S. Konno
Nagoya Math. J.
Vol. 113 (1989), 129–146

EISENSTEIN SERIES IN HYPERBOLIC 3-SPACE AND KRONECKER LIMIT FORMULA FOR BIQUADRATIC FIELD

SHUJI KONNO

§ 0. Introduction

Let $L = kK$ be the composite of two imaginary quadratic fields k and K. Suppose that the discriminants of k and K are relatively prime. For any primitive ray class character χ of L, we shall compute $L(1, \chi)$ for the Hecke L-function in L. We write $\mathfrak{f}$ for the conductor of χ and C for the ray class modulo $\mathfrak{f}$. Let $\mathfrak{c} \in C$ be any integral ideal prime to $\mathfrak{f}$. We write $\mathfrak{a} = \mathfrak{c}/(\vartheta_L \mathfrak{f}) = \mathfrak{g}\omega_1 + \mathfrak{n}\omega_2$ as $\mathfrak{g}$-module where $\mathfrak{g}, \mathfrak{n}$ and ϑ_L are, respectively, the ring of integers in k, an ideal in k and the differente of L. Let $L(s, \chi) = T(\chi)^{-1} \sum_C \bar{\chi}(C) \Psi(C, s)$ where $T(\chi)$ is the Gaussian sum and, as in (3.2),

$$\Psi(C, s) = N_{L/Q}(\mathfrak{a})^s \sum_{(\mu)\mathfrak{f}}'' e^{2\pi i Tr_{L/Q}(\mu)} |N_{L/Q}(\mu)|^{-s} .$$

In § 1, 2, for each pair of ideals $(\mathfrak{m}, \mathfrak{n})$ in k, we associate Eisenstein series in hyperbolic 3-space having characters. For this series, we show the Kronecker limit formula. In § 3, 4, we show that $\Psi(C, s)$ is written as the constant term in the Fourier expansion of the Eisenstein series with reference to the hyperbolic substitution of $SL_2(k)$ (Theorems 4.3, 4.4). In § 5, we compute the Kronecker limit formula for $\Psi(C, s)$ (Theorems 5.6, 5.7). The limit formula is written as the Fourier cosine series of $\omega + \tilde{\omega}$ $(\omega = \omega_1^{-1}\omega_2)$ whose coefficients are functions of $\omega - \tilde{\omega}$ where $\tilde{\omega}$ is the conjugate of ω over k.

NOTATIONS. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively, the ring of rational integers, the rational number field, the real number field and the complex number field. For $z \in \boldsymbol{C}$, $\bar{z}$ denotes the complex conjugate of z. We write $S(z) = z + \bar{z}$ and $|z|^2 = z\bar{z}$. For $z \in \boldsymbol{C}$, $\sqrt{z}$ means $-\pi/2 < \arg \sqrt{z} \leqq \pi/2$. For an associative ring A with identity element, $A^\times$

Received August 6, 1987.

denotes the group of invertible element of A. We write $e(x) = e^{2\pi i x}$ for $x \in \boldsymbol{R}$ and $e[z] = e(S(z))$ for $z \in \boldsymbol{C}$. We denote $K_z(2Y) = 1/2\int_0^\infty e^{-Y(t+t^{-1})}t^{z-1}dt$.

§1. Eisenstein series in the 3-dimensional hyperbolic space

We shall consider Eisenstein series with characters in the 3-dimensional hyperbolic space. Let $\boldsymbol{K} = \boldsymbol{C} + \boldsymbol{C}j$ be the Hamilton quaternion algebra with j satisfying $j^2 = -1$, $j^{-1}zj = \bar{z}$ for $z \in \boldsymbol{C}$. Let $\zeta \to \bar{\zeta}$ denote the quaternion conjugation in $\boldsymbol{K}$ and let $N(\zeta) = \zeta\bar{\zeta}$ be the quaternion norm. Let $\boldsymbol{H}$ denote the 3-dimensional hyperbolic space. We write a point $\xi \in \boldsymbol{H}$ as $\xi = z + vj$ for $z \in \boldsymbol{C}$, $v > 0$ and consider $\boldsymbol{H}$ to be contained in $\boldsymbol{K}$.

Let B_1 be the subgroup of $SL_2(\boldsymbol{C})$ consisting of elements $b = v^{-1/2}\begin{pmatrix} v & z \\ 0 & 1 \end{pmatrix}$ with $v > 0$, $z \in \boldsymbol{C}$. Then B_1 is a complete set of representatives for the space of right cosets $SL_2(\boldsymbol{C})/SU(2, \boldsymbol{C})$. We shall identify $b = v^{-1/2}\begin{pmatrix} v & z \\ 0 & 1 \end{pmatrix} \in B_1$ with the point $\xi = z + vj \in \boldsymbol{H}$ and we can view $\boldsymbol{H} = B_1 = SL_2(\boldsymbol{C})/SU(2, \boldsymbol{C})$. Let $\xi \in \boldsymbol{H}$ and $b \in B_1$ be as above. For any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\boldsymbol{C})$, we can write

$$gb = v_1^{-1/2}\begin{pmatrix} v_1 & z_1 \\ 0 & 1 \end{pmatrix}c_1 \tag{1.1}$$

where $v_1 = v/N(\gamma\xi + \delta)$, $z_1 = \{(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha\bar{\gamma}v^2\}/N(\gamma\xi + \delta)$ and

$$c_1 = N(\gamma\xi + \delta)^{-1/2}\begin{pmatrix} \overline{\gamma z + \delta} & -\bar{\gamma}v \\ \gamma v & \gamma z + \delta \end{pmatrix} \in SU(2, \boldsymbol{C})$$

with $N(\gamma\xi + \delta) = |\gamma z + \delta|^2 + |\gamma|^2v^2$. Thus the left multiplication of $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ on B_1 induces on $\boldsymbol{H}$ the transformation $\xi \to (\alpha\xi + \beta)(\gamma\xi + \delta)^{-1}$;

$$(\alpha\xi + \beta)(\gamma\xi + \delta)^{-1} = \frac{(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha\bar{\gamma}v^2}{N(\gamma\xi + \delta)} + \frac{v}{N(\gamma\xi + \delta)}j\,. \tag{1.2}$$

The group $SL_2(\boldsymbol{C})/\{\pm I\}$ act on $\boldsymbol{H}$ transitively and

$$ds^2 = v^{-2}(dv^2 + dzd\bar{z}) \tag{1.3}$$

is an invariant metric on $\boldsymbol{H}$.

Let $k = \boldsymbol{Q}(\sqrt{-d_1})$ be the imaginary quadratic field of discriminant $-d_1$. Denote by $\mathfrak{g}$ the ring of integers in k and by $\tilde{\mathfrak{g}} = \mathfrak{g}(1/\sqrt{-d_1})$ the inverse differente. Let w_k be the number of roots of unity in k. We consider k to be contained in $\boldsymbol{C}$. For an ideal $\mathfrak{a} \neq 0$, we write $(\mathfrak{a})$ for the absolute ideal class of $\mathfrak{a}$ and $\zeta_k((\mathfrak{a}), s)$ for the zeta function of $(\mathfrak{a})$ in k. Let $\mathfrak{a} \oplus \mathfrak{b}$ be the module consisting of all pairs (a, b) for $a \in \mathfrak{a}$, $b \in \mathfrak{b}$. For any non-zero (fractional) ideals $\mathfrak{m}$, $\mathfrak{n}$ in k, we define

$$E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s) = v^{2s} N_{k/Q}(\mathfrak{m}\mathfrak{n})^s \sum_{(m,n)\in\mathfrak{m}\oplus\mathfrak{n}}{}' \frac{e[-mu_1 - nu_2]}{N(n\xi + m)^{2s}}. \tag{1.4}$$

Here $\xi = z + vj \in \boldsymbol{H}$, $(u_1, u_2) \in \boldsymbol{C}^2$ and $s \in \boldsymbol{C}$; the summation is taken over all $(m, n) \in \{\mathfrak{m} \oplus \mathfrak{n}\} \backslash \{(0, 0)\}$. The series converges absolutely for $\mathrm{Re}(s) > 1$. We consider $E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s)$ to be a kind of Eisenstein series.

