c \geqq 0$, Then $\mathrm{supp}[\nu] \subset C\Omega$, so that

$$R^{(\alpha)}_{C\Omega}u + W^{(\alpha)}\mu = c + R^{(\alpha)}_{E\cup\omega}(W^{(\alpha)}\nu + W^{(\alpha)}\mu) \leqq u + R^{(\alpha)}_{\omega}W^{(\alpha)}\mu \text{ on } R^{n+1},$$

because $u - c + R^{(\alpha)}_{\omega}W^{(\alpha)}\mu \geqq 0$. Since $W^{(\alpha)}\mu$ is continuous in a certain neighborhood of $C\Omega$ and vanishes at the infinity, Lemma 3.5 shows

$$u - R^{(\alpha)}_{C\Omega}u \geqq W^{(\alpha)}\mu - W^{(\alpha)}\mu'_{C\Omega},$$

which shows Proposition 3.11.

PROPOSITION 3.12. *Let ω_1 and ω_1 be open sets in R^{n+1} with $\overline{\omega_1} \cap \overline{\omega_2} = \phi$ and $\mu \in M_\alpha$. Then $\mu'_{\omega_1} = \mu'_{\omega_1\cup\omega_2}|_{\overline{\omega_1}} + (\mu'_{\omega_1\cup\omega_2}|_{\overline{\omega_2}})'_{\omega_1}$.*

Proof. Let $(\omega_{1,m})_{m=1}^{\infty}$ be an exhaustion of ω_1 and put $\mu_1 = \mu'_{\omega_1\cup\omega_2}|_{\overline{\omega_1}}$ and $\mu'_m = (\mu_1)'_{\omega_{1,m}\cup\omega_2}$. Since $\mathrm{supp}[\mu'_m|_{\omega_1}] \subset \overline{\omega_{1,m}} \subset \omega_1$, by Proposition 3.10, (2), $\mu'_m|_{\omega_1} = (\mu'_m|_{\omega_1})'_{\omega_1}$, so that

$$W^{(\alpha)}(\mu'_m|_{\omega_1}) = R^{(\alpha)}_{\omega_1}W^{(\alpha)}(\mu'_m|_{\omega_1}) \leqq R^{(\alpha)}_{\omega_1}W^{(\alpha)}\mu_1 = W^{(\alpha)}(\mu_1)'_{\omega_1}.$$

On the other hand, by Corollary 3.9, we have $(\mu_1)'_{\omega_1\cup\omega_2} = \mu_1$, so that

$(\mu_1)'_{\omega_1\cup\omega_2}|_{\overline{\omega_1}} = \mu_1$. Since w*-$\lim_{m\to\infty}\mu'_m = (\mu_1)'_{\omega_1\cup\omega_2}$ by Lemma 2.7, it follows that

$$W^{(\alpha)}\mu_1 = W^{(\alpha)}((\mu_1)'_{\omega_1\cup\omega_2}|_{\overline{\omega_1}}) \leqq \liminf_{m\to\infty} W^{(\alpha)}(\mu'_m|_{\omega_1}) \leqq W^{(\alpha)}(\mu_1)'_{\omega_1} .$$

Thus, $\mu_1 = (\mu_1)'_{\omega_1}$, and hence $\mu'_{\omega_1} = (\mu'_{\omega_1\cup\omega_2})'_{\omega_1} = \mu_1 + (\mu'_{\omega_1\cup\omega_2}|_{\overline{\omega_2}})'_{\omega_1}$, which shows Proposition 3.12.

COROLLARY 3.13. *Let Ω and ω be open sets in R^{n+1} and $\mu \in M_\alpha$. Then* $(\mu'_\Omega|_\omega)'_{\Omega\cap\omega} = \mu'_\Omega|_\omega$.

Proof. By Proposition 3.8, we may assume that supp$[\mu]$ is compact. Let $(\omega_m)_{m=1}^\infty$ be an exhaustion of ω. By Proposition 3.6, we may assume that $((\mu'_\Omega|_\omega)'_{\Omega\cap(\omega_m\cup C\overline{\omega_{m+1}})})_{m=1}^\infty$ converges vaguely to some measure ν. By the definition of inner $W^{(\alpha)}$-swept-out measures, we have $\nu = (\mu'_\Omega|_\omega)'_\Omega = \mu'_\Omega|_\omega$ (see Corollary 3.9). Proposition 3.12 gives

$$(\mu'_\Omega|_\omega)'_{\Omega\cap(\omega_m\cup C\overline{\omega_{m+1}})} \leqq (\mu'_\Omega|_\omega)'_{\Omega\cap\omega_m} \qquad \text{on } \omega_m .$$

Letting $m\to\infty$, we have

$$\mu'_\Omega|_\omega \leqq (\mu'_\Omega|_\omega)'_{\Omega\cap\omega} \qquad \text{on } \omega .$$

Hence Proposition 3.6 shows $\mu'_\Omega|_\omega = (\mu'_\Omega|_\omega)'_{\Omega\cap\omega}$.

PROPOSITION 3.14. *Let Ω be an open set in R^{n+1}, T the projection of $C\overline{\Omega}$ to the t-axis and $X_0 = (x_0, t_0) \in R^{n+1}$. Let M be the connected component of $T\cup\{t_0\}$ satisfying $t_0 \in M$ and put $t_1 = \sup M$. If $\varepsilon'_{X_0,\Omega} \neq \varepsilon_{X_0}$, then*

$$\operatorname{supp}[\varepsilon'_{X_0,\Omega}] \supset \overline{\Omega} \cap (R^n \times (t_0, t_1)) .$$

For the proof, we use the following

LEMMA 3.15. *Let $\mu, \nu \in M_\alpha$ and $X_0 = (x_0, t_0) \in R^{n+1}\setminus\operatorname{supp}[\nu]$. Suppose that $W^{(\alpha)}\mu \geqq W^{(\alpha)}\nu$ on R^{n+1} and that* $\operatorname{supp}[\mu] \subset \{(x, t) \in R^{n+1}; t < t_0\}$. *If $W^{(\alpha)}\mu(X_0) = W^{(\alpha)}\nu(X_0)$, then* $\operatorname{supp}[\nu] \subset \{(x, t) \in R^{n+1}; t \leqq t_0\}$ *and $W^{(\alpha)}\mu = W^{(\alpha)}\nu$ on* $\{(x, t) \in R^{n+1}; t > t_0\}$.

Proof. Since $W^{(\alpha)}(\mu - \nu) \geqq 0$, $W^{(\alpha)}(\mu - \nu)(X_0) = 0$ and since $W^{(\alpha)}(\mu - \nu)$ is of class C^∞ in a neighborhood of X_0,

$$W^{(\alpha)}(\mu - \nu)(X_0) = \frac{\partial}{\partial t} W^{(\alpha)}(\mu - \nu)(X_0) = 0$$

and

$$0 = L^{(\alpha)} W^{(\alpha)}(\mu - \nu)(X_0) = -C_{n,\alpha}\int_{R^n} W^{(\alpha)}(\mu - \nu)(x_0 - y, t_0)|y|^{-n-2\alpha}\, dy$$

Then we have $W^{(\alpha)}\mu(x, t_0) = W^{(\alpha)}\nu(x, t_0)$ dx-a.e., so that for any $s > 0$ and for any $x \in R^n$, we have

$$\begin{aligned} W^{(\alpha)}\mu(x, t_0 + s) &= \int W^{(\alpha)}(x - y, s) W^{(\alpha)}\mu(y, t_0) dy \\ &= \int W^{(\alpha)}(x - y, s) W^{(\alpha)}\nu(y, t_0) dy \\ &\leqq W^{(\alpha)}\nu(x, t_0 + s) . \end{aligned}$$

Therefore $W^{(\alpha)}\mu = W^{(\alpha)}\nu$ on $\{(x, t) \in R^{n+1}; t > t_0\}$ and $\nu = 0$ on $\{(x, t) \in R^{n+1}; t > t_0\}$, which shows Lemma 3.15.