To get the Fourier expansion of $E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s)$, we put

$$D(\xi, u, s) = \sum_{m\in\mathfrak{m}} e[-mu] N(\xi + m)^{-2s} \qquad (\mathrm{Re}(s) > 1) \tag{1.5}$$

where $\xi \in \boldsymbol{H}$, $u \in \boldsymbol{C}$ and $s \in \boldsymbol{C}$. The self-dual Haar measure on $\boldsymbol{C}$, with respect to the basic character $z \to e[-z]$, is $|dz \wedge d\bar{z}| = 2dxdy$ $(z = x + yi)$. The dual lattice of $\mathfrak{m}$ in $\boldsymbol{C}$, with respect to the bicharacter $(z_1, z_2) \to e[-z_1 z_2]$, is $\tilde{\mathfrak{m}} = \mathfrak{m}^{-1}\tilde{\mathfrak{g}}$.

LEMMA 1.1. *We have the Fourier expansion*

$$D(\xi, u, s) = \delta_u v^{2-4s} \frac{2\pi\Gamma(2s-1)}{\Gamma(2s)} \frac{1}{\sqrt{d_1} N_{k/Q}(\mathfrak{m})} + \frac{2(2\pi)^{2s}}{\Gamma(2s)} \frac{1}{\sqrt{d_1} N_{k/Q}(\mathfrak{m})} \times \sum_{u \neq \ell \in \tilde{\mathfrak{m}}}{}' \left| \frac{\ell - u}{v} \right|^{2s-1} K_{2s-1}(4\pi|\ell - u|v) e[-(\ell - u)z] \tag{1.6}$$

where $\delta_u = 1$ *or* 0, *according as* $u \in \tilde{\mathfrak{m}}$ *or not.*

Proof. Let Q be the fundamental parallelogram for $\boldsymbol{C}/\mathfrak{m}$ and let $|Q| = \sqrt{d_1} N_{k/Q}(\mathfrak{m})$ be its area. Then $z \to D(\xi, u, s)e[-uz]$ is periodic with period lattice $\mathfrak{m}$. Expanding this into Fourier series, we get

$$D(\xi, u, s) = \sum_{\ell\in\tilde{\mathfrak{m}}} g_\ell(v) e[-(\ell - u)z] \tag{1.7}$$

$$g_\ell(v) = \frac{1}{|Q|} \int_{\boldsymbol{C}} \frac{e[(\ell - u)z]}{(|z|^2 + v^2)^{2s}} |dz \wedge d\bar{z}|. \tag{1.8}$$

Applying Mellin transformation to this and by $\int_{-\infty}^{+\infty} e^{-X^2/2 - iXY} dX = \sqrt{2\pi} c^{-Y^2/2}$, we get

$$\Gamma(2s)g_\ell(v) = \frac{2\pi}{\sqrt{d_1}N_{k/Q}(\mathfrak{m})}\int_0^\infty e^{-v^2t-(2\pi)^2/t|\ell-u|^2}t^{2s-2}dt\,. \tag{1.9}$$

Consequently, we have

$$g_\ell(v) = v^{2-4s}\frac{2\pi\Gamma(2s-1)}{\Gamma(2s)}\frac{1}{\sqrt{d_1}N_{k/Q}(\mathfrak{m})} \qquad (\ell = u)\,, \tag{1.10}$$

$$g_\ell(v) = \frac{2(2\pi)^{2s}}{\Gamma(2s)}\frac{1}{\sqrt{d_1}N_{k/Q}(\mathfrak{m})}\left|\frac{\ell-u}{v}\right|^{2s-1}K_{2s-1}(4\pi|\ell-u|v) \qquad (\ell \neq u)\,. \tag{1.11}$$

Substituting (1.10) and (1.11) in (1.7), we obtain (1.6).

Let $\mathfrak{a}$ be any non-zero ideal in k. For $u \in \boldsymbol{C}$ and $s \in \boldsymbol{C}$, we define

$$G_\mathfrak{a}(s, u) = \sum_{0\neq a\in\mathfrak{a}}{}' e[-au]|N_{k/Q}(a)|^{-s}\,. \tag{1.12}$$

PROPOSITION 1.2. *We have the Fourier expansion*

$$E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s) = A(s) + B(s) + C(s)\,; \tag{1.13}$$

$$A(s) = v^{2s}N_{k/Q}(\mathfrak{m}\mathfrak{n})^sG_\mathfrak{m}(2s, u_1)\,,$$

$$B(s) = v^{2-2s}\frac{2\pi\Gamma(2s-1)}{\Gamma(2s)}\frac{N_{k/Q}(\mathfrak{m})^{s-1}N_{k/Q}(\mathfrak{n})^s}{\sqrt{d_1}}G_\mathfrak{n}(2s-1, u_2) \qquad \textit{for } u_1 \in \tilde{\mathfrak{m}}\,;$$

$$= 0 \qquad \textit{for } u_1 \notin \tilde{\mathfrak{m}}\,,$$

$$C(s) = \frac{2(2\pi)^{2s}}{\Gamma(2s)}\frac{N_{k/Q}(\mathfrak{m})^{s-1}N_{k/Q}(\mathfrak{n})^s}{\sqrt{d_1}}\sum_{0\neq n\in\mathfrak{n}}{}'\sum_{u_1\neq\ell\in\tilde{\mathfrak{m}}}{}'\left|\frac{\ell-u_1}{n}\right|^{2s-1}$$
$$\times\, vK_{2s-1}(4\pi|n(\ell-u_1)|v)e[-n(\ell-u_1)z - nu_2]\,.$$

Proof. Since

$$E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s) = v^{2s}N_{k/Q}(\mathfrak{m}\mathfrak{n})^s\sum_{0\neq m\in\mathfrak{m}}{}' e[-mu_1]|N_{k/Q}(m)|^{-2s}$$
$$+\, v^{2s}N_{k/Q}(\mathfrak{m}\mathfrak{n})^s\sum_{0\neq n\in\mathfrak{n}}{}' e[-nu_2]D(n\xi, u_1, s)\,,$$

by Lemma 1.1 and by (1.12), we obtain the proof.

The function $E_{\mathfrak{m},\mathfrak{n}}(\xi, 0, 0, s)$ also satisfies a functional equation. Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero ideals in k. For $c \in k^\times$ and $s \in \boldsymbol{C}$, we define

$$\tau_s(\mathfrak{a}, \mathfrak{b}, c) = N_{k/Q}(\mathfrak{a})^{s-1/2}N_{k/Q}(\mathfrak{b})^{s+1/2}\sum_b N_{k/Q}(cb^{-2})^s\,. \tag{1.14}$$

The summation is taken over all $b \in \mathfrak{b}\setminus\{0\}$ such that $cb^{-1} \in \mathfrak{a}^{-1}$. It is a finite sum and we see that $\tau_s(\mathfrak{a}, \mathfrak{b}, c) = 0$ unless $c \in \mathfrak{a}^{-1}\mathfrak{b}$. By a little computations we find that

(1.15) $$\tau_s(\mathfrak{a}, \mathfrak{b}, c) = \tau_{-s}(\mathfrak{b}^{-1}, \mathfrak{a}^{-1}, c)\,.$$

THEOREM 1.3. *Let* $E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s)$ *be as in* (1.4). *Then*

$$\mathscr{E}_{\mathfrak{m},\mathfrak{n}}(\xi, s) = \Gamma(2s)(2\pi/\sqrt{d_1})^{-2s}E_{\mathfrak{m},\mathfrak{n}}(\xi, 0, 0, s)$$

is continued to the whole s-plane meromorphically and satisfies

(1.16) $$\mathscr{E}_{\mathfrak{m},\mathfrak{n}}(\xi, s) = \mathscr{E}_{\mathfrak{n}^{-1},\mathfrak{m}^{-1}}(\xi, 1 - s)\,.$$