Proof of Proposition 3.14. Put

$$s = \sup\{t \geqq t_0; \operatorname{supp}[\varepsilon'_{X_0, \Omega}] \cap (R^n \times \{t\}) \neq \phi\} .$$

Then Lemma 3.15 yields $s > t_0$ and

$$\operatorname{supp}[\varepsilon'_{X_0, \Omega}] = \overline{\Omega \cap (R^n \times (t_0, s))} .$$

Suppose that $s < t_1$ and $\Omega \cap (R^n \times (s, \infty)) \neq \phi$; we can take a non-empty open set ω in R^n and a positive number $\delta > 0$ such that $t_0 < s - \delta$ and

$$D_\delta = \omega \times (s - \delta, s) \subset C\overline{\Omega} .$$

Put $\nu_\delta = \varepsilon'_{X_0, \Omega \cup D_\delta}|_{\overline{D_\delta}}$. If $\nu_\delta = 0$, then Proposition 3.12 gives $\varepsilon'_{X_0, \Omega} = \varepsilon'_{X_0, \Omega \cup D_\delta}$, so that Lemma 3.15 shows that $\varepsilon'_{X_0, \Omega}$ vanishes on $R^n \times (s - \delta, \infty)$, which is a contradiction. Hence $\nu_\delta \neq 0$ for every sufficiently small $\delta > 0$. By Lemma 3.15, there exists $s' > s$ such that

$$W^{(\alpha)}\nu_\delta = W^{(\alpha)}\nu'_{\delta, \Omega} \qquad \text{on } R^n \times [s', \infty) .$$

Since Proposition 3.12 shows $\operatorname{supp}[\nu_\delta + \nu'_{\delta, \Omega}] \subset R^n \times (-\infty, s]$, for any $s < t < s'$, we have

$$\begin{aligned} 0 &= W^{(\alpha)}\nu_\delta(0, s') - W^{(\alpha)}\nu'_{\delta, \Omega}(0, s') \\ &= \int_{R^n} (W^{(\alpha)}\nu_\delta(x, t) - W^{(\alpha)}\nu'_{\delta, \Omega}(x, t)) W^{(\alpha)}(-x, s' - t) dx . \end{aligned}$$

Since $W^{(\alpha)}\nu_\delta \geqq W^{(\alpha)}\nu'_{\delta, \Omega}$ on R^{n+1}, Lemma 2.4, (3) shows

$$W^{(\alpha)}\nu_\delta \geqq W^{(\alpha)}\nu'_{\delta, \Omega} \qquad \text{on } R^n \times (s, \infty) .$$

We may assume that $\left(\nu_\delta \Big/ \int d\nu_\delta\right)_{\delta>0}$ and $\left(\nu'_{\delta, \Omega} \Big/ \int d\nu_\delta\right)_{\delta>0}$ converges vaguely as $\delta \to 0$. Put

$$\nu = \text{w*-}\lim_{\delta\downarrow 0}\left(\nu_\delta \Big/ \int d\nu_\delta\right) \quad \text{and} \quad \nu' = \text{w*-}\lim_{\delta\downarrow 0}\left(\nu'_{\delta,\Omega} \Big/ \int d\nu_\delta\right);$$

then Proposition 3.12 gives $\text{supp}[\nu'] \subset \bar{\Omega} \cap (R^n \times [t_0, s])$, $W^{(\alpha)}\nu \geqq W^{(\alpha)}\nu'$ on R^{n+1} and $W^{(\alpha)}\nu = W^{(\alpha)}\nu'$ on $R^n \times (s, \infty)$. Since $\text{supp}[\nu] \subset R^n \times \{s\}$, $W^{(\alpha)}\nu = 0$ on $R^n \times (-\infty, s]$. Hence $W^{(\alpha)}\nu = W^{(\alpha)}\nu'$ on R^{n+1}, which implies $\nu = \nu'$. But this contradicts $\text{supp}[\nu] \subset C\bar{\Omega}$ and $\text{supp}[\nu'] \subset \bar{\Omega}$. Thus Proposition 3.14 is shown.

§ 4. $L^{(\alpha)}$-regular points and a Poincaré type condition

As in the classical potential theory, we define $L^{(\alpha)}$-regular points for Dirichlet problem.

DEFINITION 2. Let Ω be an open set in R^{n+1} and $X_0 \in \partial\Omega$. Then X_0 is said to be regular for the Dirichlet problem of $L^{(\alpha)}$ on Ω if

$$\text{w*-}\lim_{X\in\Omega, X\to X_0} \varepsilon''_{X, C\Omega} = \varepsilon_{X_0}.$$

PROPOSITION 4.1. *Let Ω and Ω' be open sets in R^{n+1} and $X_0 \in \partial\Omega \cap \partial\Omega'$. If there exists a neighborhood V of X_0 such that $\Omega \cap V = \Omega' \cap V$ and if X_0 is regular for the Dirichlet problem of $L^{(\alpha)}$ on Ω', then X_0 is so on Ω.*

Proof. Let U be an open neighborhood of X_0 with $\bar{U} \subset V$. Then $\text{w*-}\lim_{X\in\Omega, X\to X_0} \varepsilon''_{X, C\Omega'}|_U = \varepsilon_{X_0}$. For any $X \in \Omega$, Lemma 3.5 and the domination principle of $\tilde{W}^{(\alpha)}$ (Proposition 3.11) show

$$\tilde{W}^{(\alpha)}(\varepsilon''_{X, C\Omega'}|_U) \leqq \tilde{\tilde{R}}^{(\alpha)}_{C\Omega}\tilde{W}^{(\alpha)}\varepsilon_X = \tilde{R}^{(\alpha)}_{C\Omega}\tilde{W}^{(\alpha)}\varepsilon_X = \tilde{W}^{(\alpha)}(\varepsilon''_{X, C\Omega}) \leqq \tilde{W}^{(\alpha)}\varepsilon_X.$$

Let $(X_m)_{m=1}^\infty$ be an arbitrary sequence in Ω with $\lim_{m\to\infty} X_m = X_0$. Since $\int d\varepsilon''_{X_m, C\Omega} \leqq 1$, it suffices to show $\text{w*-}\lim_{m\to\infty} \varepsilon''_{X_m, C\Omega} = \varepsilon_{X_0}$ in the case that $(\varepsilon''_{X_m, C\Omega})_{m=1}^\infty$ converges vaguely. Put $\mu = \text{w*-}\lim_{m\to\infty} \varepsilon''_{X_m, C\Omega}$. Since for any non-negative $f \in C_K(R^{n+1})$, $W^{(\alpha)}(f dX)$ is finite continuous and vanishes at the infinity,

$$\begin{aligned}\int \tilde{W}^{(\alpha)}\varepsilon_{X_0} f\,dX &= \lim_{m\to\infty}\int \tilde{W}^{(\alpha)}(\varepsilon''_{X_m, C\Omega'}|_U) f\,dX \\ &\leqq \lim_{m\to\infty}\int \tilde{W}^{(\alpha)}(\varepsilon''_{X_m, C\Omega})\, f\,dX = \int \tilde{W}^{(\alpha)}\mu\cdot f\,dX \\ &\leqq \int \tilde{W}^{(\alpha)}\varepsilon_{X_0} f\,dX.\end{aligned}$$

Therefore $\tilde{W}^{(\alpha)}\varepsilon_{X_0} = \tilde{W}^{(\alpha)}\mu$ a.e., so that $\tilde{W}^{(\alpha)}\varepsilon_{X_0} = \tilde{W}^{(\alpha)}\mu$, which gives $\mu = \varepsilon_{X_0}$. This shows that X_0 is regular for the Dirichlet problem of $L^{(\alpha)}$ on Ω.

PROPOSITION 4.2. *Let Ω be an open set in R^{n+1} and $X_0 = (x_0, t_0) \in \partial\Omega$ such that for any neighborhood V of X_0,*

$$V \cap \Omega \cap \{(x, t); t < t_0\} \neq \phi .$$

Then the following four conditions are equivalent:

(1) *X_0 is regular for the Dirichlet problem of $L^{(\alpha)}$ on Ω.*

(2) *For any $u \in S_\alpha$, $R^{(\alpha)}_{C\Omega} u(X_0) = u(X_0)$.*

(3) *There exist $u \in S_\alpha$ and $\{(x_m, t_m)\}_{m=1}^{\infty} \subset R^{n+1}$ such that $t_m < t_0$, $\lim_{m\to\infty} t_m = t_0$, $R^{(\alpha)}_{C\Omega} u(x_m, t_m) \neq u(x_m, t_m)$ and that $R^{(\alpha)}_{C\Omega} u(X_0) = u(X_0)$.*

(4) $\varepsilon''_{X_0, C\Omega} = \varepsilon_{X_0}$.

Proof. Proposition 3.8 shows for any $\mu \in M_\alpha$, $R^{(\alpha)}_{C\Omega} W^{(\alpha)}\mu(X_0) = \int W^{(\alpha)}\mu \, d\varepsilon''_{X_0, C\Omega}$, so that (2)$\leftrightarrow$(4) holds.