Proof. Let $u_1 = u_2 = 0$ and $\ell = m/\sqrt{-d_1}$ in (1.13). We see

(1.17) $$\begin{cases} A(s) = w_k(v^2N_{k/Q}(\mathfrak{m}^{-1}\mathfrak{n}))^s\zeta_k((\mathfrak{m}^{-1}), 2s) \\ B(s) = w_k\dfrac{\Gamma(2s-1)}{\Gamma(2s)}\dfrac{2\pi}{\sqrt{d_1}}(v^2N_{k/Q}(\mathfrak{m}^{-1}\mathfrak{n}))^{1-s}\zeta_k((\mathfrak{n}^{-1}), 2s-1) \\ C(s) = \dfrac{2}{\Gamma(2s)}\left(\dfrac{2\pi}{\sqrt{d_1}}\right)^{2s}\displaystyle\sum_{0\neq n\in\mathfrak{m}^{-1}\mathfrak{n}}{}' \tau_{s-1/2}(\mathfrak{m}, \mathfrak{n}, n)v \\ \qquad \times K_{2s-1}(4\pi|n|v/\sqrt{d_1})e[-nz/\sqrt{-d_1}]\,. \end{cases}$$

For any non-zero ideal $\mathfrak{a}$ in k, $Z((\mathfrak{a}^{-1}), s) = \Gamma(s)(2\pi/\sqrt{d_1})^{-s}\zeta_k((\mathfrak{a}^{-1}), s)$ is continued to the whole s-plane meromorphically and satisfies $Z((\mathfrak{a}^{-1}), s) = Z((\mathfrak{a}), 1 - s)$. Moreover $\tau_{s-1/2}$ and K_{2s-1} are holomorphic in the whole s-plane, they satisfy (1.15) and $K_{2s-1} = K_{1-2s}$. From these we obtain the proof.

§ 2. Kronecker limit formula for Eisenstein series

Let $E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s) = A(s) + B(s) + C(s)$ be as in Proposition 1.2. We discuss the following two cases respectively. Case (a) $(u_1, u_2) \in \mathfrak{m}^{-1}\mathfrak{g} \oplus \mathfrak{n}^{-1}\mathfrak{g}$, case (b) $(u_1, u_2) \notin \mathfrak{m}^{-1}\mathfrak{g} \oplus \mathfrak{n}^{-1}\mathfrak{g}$.

Case (a). In this case by (1.4), we may assume that $u_1 = u_2 = 0$.

THEOREM 2.1. *The function* $E_{\mathfrak{m},\mathfrak{n}}(\xi, 0, 0, s)$ *is continued holomorphically to* $\mathrm{Re}(s) > 1/2$ *except for the simple pole at* $s = 1$. *At* $s = 1$, $E_{\mathfrak{m},\mathfrak{n}}(\xi, 0, 0, s)$ *has the expansion*

(2.1) $$E_{\mathfrak{m},\mathfrak{n}}(\xi, 0, 0, s) = \frac{2\pi^2}{d_1}\frac{1}{s-1} + \frac{2\pi^2}{d_1}\left\{\frac{w_k\sqrt{d_1}}{\pi}\alpha_0(\mathfrak{n}^{-1}) - 2 - \log N_{k/Q}(\mathfrak{m}^{-1}\mathfrak{n}) - \log v^2 + h_{\mathfrak{m},\mathfrak{n}}(\xi)\right\} + O(|s-1|)$$

where

(2.2) $$\alpha_0(\mathfrak{n}^{-1}) = \lim_{s\to 1}\left\{\zeta_k((\mathfrak{n}^{-1}), s) - \frac{2\pi}{w_k\sqrt{d_1}}\frac{1}{s-1}\right\}.$$

The function $h_{\mathfrak{m},\mathfrak{n}}(\xi)$ *is defined by*

$$h_{\mathfrak{m},\mathfrak{n}}(\xi)=\frac{w_k d_1}{2\pi^2}N_{k/Q}(\mathfrak{m}^{-1}\mathfrak{n})\zeta_k((\mathfrak{m}^{-1}),2)v^2 + 4\sum_{0\neq n\in\mathfrak{m}^{-1}\mathfrak{n}}{}'\tau_{1/2}(\mathfrak{m},\mathfrak{n},n)vK_1(4\pi|n|v/\sqrt{d_1})e[-nz/\sqrt{-d_1}]\,. \tag{2.3}$$

Proof can be done as in [1], [3], using Proposition 1.2.

Case (b). In this case we have

THEOREM 2.2. *Suppose* $(u_1,u_2)\notin\mathfrak{m}^{-1}\mathfrak{g}\oplus\mathfrak{n}^{-1}\mathfrak{g}$. *Then* $E_{\mathfrak{m},\mathfrak{n}}(\xi,u_1,u_2,s)$ *is holomorphic in* $\mathrm{Re}(s)>1/2$ *and we have*

$$\begin{aligned}E_{\mathfrak{m},\mathfrak{n}}(\xi,u_1,u_2,1)&=b(u_1,u_2)+N_{k/Q}(\mathfrak{m}\mathfrak{n})G_{\mathfrak{m}}(2,u_1)v^2\\&+\frac{8\pi^2}{\sqrt{d_1}}N_{k/Q}(\mathfrak{n})\sum_{0\neq n\in\mathfrak{n}}{}'\sum_{u_1\neq m\in\mathfrak{m}^{-1}\mathfrak{g}}{}'\left|\frac{m-u_1}{n}\right|vK_1(4\pi|n(m-u_1)|v)\\&\times e[-n(m-u_1)z-nu_2]\end{aligned}\tag{2.4}$$

where $b(u_1,u_2)$ *is given by*

$$b(u_1,u_2)=\begin{cases}0 & \text{if } u_1\notin\mathfrak{m}^{-1}\mathfrak{g}\\ \dfrac{2\pi}{\sqrt{d_1}}N_{k/Q}(\mathfrak{n})G_{\mathfrak{n}}(1,u_2) & \text{if } u_1\in\mathfrak{m}^{-1}\mathfrak{g}\ \text{ and }\ u_2\notin\mathfrak{n}^{-1}\mathfrak{g}\,.\end{cases}\tag{2.5}$$

Proof. In Proposition 1.2, $A(s)$ and $C(s)$ are holomorphic in $\mathrm{Re}(s)>1/2$. As to $B(s)$, it is 0 when $u_1\notin\mathfrak{m}^{-1}\mathfrak{g}$; it is holomorphic when $u_1\in\mathfrak{m}^{-1}\mathfrak{g}$ and $u_2\notin\mathfrak{n}^{-1}\mathfrak{g}$ ([12], p. 77, § 10). Again by Proposition 1.2, we obtain the proof.

As an analogy of $\log|\vartheta_1(w,z)/\eta(z)|^2$ for the Kronecker's second limit formula, we write $\psi(\zeta,\xi)$ for the right hand side of (2.4). For any $\xi\in\boldsymbol{H}$, let $\mathscr{L}_\xi=\mathfrak{m}^{-1}\mathfrak{g}\xi+\mathfrak{n}^{-1}\mathfrak{g}$ be the $\mathfrak{g}$-lattice in $\boldsymbol{K}$. Let $\zeta=\zeta_1+\zeta_2 j\in\boldsymbol{K}$, $(\zeta_1,\zeta_2\in\boldsymbol{C})$ and $\xi=z+vj\in\boldsymbol{H}$. When $\zeta\notin\mathscr{L}_\xi$, we define

$$\begin{aligned}\psi_{\mathfrak{m},\mathfrak{n}}(\zeta,\xi)&=b\Big(-\frac{1}{v}\zeta_2,\zeta_1-\frac{z}{v}\zeta_2\Big)+N_{k/Q}(\mathfrak{m}\mathfrak{n})G_{\mathfrak{m}}\Big(2,-\frac{1}{v}\zeta_2\Big)v^2\\&+\frac{8\pi^2}{\sqrt{d_1}}K_{k/Q}(\mathfrak{n})\sum_{0\neq n\in\mathfrak{n}}{}'\sum_{\substack{m\in\mathfrak{m}^{-1}\mathfrak{g}\\ mv+\zeta_2\neq0}}{}'\left|\frac{mv+\zeta_2}{n}\right|K_1(4\pi|n(mv+\zeta_2)|)\\&\times e[-n(mz+\zeta_1)]\,.\end{aligned}\tag{2.6}$$

Then we have

$$E_{\mathfrak{m},\mathfrak{n}}(\xi,u_1,u_2,1)=\psi_{\mathfrak{m},\mathfrak{n}}(-u_1\xi+u_2,\xi)\,.\tag{2.7}$$

We see easily that

$$\psi_{\mathfrak{m},\mathfrak{n}}(\zeta+\zeta_0, \xi) = \psi_{\mathfrak{m},\mathfrak{n}}(\zeta, \xi) \qquad \text{for } \zeta_0 \in \mathscr{L}_\xi . \tag{2.8}$$

Let Γ be the subgroup of $SL_2(k)$ defined by

$$\Gamma = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(k) \,\middle|\, (\mathfrak{n} \oplus \mathfrak{m}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \mathfrak{n} \oplus \mathfrak{m} \right\} . \tag{2.9}$$

Then $\Gamma/\{\pm I\}$ is a discrete subgroup of $SL_2(\boldsymbol{C})/\{\pm I\}$ and act on $\boldsymbol{H}$ properly discontinuously. For $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, we write $\hat{u}_2 = \alpha u_2 + \beta u_1$ and $\hat{u}_1 = \gamma u_2 + \delta u_1$. Then $(\hat{u}_1, \hat{u}_2) \in \mathfrak{m}^{-1}\mathfrak{g} \oplus \mathfrak{n}^{-1}\mathfrak{g}$ if and only if $(u_1, u_2) \in \mathfrak{m}^{-1}\mathfrak{g} \oplus \mathfrak{n}^{-1}\mathfrak{g}$. Furthermore, we see that

$$E_{\mathfrak{m},\mathfrak{n}}((\alpha\xi+\beta)(\gamma\xi+\delta)^{-1}, \hat{u}_1, \hat{u}_2, s) = E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s) . \tag{2.10}$$

PROPOSITION 2.3. *For any* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, *we have*

(i) $h_{\mathfrak{m},\mathfrak{n}}((\alpha\xi+\beta)(\gamma\xi+\delta)^{-1}) = h_{\mathfrak{m},\mathfrak{n}}(\xi) - \log N(\gamma\xi+\delta)^2$

(ii) $\psi_{\mathfrak{m},\mathfrak{n}}(\zeta(\gamma\xi+\delta)^{-1}, (\alpha\xi+\beta)(\gamma\xi+\delta)^{-1}) = \psi_{\mathfrak{m},\mathfrak{n}}(\zeta, \xi)$.

Proof. (i) It is well known ([1], [3]). (ii) For any $\zeta \in \boldsymbol{K}$, we write $\zeta = -u_1\xi + u_2$ with $u_1, u_2 \in \boldsymbol{C}$. Let $\hat{u}_2 = \alpha u_2 + \beta u_1$ and $\hat{u}_1 = \gamma u_2 + \delta u_1$ be as above. Since $-\hat{u}_1(\alpha\xi+\beta)(\gamma\xi+\delta)^{-1} + \hat{u}_2 = \zeta(\gamma\xi+\delta)^{-1}$, we obtain the proof.

§ 3. Reduction of the problem

Let $L = kK$ be the biquadratic field composed of two imaginary quadratic fields k and K with discriminants $-d_1$ and $-d_2$ respectively. We assume that d_1 and d_2 are relatively prime. Denote by $\mathfrak{o}_L$ the ring of integers in L and by ϑ_L the differente of L. Let $\mathfrak{f}$ be any integral ideal in L. Denote by $E_L(\mathfrak{f})$ the group consisting of units in L which satisfy $\equiv 1 \bmod \mathfrak{f}$. Let χ be any primitive ray class character modulo $\mathfrak{f}$ in L. For any $\alpha \in \mathfrak{o}_L$ satisfying $((\alpha), \mathfrak{f}) = 1$, we can write $\chi((\alpha)) = \chi_1(\alpha)$ where χ_1 is a character of $(\mathfrak{o}_L/\mathfrak{f})^\times$. We write χ for χ_1. Let $L(s, \chi)$ be the Hecke L-series. Our aim is to compute $L(1, \chi)$.

Let $\gamma_0 \in L^\times$ be such that $(\gamma_0) = \mathfrak{h}/(\vartheta_L\mathfrak{f})$ with an integral ideal $\mathfrak{h}$ which is prime to $\mathfrak{f}$. We define

$$T(\chi) = \bar{\chi}(\mathfrak{h}) \sum_{\rho \bmod \mathfrak{f}} \bar{\chi}(\rho) e(\mathrm{Tr}_{L/Q}(\rho\gamma_0)) .$$

Note that $T(\chi) \neq 0$ since χ is primitive. Let C be any ray class modulo $\mathfrak{f}$ in L and let $\mathfrak{c} \in C$ be an integral ideal which is prime to $\mathfrak{f}$. For $\mathfrak{a} = \mathfrak{c}/(\vartheta_L\mathfrak{f})$, we put

(3.1) $$\Psi_1(\mathfrak{a}, s) = N_{L/Q}(\mathfrak{a})^s \sum_{(\mu)_\mathfrak{f}}'' e(\mathrm{Tr}_{L/Q}(\mu))|N_{L/Q}(\mu)|^{-s} \qquad (\mathrm{Re}(s) > 1)\,.$$

The summation is taken over all non-associated classes $(\mu)_\mathfrak{f}$ in $\mathfrak{a}\setminus\{0\}$ with respect to $E_L(\mathfrak{f})$. Then $\Psi_1(\mathfrak{a}, s)$ depends only on C but not on the choice of $\mathfrak{c}$. Therefore we define

(3.2) $$\Psi(C, s) = \Psi_1(\mathfrak{a}, s)\,.$$

It is known that

(3.3) $$L(s, \chi) = T(\chi)^{-1} \sum_C \bar{\chi}(C)\Psi(C, s)$$

where the summation is taken over all ray classes modulo $\mathfrak{f}$, ([10]). Thus to obtain $L(1, \chi)$, we compute the limit formula for $\Psi(C, s)$.

§4. Limit formula for $\Psi(C, s)$

Let $M = \boldsymbol{Q}(\sqrt{d_1 d_2})$ be the real quadratic subfield of L. Let $x \to \tilde{x}$ be the non-trivial automorphism of L over k. If $y \in M$, we write y' for $\tilde{y}$. We write $\mathfrak{o}_M$ for the ring of integers in M. Put $\mathfrak{f}_0 = \mathfrak{f} \cap \mathfrak{o}_M$ and let $E_M^+(\mathfrak{f}_0)$ be the group consisting of units $x \in M$ with $x \equiv 1 \bmod \mathfrak{f}_0$ and totally positive. Let $\varepsilon > 1$ be the generating element of $E_M^+(\mathfrak{f}_0)$. Note that $\varepsilon > 1 > \varepsilon' > 0$. Let ε_0 be a generating element of $E_L(\mathfrak{f})$ modulo the torsion subgroup. We choose ε_0 such that $|\varepsilon_0| > 1$ and fix once and for all. Since $\varepsilon_0\bar{\varepsilon}_0 \in E_M^+(1)$, let e be the least positive integer such that $(\varepsilon_0\bar{\varepsilon}_0)^e \in E_M^+(\mathfrak{f}_0)$.

LEMMA 4.1. *We have* $(\varepsilon_0\bar{\varepsilon}_0)^e = \varepsilon^g$ *for* $g = 1$ *or* 2.