(1)$\rightarrow$(3): Choose $f \in C_K(R^{n+1})$ such that $f \geqq 0$ and that $f > 0$ on a neighborhood of X_0. Then $W^{(\alpha)}(fdX)$ is a required function. In fact, since $W^{(\alpha)}(fdX)$ is finite continuous and vanishes at the infinity, we have

$$\lim_{Y \in \Omega, Y \to X_0} \int W^{(\alpha)}(fdX) d\varepsilon''_{Y, C\Omega} = W^{(\alpha)}(fdX)(X_0) ,$$

so that Proposition 3.8 gives

$$\lim_{Y \in \Omega, Y \to X_0} R^{(\alpha)}_{C\Omega} W^{(\alpha)}(fdX)(Y) = W^{(\alpha)}(fdX)(X_0) .$$

Since

$$Q^{(\alpha)}_{C\Omega} W^{(\alpha)}(fdX)(Y) = W^{(\alpha)}(fdX)(Y) \qquad \text{on } C\Omega ,$$

we have

$$R^{(\alpha)}_{C\Omega} W^{(\alpha)}(fdX)(X_0) = W^{(\alpha)}(fdX)(X_0) .$$

Assume $W^{(\alpha)}(fdX) = R^{(\alpha)}_{C\Omega} W^{(\alpha)}(fdX)$ on $R^n \times (t, t_0)$ with some $t < t_0$ and denote by f_1 the restriction of f to $R^n \times (t, t_0)$. Then Propositions 3.8, 3.11 show

$$W^{(\alpha)}(f_1 dX) = R^{(\alpha)}_{C\Omega} W^{(\alpha)}(f_1 dX) \qquad \text{on } R^{n+1} ,$$

which contradicts Proposition 2.10. Thus (3) holds.

(3)$\rightarrow$(4): By Proposition 3.7, $u - R^{(\alpha)}_{C\Omega} u$ is lower semi-continuous on Ω. Furthermore for any $\delta > 0$,

$$\{X \in \Omega ; u(X) > R^{(\alpha)}_{C\Omega} u(X)\} \cap (R^n \times (t_0 - \delta, t_0)) \neq \phi .$$

In fact, if $u(X) = R^{(\alpha)}_{C\Omega} u(X)$ on $\Omega \cap (R^n \times (t_0 - \delta, t_0))$, $L^{(\alpha)} u = 0$ on $\Omega \cap$

$(R^n \times (t_0 - \delta, t_0))$ (in the sense of distributions), because $L^{(\alpha)}(R^{(\alpha)}_{C\Omega} u) = 0$ on Ω, and hence for any $(x, t) \in \Omega \cap (R^n \times (t_0 - \delta, t_0))$,

$$\int_{R^n} (u - R^{(\alpha)}_{C\Omega} u)(x + y, t) |y|^{-n-2\alpha} dy = 0 ,$$

that is, $u = R^{(\alpha)}_{C\Omega} u$ on $R^n \times (t_0 - \delta, t_0)$ (see Lemma 2.4, (3)), which contradicts (3). Hence we can choose $\mu_\delta \in M_\alpha$ such that $\mu_\delta \neq 0$, $\mathrm{supp}[\mu_\delta] \subset \Omega \cap (R^n \times (t_0 - \delta, t_0))$ and that $u - R^{(\alpha)}_{C\Omega} u \geqq W^{(\alpha)} \mu_\delta$ on a certain neighborhood of $\mathrm{supp}[\mu_\delta]$. Then Proposition 3.11 gives

$$u - R^{(\alpha)}_{C\Omega} u \geqq W^{(\alpha)} \mu_\delta - W^{(\alpha)} \mu'_{\delta, C\Omega} \qquad \text{on } R^{n+1} ,$$

so that by Proposition 3.8, and the assumption that $u(X_0) = R^{(\alpha)}_{C\Omega} u(X_0)$,

$$\tilde{W}^{(\alpha)} \varepsilon_{X_0} = \tilde{W}^{(\alpha)} \varepsilon''_{X_0, C\Omega} \qquad \mu_\delta\text{-a.e.},$$

which implies $\tilde{W}^{(\alpha)} \varepsilon_{X_0} = \tilde{W}^{(\alpha)} \varepsilon''_{X_0, C\Omega}$ on $R^n \times (-\infty, t_0 - \delta)$ by Lemma 3.15 for $\tilde{L}^{(\alpha)}$. Therefore let $\delta \to 0$; then Proposition 2.10 yields

$$\varepsilon_{X_0} = \varepsilon''_{X_0, C\Omega} ,$$

which shows (4).