Proof. We can write $(\varepsilon_0\bar{\varepsilon}_0)^e = \varepsilon^g$ for $g \geqq 1$. Suppose $g > 2$. This implies $|\varepsilon_0^e\varepsilon^{-1}|^2 = \varepsilon^{g-2} > 1$. As an element of $E_L(\mathfrak{f})$, we write $\varepsilon = \zeta\varepsilon_0^q$ where $q \geqq 1$ and ζ is a root of unity. From $1 < |\varepsilon_0^e\varepsilon^{-1}|^q = \varepsilon^{e-q}$, we see $e > q$. Since $\varepsilon^g = |\varepsilon_0|^{2q}|\varepsilon_0|^{2(e-q)} = \varepsilon^2|\varepsilon_0|^{2(e-q)}$, we get $(\varepsilon_0\bar{\varepsilon}_0)^{e-q} \in E_M^+(\mathfrak{f}_0)$. This is a contradiction.

Let C be any ray class modulo $\mathfrak{f}$ and let $\mathfrak{c} \in C$ be an integral ideal prime to $\mathfrak{f}$. We write $\mathfrak{a} = \mathfrak{c}/(\vartheta_L\mathfrak{f})$ as $\mathfrak{g}$-module;

(4.1) $$\mathfrak{a} = \mathfrak{g}\omega_1 + \mathfrak{n}\omega_2\,.$$

Here $\{\omega_1, \omega_2\}$ $(\omega_j \in L; j = 1, 2)$ are linearly independent over k and $\mathfrak{n}$ is a non-zero (fractional) ideal in k. We shall fix the expression (4.1) and we write $\omega = \omega_1^{-1}\omega_2$.

LEMMA 4.2. *We can find an element* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k)$ *such that*

(i) $\begin{pmatrix} \omega & \tilde{\omega} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega & \tilde{\omega} \\ 1 & 1 \end{pmatrix}$,

(ii) $(\mathfrak{n} \oplus \mathfrak{g})\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathfrak{n} \oplus \mathfrak{g}$.

In particular, we have $\varepsilon = c\omega + d$ *and* $\varepsilon' = c\tilde{\omega} + d$.

Proof. Take non-zero $n \in \mathfrak{n}$. Since, $\omega_1\varepsilon$, $n\omega_2\varepsilon \in \mathfrak{a}$, we find $\alpha, \gamma \in \mathfrak{n}$ and $\beta, \delta \in \mathfrak{g}$ such that $n\omega_2\varepsilon = \alpha\omega_2 + \beta\omega_1$ and $\omega_1\varepsilon = \gamma\omega_2 + \delta\omega_1$. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ satisfies (i) and (ii).

Let Γ be the group defined by (2.9) with $\mathfrak{m} = \mathfrak{g}$. By Lemma 4.2, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a hyperbolic element of Γ, it generates an infinite cyclic subgroup of Γ, moreover it has two fixed points $\omega, \tilde{\omega}$ in $\boldsymbol{C}$. From now on, we deal $E_{\mathfrak{m},\mathfrak{n}}(\xi, u_1, u_2, s)$ with $\mathfrak{m} = \mathfrak{g}$, $\mathfrak{n}$ being as in (4.1) and

$$u_j = \mathrm{Tr}_{L/k}(\omega_j) \qquad (j = 1, 2). \tag{4.2}$$

To be precise;

$$E_{\mathfrak{n}}(\xi, u_1, u_2, s) = v^{2s} N_{k/\boldsymbol{Q}}(\mathfrak{n})^s \sum_{(m,n)\in\mathfrak{g}\oplus\mathfrak{n}}{}' \frac{e[-mu_1 - nu_2]}{N(n\xi + m)^{2s}}. \tag{4.3}$$

Then (u_1, u_2) is of case (a) if and only if $\mathfrak{f} = (1)$. We write

$$\xi^* = (a\xi + b)(c\xi + d)^{-1}, \qquad (u_2^*, u_1^*) = (u_2, u_1)\begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{4.4}$$

Then we find that

$$E_{\mathfrak{n}}(\xi^*, u_1^*, u_2^*, s) = E_{\mathfrak{n}}(\xi^*, u_1, u_2, s) = E_{\mathfrak{n}}(\xi, u_1, u_2, s). \tag{4.5}$$

Let ρ_ω denote the semi-circle in $\boldsymbol{H}$ which is defined by

$$\rho(t) = z(t) + v(t)j\ ; \quad z(t) = \frac{t^2\omega + \tilde{\omega}}{t^2 + 1}, \quad v(t) = \frac{t|\omega - \tilde{\omega}|}{t^2 + 1}, \tag{4.6}$$

where t is a positive parameter. We see that $\rho(t)^* = \rho(t\varepsilon^2)$ ([6]).

THEOREM 4.3. *Notations being as above. Let* $w_L(\mathfrak{f})$ *be the number of roots of unity in* $E_L(\mathfrak{f})$ *and* $R_L(\mathfrak{f}) = 2\log|\varepsilon_0|$ *be the regulator of* $E_L(\mathfrak{f})$. *Then we have*

$$\Psi(C, s) = \frac{\Gamma(2s)}{\Gamma(s)^2}\frac{R_L(\mathfrak{f})}{w_L(\mathfrak{f})d_2^s}\frac{1}{\log\varepsilon}\int_{t_0}^{t_0\varepsilon^2} E_{\mathfrak{n}}(\rho(t), u_1, u_2, s)\frac{dt}{t} \tag{4.7}$$

where $t_0 > 0$ *is any real number.*

Proof. We put

$$c_0 = \int_{t_0}^{t_0\varepsilon^2} E_{\mathfrak{n}}(\rho(t), u_1, u_2, s)\frac{dt}{t}.$$

By (4.5), the integrand is invariant by $t \to t\varepsilon^2$. For $(n, m) \in \mathfrak{n} \oplus \mathfrak{g}\backslash\{(0, 0)\}$, we write $-\beta = n\omega + m$ and $-\tilde{\beta} = n\tilde{\omega} + m$. Then β runs over the set $\mathfrak{a}\omega_1^{-1}\backslash\{0\}$ as (n, m) runs over the set $\mathfrak{n} \oplus \mathfrak{g}\backslash\{(0, 0)\}$. By (4.2), (4.6) we see that $e[-mu_1 - nu_2] = e(\mathrm{Tr}_{L/Q}(\beta\omega_1))$ and $N(n\rho(t) + m) = (t^2|\beta|^2 + |\tilde{\beta}|^2)/(t^2 + 1)$. Substituting $t = |\tilde{\beta}/\beta|t_1^{1/2}$, we get

$$c_0 = \frac{|\omega - \tilde{\omega}|^{2s}}{2} N_{k/Q}(\mathfrak{n})^s \sum_{0 \neq \beta \in \mathfrak{a}\omega_1^{-1}}{}' \frac{e(\mathrm{Tr}_{L/Q}(\beta\omega_1))}{|N_{L/Q}(\beta)|^s} \int_A^B \frac{t_1^{s-1}}{(t_1 + 1)^{2s}} dt_1 \tag{4.8}$$

with $A = |\beta/\tilde{\beta}|^2 t_0^2$ and $B = A\varepsilon^4$. Any $\beta \in \mathfrak{a}\omega_1^{-1}\backslash\{0\}$ is written as $(\beta)_{\mathfrak{f}}\varepsilon_0^j\zeta$ where $\{(\beta)_{\mathfrak{f}}\}$ are complete set of representatives for the non-associated classes of $\mathfrak{a}\omega_1^{-1}\backslash\{0\}$ modulo $E_L(\mathfrak{f})$, $j \in Z$ and ζ is a root of unity in $E_L(\mathfrak{f})$. Note that $e(\mathrm{Tr}_{L/Q}(\beta\omega_1))|N_{L/Q}(\beta)|^{-s}$ is invariant when β is replaced by $\beta\alpha$ with $\alpha \in E_L(\mathfrak{f})$. Thus we get

$$c_0 = \frac{w_L(\mathfrak{f})|\omega - \tilde{\omega}|^{2s}}{2} N_{k/Q}(\mathfrak{n})^s \sum_{(\beta)_{\mathfrak{f}}}{}'' \frac{e(\mathrm{Tr}_{L/Q}(\beta\omega_1))}{|N_{L/Q}(\beta)|^s} \sum_{j=-\infty}^{\infty} \int_{A_j}^{B_j} \frac{t_1^{s-1}}{(t_1 + 1)^{2s}} dt_1 \tag{4.9}$$

with $A_j = |(\beta\varepsilon_0^j)/(\tilde{\beta}\tilde{\varepsilon}_0^j)|^2 t_0^2$ and $B_j = A_j\varepsilon^4$ for $j \in Z$. By Lemma 4.1, we see that $A_j = |\beta/\tilde{\beta}|^2 t_0^2 \varepsilon^{(2g/e)j}$ with $g = 1$ or 2 and hence

$$\sum_{j=-\infty}^{\infty} \int_{A_j}^{B_j} \frac{t_1^{s-1}}{(t_1 + 1)^{2s}} dt_1 = \frac{2e}{g} \frac{\Gamma(s)^2}{\Gamma(2s)}. \tag{4.10}$$

Since $\begin{Vmatrix} \omega_1 & \omega_2 \\ \tilde{\omega}_1 & \tilde{\omega}_2 \end{Vmatrix}^2 = d_2 N_{k/Q}(\mathfrak{n})^{-1} N_{L/Q}(\mathfrak{a})$, we get

$$|\omega - \tilde{\omega}|^2 = \frac{d_2}{N_{k/Q}(\mathfrak{n})} \frac{N_{L/Q}(\mathfrak{a})}{|N_{L/Q}(\omega_1)|}. \tag{4.11}$$

Substituting (4.10), (4.11) in (4.9), we find

$$c_0 = \frac{\Gamma(s)^2}{\Gamma(2s)} \frac{e w_L(\mathfrak{f}) d_2^s}{g} \Psi(C, s).$$

Recalling $R_L(\mathfrak{f}) = (g/e)\log\varepsilon$, we obtain (4.7).

Consequently, combining Theorems 2.1, 2.2 with Theorem 4.3, we get

THEOREM 4.4. *Let C be any ray class modulo $\mathfrak{f}$ in L and let $\mathfrak{c} \in C$ be an integral ideal prime to $\mathfrak{f}$. We write $\mathfrak{a} = \mathfrak{c}/(\vartheta_L\mathfrak{f}) = \mathfrak{g}\omega_1 + \mathfrak{n}\omega_2$ as $\mathfrak{g}$-module where $\mathfrak{n}$ is an ideal in k. Put $\omega = \omega_1^{-1}\omega_2$ and $u = \mathrm{Tr}_{L/k}(\omega_j)$ $(j = 1, 2)$.*

Let $\Psi(C, s)$ *be as in* (3.2) *and let* $\rho(t)$ *be the curve defined by* (4.6).

(i) *If* $\mathfrak{f} = (1)$, *we have*

$$\lim_{s\to 1}\left\{\Psi(C, s) - \frac{4\pi^2 R_L(1)}{w_L(1)d_1d_2}\frac{1}{s-1}\right\} \tag{4.12}$$
$$= \frac{4\pi^2 R_L(1)}{w_L(1)d_1d_2}\left\{\frac{w_k\sqrt{d_1}}{\pi}\alpha_0(\mathfrak{n}^{-1}) - \log d_2 - \log N_{k/Q}(\mathfrak{n})\right.$$
$$\left. - \frac{1}{2\log\varepsilon}\int_{t_0}^{t_0\varepsilon^2}\{\log v(t)^2 - h_{\mathfrak{g},\mathfrak{n}}(\rho(t))\}\frac{dt}{t}\right\}$$

where $h_{\mathfrak{g},\mathfrak{n}}(\xi)$ *is given by* (2.3) *with* $\mathfrak{m} = \mathfrak{g}$.

(ii) *If* $\mathfrak{f} \neq (1)$, *we have*

$$\Psi(C, 1) = \frac{R_L(\mathfrak{f})}{w_L(\mathfrak{f})d_2}\frac{1}{\log\varepsilon}\int_{t_0}^{t_0\varepsilon^2}\psi_{\mathfrak{g},\mathfrak{n}}(-u_1\rho(t) + u_2, \rho(t))\frac{dt}{t} \tag{4.13}$$

where $\psi_{\mathfrak{g},\mathfrak{n}}(\zeta, \xi)$ *is given by* (2.6) *with* $\mathfrak{m} = \mathfrak{g}$. *In the above,* $t_0 > 0$ *is any real number.*

§ 5. Computations of the integral

In this section we shall compute the integrals in Theorem 4.4. To proceed the computations, we take $t_0 = \varepsilon'$, $t_0\varepsilon^2 = \varepsilon$. Put

$$I_1 = \int_{\varepsilon'}^{\varepsilon}\{\log v(t)^2 - h_{\mathfrak{g},\mathfrak{n}}(\rho(t))\}\frac{dt}{t} \tag{5.1}$$

$$I_2 = \int_{\varepsilon'}^{\varepsilon}\psi_{\mathfrak{g},\mathfrak{n}}(-u_1\rho(t) + u_2, \rho(t))\frac{dt}{t} \tag{5.2}$$

where $\rho(t) = z(t) + v(t)j$ $(t > 0)$ is given by (4.6). We write $\nu = (1/2)(\omega - \tilde{\omega})$ and for any $p \in \boldsymbol{C}^\times$, $q \in \boldsymbol{C}$, we define

$$H(p, q) = \int_{\varepsilon'}^{\varepsilon} v(t)K_1(4\pi|p|v(t))(e[-pz(t) - q] + e[pz(t) + q])\frac{dt}{t}. \tag{5.3}$$

Step 1. We show that the problem is reduced to the computation of $H(p, q)$. It is easy to see that

$$\int_{\varepsilon'}^{\varepsilon} v(t)^2\frac{dt}{t} = 2|\nu|^2\frac{\varepsilon^2 - 1}{\varepsilon^2 + 1}. \tag{5.4}$$

LEMMA 5.1. *We have*

$$\int_{\varepsilon'}^{\varepsilon}\log v(t)^2\frac{dt}{t} = \log(4|\nu|^2)\cdot\log\varepsilon^2 - 2(\log\varepsilon)^2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(1 - \varepsilon^{-2n}). \tag{5.5}$$

Proof. Since $v(t) = 2|\nu|t/(t^2+1)$, we write $\int_{\varepsilon'}^{\varepsilon} \log v(t)^2(dt/t) = 2\log(2|\nu|) \times \int_{\varepsilon'}^{\varepsilon}(dt/t) + 2\int_{\varepsilon'}^{\varepsilon}\log t(dt/t) - 2\int_{\varepsilon'}^{\varepsilon}\log(1+t^2)(dt/t)$. The first (second) term is $\log(4|\nu|^2)\cdot\log\varepsilon^2$ (0, respectively). As to the third term, we write $\int_{\varepsilon'}^{\varepsilon} = \int_{\varepsilon'}^{1} + \int_{1}^{\varepsilon}$. Replacing t^{-1} for t in $\int_{1}^{\varepsilon}$, we get

$$2\int_{\varepsilon'}^{\varepsilon}\log(1+t^2)\frac{dt}{t} = 4\int_{\varepsilon'}^{1}\log(1+t^2)\frac{dt}{t} - 4\int_{\varepsilon'}^{1}\log t\frac{dt}{t}.$$

Since $\log(1+X) = \sum_{n=1}^{\infty}((-1)^{n-1}/n)X^n$ (uniformly convergent for $0 \leqq X \leqq 1$), we obtain

$$2\int_{\varepsilon'}^{\varepsilon}\log(1+t^2)\frac{dt}{t} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}(1-\varepsilon^{-2n}) + 2(\log\varepsilon)^2.$$

This proves (5.5).