(2)$\to$(1): Let $(X_m)_{m=1}^\infty$ be an arbitrary sequence in Ω with $\lim_{m\to\infty} X_m = X_0$. To show $\text{w*-}\lim_{m\to\infty} \varepsilon''_{X_m, C\Omega} = \varepsilon_{X_0}$, we may assume that $(\varepsilon''_{X_m, C\Omega})_{m=1}^\infty$ converges vaguely. Put $\nu = \text{w*-}\lim_{m\to\infty} \varepsilon''_{X_m, C\Omega}$. For any $\mu \in M_{\alpha, c}$ whose support is compact, we have

$$\begin{aligned} \int \tilde{W}^{(\alpha)} \nu \, d\mu &= \lim_{m\to\infty} \int W^{(\alpha)} \mu \, d\varepsilon''_{X_m, C\Omega} = \lim_{m\to\infty} W^{(\alpha)} \mu'_{C\Omega}(X_m) \\ &\geqq W^{(\alpha)} \mu'_{C\Omega}(X_0) = W^{(\alpha)} \mu(X_0) = \int \tilde{W}^{(\alpha)} \varepsilon_{X_0} \, d\mu , \end{aligned}$$

so that $\tilde{W}^{(\alpha)} \nu \geqq \tilde{W}^{(\alpha)} \varepsilon_{X_0}$ a.e., that is, $\tilde{W}^{(\alpha)} \nu = \tilde{W}^{(\alpha)} \varepsilon_{X_0}$, which shows $\nu = \varepsilon_{X_0}$. Thus X_0 is regular. This completes the proof.

For any $(x, t) \in R^{n+1}$ and $k \in R$, we set

$$\tau_k^{(\alpha)}(x, t) = (2^k x, 2^{2\alpha k} t) .$$

Remark 4.3. Let $u \in S_\alpha$ and $k \in R$ and put $v(X) = u(\tau_k^{(\alpha)} X)$. Then $v \in S_\alpha$.

We shall prove the following main theorem.

THEOREM. *Let Ω be an open set in R^{n+1} and $X_0 \in \partial\Omega$. If there exists a non-empty open set ω in R^n such that α-tusk $T^{(\alpha)}_{X_0}(\omega)$ of ω at X_0 is in*

$C\Omega$, then X_0 is regular for the Dirichlet problem of $L^{(\alpha)}$ on Ω.

Proof. We may assume that X_0 is the origin 0 of R^{n+1}. By Proposition 4.1, we may assume that

$$T_0^{(\alpha)}(\omega) = \{(px, -p^{2\alpha});\ x \in \omega,\ 0 < p < \infty\}.$$

Put

$$\begin{aligned} V &= \{(x, t);\ -1 < t < 1, |x| < 1\}, \\ V_k &= \{\tau_k^{(\alpha)}(X);\ X \in V\}, \\ D &= V \backslash \overline{T_0^{(\alpha)}(\omega)} \text{ and } D_k = V_k \cap D \ (k\text{: integer}). \end{aligned}$$

By Propositions 4.1 and 4.2, it suffices to show that 0 is regular on D.

For any $\delta > 0$, we can choose a positive integer k such that

$$\sup_{X \in V} \int_{CV_k} d\varepsilon''_{X, C\bar{D}} < \delta, \tag{4.1}$$

because for any $X \in \bar{D}$, $\varepsilon''_{X, C\bar{D}} - \varepsilon_X = \tilde{L}^{(\alpha)}(\tilde{W}^{(\alpha)}\varepsilon''_{X, C\bar{D}} - \tilde{W}^{(\alpha)}\varepsilon_X)$ in the sense of distribution, that is,

$$\varepsilon''_{X, C\bar{D}} = C_{n,\alpha}\left(\int_{R^n} (\tilde{W}^{(\alpha)}\varepsilon_X(y - z, t) - \tilde{W}^{(\alpha)}\varepsilon''_{X, C\bar{D}}(y - z, t))|z|^{-n-2\alpha}dz\right)dydt \tag{4.2}$$

in $C\bar{D}$. Put

$$u_1(X) = \int_{CV} d\varepsilon''_{X, C\bar{D}},$$

$$\beta = \sup_{X \in V_{-1}} \int_{CV} d\varepsilon''_{X, C\bar{D}},$$

and

$$u_2(X) = \beta \int_{CV_{-k-1}} d\varepsilon''_{X, C\overline{D_{-k-1}}} + (1 - \beta) \int_{CV_{-1}} d\varepsilon''_{X, C\overline{D_{-k-1}}}.$$