Note that $\tau_{1/2}(\mathfrak{g}, \mathfrak{n}, n) = \tau_{1/2}(\mathfrak{g}, \mathfrak{n}, -n)$. In (2.3), let $\mathfrak{m} = \mathfrak{g}$ and take the summation "$0 \neq n \in \mathfrak{n}/\{\pm 1\}$" for "$0 \neq n \in \mathfrak{n}$". By (5.1), (5.3), (5.4), (5.5), we get

$$\begin{aligned}(5.6)\qquad I_1 = {} & 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(1-\varepsilon^{-2n}) - 2(\log\varepsilon)^2 \\ & + \log\varepsilon^2\cdot\log(4|\nu|^2) - \frac{w_k d_1}{\pi^2}\frac{\varepsilon^2-1}{\varepsilon^2+1}N_{k/Q}(\mathfrak{n})\zeta_k((\mathfrak{g}), 2)|\nu|^2 \\ & - 4\sum_{0\neq n\in\mathfrak{n}/\{\pm1\}}{}' \tau_{1/2}(\mathfrak{g}, \mathfrak{n}, n)H(n/\sqrt{-d_1}, 0).\end{aligned}$$

Similarly, taking $\mathfrak{m} = \mathfrak{g}$ and $\zeta = -u_1\xi + u_2$ in (2.6), we get

$$\begin{aligned}(5.7)\qquad I_2 = {} & \log\varepsilon^2\cdot b(u_1, u_2) + 2\frac{\varepsilon^2-1}{\varepsilon^2+1}N_{k/Q}(\mathfrak{n})G_{\mathfrak{g}}(2, u_1)|\nu|^2 \\ & + \frac{8\pi^2}{\sqrt{d_1}}N_{k/Q}(\mathfrak{n})\sum_{0\neq n\in\mathfrak{n}/\{\pm1\}}{}'\sum_{u_1\neq m\in\mathfrak{g}}{}'\left|\frac{m-u_1}{n}\right|H(n(m-u_1), nu_2).\end{aligned}$$

Thus it is sufficient to compute $H(p, q)$. To this purpose, we consider the differential form on $\boldsymbol{H}$ whose integral along the path $\rho(t)$ ($\varepsilon' \leqq t \leqq \varepsilon$) contains $H(p, q)$.

Step 2. We construct certain closed form on $\boldsymbol{H}$. Let B_1 be as in §1 and let $\{-v^{-1}dz, v^{-1}dv, v^{-1}d\bar{z}\}$ be a basis for the left B_1 invarinat forms on $\boldsymbol{H}$. We write

$$(5.8) \qquad \eta = K_1(4\pi v)e[-z]\frac{dz}{v} - 2iK_2(4\pi v)e[-z]\frac{dv}{v} + K_1(4\pi v)e[-z]\frac{d\bar{z}}{v}.$$

Since $(d/dX)(X^{-1}K_1(X)) = -X^{-1}K_2(X)$, η is a closed form. For $p \in \boldsymbol{C}^{\times}$, $q \in \boldsymbol{C}$, let $\varphi_{p,q}$ be the transformation $\xi \to p^{-1/2}\begin{pmatrix} p & q \\ 0 & 1 \end{pmatrix}(\xi)$ on $\boldsymbol{H}$. Let $(\varphi_{p,q})^*$ be the linear map of the cotangent space on $\boldsymbol{H}$ induced by $\varphi_{p,q}$. We get

$$(5.9) \qquad (\varphi_{p,q})^*(\eta) = \frac{p}{|p|}K_1(4\pi|p|v)e[-pz-q]\frac{dz}{v} + \frac{\bar{p}}{|p|}K_1(4\pi|p|v)e[-pz-q]\frac{d\bar{z}}{v} - 2iK_2(4\pi|p|v)e[-pz-q]\frac{dv}{v}.$$

Then $(\varphi_{p,q})^*(\eta) - (\varphi_{-p,-q})^*(\eta)$ is the closed form what we wanted.

Let us now compute

$$(5.10) \qquad J = \int_{\rho(\varepsilon')}^{\rho(\varepsilon)} (\varphi_{p,q})^*(\eta) - (\varphi_{-p,-q})^*(\eta).$$

As we have seen above J does not depend on the choice of the path joining $\rho(\varepsilon')$ and $\rho(\varepsilon)$. We write $\rho(\varepsilon') = x_0 + y_0 i + v_0 j$, $\rho(\varepsilon) = x_0^* + y_0^* i + v_0 j$, $z_0 = x_0 + y_0 i$ and $z_0^* = x_0^* + y_0^* i$. Let κ be the broken line joining $\rho(\varepsilon') \to x_0^* + y_0 i + v_0 j \to \rho(\varepsilon)$.

Step 3. We compute J along κ.

LEMMA 5.2. *We have*

$$(5.11) \qquad J = \int_{\kappa} (\varphi_{p,q})^*(\eta) - (\varphi_{-p,-q})^*(\eta) = \frac{2}{\pi|p|v_0}K_1(4\pi|p|v_0)\sin\left(2\pi S(p\nu)\frac{\varepsilon^2-1}{\varepsilon^2+1}\right)\cos(\pi S(p\omega + p\tilde{\omega} + 2q)).$$

Proof. The choice of κ implies that

$$J = \frac{2}{|p|v_0}K_1(4\pi|p|v_0)\int_{\kappa}\cos(2\pi S(pz+q))p\,dz + \cos(2\pi S(pz+q))\bar{p}\,d\bar{z}.$$

Substitute $p = p_1 + p_2 i$, $q = q_1 + q_2 i$ and $z = x + yi$ with $p_j, q_j, x, y \in \boldsymbol{R}$ $(j = 1, 2)$. By a direct computations, we get

$$J = \frac{2}{\pi|p|v_0}K_1(4\pi|p|v_0)\sin(\pi S(pz_0^* - pz_0))\cos(\pi S(pz_0^* + pz_0 + 2q)).$$

Note that $z_0 = (\varepsilon^2\tilde{\omega} + \omega)/(\varepsilon^2 + 1)$, $z_0^* = (\varepsilon^2\omega + \tilde{\omega})/(\varepsilon^2 + 1)$, $S(pz_0^* - pz_0) = 2S(p\nu)(\varepsilon^2 - 1)/(\varepsilon^2 + 1)$ and $S(pz_0^* + pz_0 + 2q) = S(p\omega + p\tilde{\omega} + 2q)$. From this we find (5.11).

Step 4. We obtain another expression for J which contains $H(p, q)$. Regarding $\rho = \rho(t)$ as the C^∞-map of $\boldsymbol{R}^+$ into $\boldsymbol{H}$, let ρ^* be the associated linear map from the cotangent space on $\boldsymbol{H}$ to that on $\boldsymbol{R}^+$. By a little computation, we get

LEMMA 5.3. *We have*

$$\begin{aligned} &\rho^*(v^{-2}dz) = (\bar{\nu}t)^{-1}dt\,, \qquad \rho^*(v^{-2}d\bar{z}) = (\nu t)^{-1}dt \\ &\rho^*(v^{-2}dv) = (1 - t^2)(2|\nu|t^2)^{-1}dt\,. \end{aligned} \tag{5.12}$$

By (5.9) and Lemma 5.3, we get

$$\begin{aligned} &\rho^*((\varphi_{p,q})^*(\eta) - (\varphi_{-p,-q})^*(\eta)) \\ &= \Big(\frac{p}{\bar{\nu}|p|} + \frac{\bar{p}}{\nu|p|}\Big)v(t)K_1(4\pi|p|v(t))(e[-pz(t) - q] + e[pz(t) + q])\frac{dt}{t} \\ &\quad - 2iv(t)K_2(4\pi|p|v(t))(e[-pz(t) - q] - e[pz(t) + q])\frac{1 - t^2}{2|\nu|t^2}dt\,. \end{aligned}$$