Then $\beta < 1$. In fact, we take a sequence $(X_m)_{m=1}^{\infty} \subset V_{-1} \cap \bar{D}$ such that $\lim_{m \to \infty} \int_{CV} d\varepsilon''_{X_m, C\bar{D}} = \beta$. We may assume that $(\varepsilon''_{X_m, C\bar{D}})_{m=1}^{\infty}$ converges vaguely to some $\nu \in \tilde{M}_\alpha$ as $m \to \infty$ and that $(X_m)_{m=1}^{\infty}$ converges to some point $X_\infty = (x_\infty, t_\infty)$. Then

$$\tilde{W}^{(\alpha)}\varepsilon_{X_\infty} \geqq \tilde{W}^{(\alpha)}\nu \qquad \text{on } R^{n+1}.$$

Since the family of the density of $\varepsilon''_{X, C\bar{D}}$ $(X \in \bar{D})$ with respect to dX is uniformly bounded on every compact set in $C\bar{D}$ (see (4.2) in this proof), we have

$$\tilde{W}^{(\alpha)}\varepsilon_{X_\infty} = \tilde{W}^{(\alpha)}\nu \text{ on } C\bar{D} \quad \text{and} \quad \int d\nu = 1\,.$$

Assume that $\beta = 1$; $\int_V d\nu \leqq \liminf_{m\to\infty} \int_V d\varepsilon''_{X_m, C\bar{D}} = 0$ and hence $\mathrm{supp}[\nu] \subset CV$. Since for any $\lambda \in M_\alpha$, $W^{(\alpha)}\lambda'_{C\bar{D}}$ is continuous on D, the function $\int W^{(\alpha)}\lambda\, d\varepsilon''_{X, C\bar{D}}$ of X is continuous on D (see Proposition 3.8), so that the mapping $D \ni X \to \varepsilon''_{X, C\bar{D}}$ is vaguely continuous. Therefore Proposition 3.14 gives $X_\infty \in \overline{V_{-1}} \cap \partial D$, because if $X_\infty \in D$, then $\nu = \varepsilon''_{X_\infty, C\bar{D}}$, which contradicts $\mathrm{supp}[\nu] \subset CV$ and Proposition 3.14. By Lemma 3.15, $\tilde{W}^{(\alpha)}\varepsilon_{X_\infty} = \tilde{W}^{(\alpha)}\nu$ on $\{(x, t); t < t_\infty\}$. Proposition 2.10 gives $\nu = \varepsilon_{X_\infty}$, which contradicts $\mathrm{supp}[\nu] \subset CV$. Thus $\beta < 1$.

Let $(\phi_m)_{m=1}^\infty$ be an increasing sequence in $C_K^\infty(R^{n+1})$ such that $0 \leqq \phi_m \leqq 1$, $\lim_{m\to\infty} \phi_m = 1$ on CV and that $\phi_m = 0$ on V_{-1}. We write $\phi_m = W^{(\alpha)}\lambda_m$ with some signed measure λ_m. Then

$$\int_{V_{-k-1}} \left(\int_{CV} d\varepsilon''_{Y, C\bar{D}}\right) d\varepsilon''_{X, C\overline{D_{-k-1}}}(Y) \leqq \lim_{m\to\infty} \int_{V_{-k-1}} W^{(\alpha)}\lambda'_{m, C\bar{D}}\, d\varepsilon''_{X, C\overline{D_{-k-1}}}\,,$$

where $\lambda'_{m, C\bar{D}} = (\lambda_m^+)'_{C\bar{D}} - (\lambda_m^-)'_{C\bar{D}}$. Since Corollary 3.13 gives

$$(\varepsilon''_{X, C\overline{D_{-k-1}}})''_{C\bar{D}} = \varepsilon''_{X, C\overline{D_{-k-1}}}|_{V_{-k-1}} + (\varepsilon''_{X, C\overline{D_{-k-1}}}|_{CV_{-k-1}})''_{C\bar{D}}\,,$$

we have

$$\int_{V_{-k-1}} W^{(\alpha)}\lambda'_{m, C\bar{D}}\, d\varepsilon''_{X, C\overline{D_{-k-1}}} = \int_{V_{-k-1}} W^{(\alpha)}\lambda_m\, d(\varepsilon''_{X, C\overline{D_{-k-1}}}) = 0\,.$$

Let $(\phi_m)_{m=1}^\infty$ be a sequence in $C_K^\infty(R^{n+1})$ such that $0 \leqq \phi_m \leqq 1$ and $\lim_{m\to\infty} \phi_m(X) = 1$ on CV and $= 0$ on V. Since $\phi_m = W^{(\alpha)}\lambda_m$ with some signed measure λ_m, by Proposition 3.8, we have, for $X \in V_{-k-1}$,

$$\begin{aligned} u_1(X) &= \lim_{m\to\infty} \int \phi_m\, d\varepsilon''_{X, C\bar{D}} = \lim_{m\to\infty} \iint \phi_m\, d\varepsilon''_{Y, C\bar{D}}\, d\varepsilon''_{X, C\overline{D_{-k-1}}}(Y) \\ &= \int \left(\int_{CV} d\varepsilon''_{Y, C\bar{D}}\right) d\varepsilon''_{X, C\overline{D_{-k-1}}}(Y) \\ &= \int_{CV_{-k-1}} \left(\int_{CV} d\varepsilon''_{Y, C\bar{D}}\right) d\varepsilon''_{X, C\overline{D_{-k-1}}}(Y) \\ &\leqq u_2(X) \leqq \beta u_1(\tau_{k+1}^{(\alpha)}(X)) + (1 - \beta)\delta\,. \end{aligned}$$

Thus we obtain inductively

$$\limsup_{X\to 0} u_1(X) \leqq \sum_{k=0}^\infty \beta^k (1 - \beta)\delta = \delta\,,$$

which gives

$$\lim_{X\to 0} u_1(X) = 0 .$$

By Proposition 3.14, we can choose $f \in C_K(R^{n+1})$ such that $0 \leqq f \leqq 1$, $\mathrm{supp}[f] \subset CV$ and that

$$u(X) = \int f(Y) d\varepsilon''_{X, C\bar{D}}(Y) > 0 \qquad \text{on } D .$$

Take $\phi \in C_K(R^{n+1})$ such that $\mathrm{supp}[\phi] \subset D$, $\phi \geqq 0$, $\mathrm{Int}(\mathrm{supp}[\phi]) \cap \{(x, t); t<0\} \neq \phi$ and that $W^{(\alpha)}(\phi dX) \leqq u$ on $\mathrm{supp}[\phi]$. For any open set $\omega \supset CD$, we put $\omega_0 = \{X; \phi(X) > 0\} \cup \omega$. Let $(\omega_m)_{m=1}^{\infty}$ be an exhaustion of ω_0. Then we have

$$\begin{aligned} W^{(\alpha)}&(\phi dX)(X) - W^{(\alpha)}(\phi dX)'_{\omega}(X) \\ &= \int (W^{(\alpha)}(\phi dX) - W^{(\alpha)}(\phi dX)'_{\omega}) d\varepsilon''_{X, \omega_0} \\ &= \lim_{m\to\infty} \int (W^{(\alpha)}(\phi dX) - W^{(\alpha)}(\phi dX)'_{\omega}) d\varepsilon''_{X, \omega_m} \\ &\leqq \lim_{m\to\infty} \int u \, d\varepsilon''_{X, \omega_m} = \lim_{m\to\infty} \int \left(\int f d\varepsilon''_{Y, C\bar{D}}\right) d\varepsilon''_{X, \omega_m}(Y) = u(X) . \end{aligned}$$

By Lemma 3.5, we have

$$W^{(\alpha)}(\phi dX)(X) - W^{(\alpha)}(\phi dX)'_{CD}(X) \leqq u(X) \qquad \text{on } R^{n+1} ,$$

which implies

$$\lim_{X \in \Omega, X\to 0} W^{(\alpha)}(\phi dX)'_{CD}(X) = W^{(\alpha)}(\phi dX)(0) .$$

Hence

$$W^{(\alpha)}(\phi dX)'_{CD}(0) = W^{(\alpha)}(\phi dX)(0)$$

(see, for example, the proof of (1)$\to$(3) in Proposition 4.2). By Proposition 4.2, (3), 0 is regular for the Dirichlet problem of $L^{(\alpha)}$ on D. This completes the proof.

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