Note that $p/\bar{\nu}|p| + \bar{p}/\nu|p| = |p|S((p\nu)^{-1})$. By (5.3), the integral (5.10) taken along the path $\rho(t)$ ($\varepsilon' \leqq t \leqq \varepsilon$) is given by

$$J = |p|S((p\nu)^{-1})H(p, q) - J_1 \tag{5.13}$$

where J_1 is

$$J_1 = 4\int_{\varepsilon'}^{\varepsilon} v(t)K_2(4\pi|p|v(t))\sin(2\pi S(pz(t) + q))\frac{1 - t^2}{2|\nu|t^2}dt\,. \tag{5.14}$$

Step 5. Computation of J_1. We write $J_1 = 4\Big(\int_{\varepsilon'}^{1} + \int_{1}^{\varepsilon}\Big)$. Replacing t by t^{-1} in $\int_1^\varepsilon$, we find that

$$\begin{aligned} J_1 = 4&\int_{\varepsilon'}^{1} v(t)K_2(4\pi|p|v(t)) \\ &\times \{\sin(2\pi S(pz(t) + q)) - \sin(2\pi S(pz(t^{-1}) + q))\}\frac{1 - t^2}{2|\nu|t^2}dt\,. \end{aligned}$$

Since $z(t) + z(t^{-1}) = \omega + \tilde{\omega}$ and $z(t) - z(t^{-1}) = -2\nu(1 - t^2)/(1 + t^2)$, we get

$$\begin{aligned} J_1 = &-8\cos(\pi S(p\omega + p\tilde{\omega} + 2q))\int_{\varepsilon'}^{1} v(t)K_2(4\pi|p|v(t)) \\ &\times \sin\Big(2\pi S(p\nu)\frac{1 - t^2}{1 + t^2}\Big)\frac{1 - t^2}{2|\nu|t^2}dt\,. \end{aligned}$$

For $0 < t \leqq 1$, $v(t) = 2|\nu|t/(1+t^2)$ is the increasing function and we see that $(1-t^2)/(1+t^2) = \sqrt{1-(v(t)/|\nu|)^2}$. Hence we can rewrite J_1 as an integral in v. Furthermore, replacing $4\pi|p|v$ by v, we get

$$(5.15)\qquad J_1 = -8\cos(\pi S(p\omega + p\tilde{\omega} + 2q)) \times \int_{4\pi|p|v_0}^{4\pi|p\nu|} v^{-1}K_2(v)\sin\left(2\pi S(p\nu)\sqrt{1-\left(\frac{v}{4\pi|p\nu|}\right)^2}\right)dv$$

where $v_0 = v(\varepsilon) = v(\varepsilon') = 2|\nu|\varepsilon/(1+\varepsilon^2)$.

LEMMA 5.4. *Let α and β be real numbers with $\beta > 0$. Let $F(v, \alpha, \beta)$ be the indefinite integral of the function $f(v) = v^{-1}K_2(v)\sin(\alpha\sqrt{1-(\beta v)^2})$ for $0 < v \leqq \beta^{-1}$. Then we have*

$$(5.16)\qquad F(v, \alpha, \beta) = -\sin\alpha\cdot v^{-1}K_1(v) + \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j-1)!}\alpha^{2j-1}\sum_{k=1}^{\infty}\binom{j-1/2}{k}\beta^{2k}(2kvK_2(v)\cdot iS_{2k-2,1}(iv) + vK_1(v)\cdot S_{2k-1,2}(iv))$$

where $S_{m,n}(Z)$ are the Lommel's functions satisfying inhomogeneous Bessel differential equations

$$(5.17)\qquad Z^2\frac{d^2S}{dZ^2} + Z\frac{dS}{dZ} + (Z^2 - n^2)S = Z^{m+1} \qquad ([8], \text{ p. } 108\text{–}109).$$

Proof. By the Taylor expansion of $\sin(\alpha\sqrt{1-(\beta v)^2})$, we see that

$$(5.18)\qquad f(v) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j-1)!}\alpha^{2j-1}v^{-1}K_2(v)(1-(\beta v)^2)^{j-1/2}.$$

The series converges uniformly on any closed interval $[A, B]$ in $(0, \beta^{-1}]$. The integration $\int_B^A f(v)dv$ can be done term by term. Since $j \geqq 1$, $\sum_{k=0}^{\infty}(-1)^k\binom{j-1/2}{k}(\beta v)^{2k}$ converges uniformly to $(1-(\beta v)^2)^{j-1/2}$ $(0 \leqq v \leqq \beta^{-1})$ by Abel's theorem. Thus, for any $[A, B] \subset (0, \beta^{-1}]$, we get

$$(5.19)\qquad \int_A^B v^{-1}K_2(v)(1-(\beta v)^2)^{j-1/2}dv = \sum_{k=0}^{\infty}(-1)^k\binom{j-1/2}{k}\beta^{2k}\int_A^B v^{2k-1}K_2(v)dv.$$

Recall that

$$(5.20)\qquad \int_A^B v^{-1}K_2(v)dv = -v^{-1}K_1(v)\Big|_A^B$$

$$(5.21)\qquad \int_A^B v^{2k-1}K_2(v)dv = (-1)^k\{2kvK_2(v)\cdot iS_{2k-2,1}(iv) + vK_1(v)\cdot S_{2k-1,2}(iv)\}\Big|_A^B \qquad \text{for } k \geqq 1$$

([8], p. 87). By (5.18), (5.19), (5.20), (5.21), we find that

$$\int_A^B f(v)dv = F(B, \alpha, \beta) - F(A, \alpha, \beta).$$

Let $F(v, \alpha, \beta)$ be as in Lemma 5.4. For any $\lambda \in C^\times$ and for any v satisfying $0 < v \leqq 4\pi|\lambda|$, we define $F_\lambda(v)$ by putting

$$(5.22)\quad \begin{aligned} F_\lambda(v) &= F(v, 2\pi S(\lambda), (4\pi|\lambda|)^{-1}) \\ &= -\sin(2\pi S(\lambda))\cdot v^{-1}K_1(v) + \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j-1)!}(2\pi S(\lambda))^{2j-1} \\ &\quad \times \sum_{k=1}^{\infty}\binom{j-1/2}{k}(4\pi|\lambda|)^{-2k}\{2kvK_2(v)\cdot iS_{2k-2,1}(iv) + vK_1(v)\cdot S_{2k-1,2}(iv)\}. \end{aligned}$$

Then, in view of (5.15), Lemma 5.4 and (5.22), we get

$$(5.23)\quad J_1 = -8\left\{F_{p\nu}(4\pi|p\nu|) - F_{p\nu}\left(\frac{8\varepsilon\pi|p\nu|}{\varepsilon^2+1}\right)\right\}\cos(\pi S(p\omega + p\tilde{\omega} + 2q)).$$

Consequently, by (5.11), (5.13), (5.23), we obtain

PROPOSITION 5.5. *Notations being as above. Then we have*

$$(5.24)\quad \begin{aligned} H(p, q) = \frac{1}{|p|S(1/p\nu)}&\left\{\frac{\varepsilon^2+1}{\varepsilon\pi|p\nu|}K_1\left(\frac{8\varepsilon\pi|p\nu|}{\varepsilon^2+1}\right)\sin\left(2\pi S(p\nu)\frac{\varepsilon^2-1}{\varepsilon^2+1}\right)\right. \\ &\left. - 8F_{p\nu}(4\pi|p\nu|) + 8F_{p\nu}\left(\frac{8\varepsilon\pi|p\nu|}{\varepsilon^2+1}\right)\right\}\cos(\pi S(p\omega + p\tilde{\omega} + 2q)). \end{aligned}$$

In particular, if $\mathfrak{f} = (1)$ *and* $0 \neq n \in \mathfrak{n}$, *then we have*

$$(5.25)\quad \begin{aligned} H(n/\sqrt{-d_1}, 0) = \frac{\sqrt{d_1}}{|n|S(\sqrt{-d_1}/(n\nu))}&\left\{\frac{\sqrt{d_1}(\varepsilon^2+1)}{\varepsilon\pi|n\nu|}K_1\left(\frac{8\varepsilon\pi|n\nu|}{\sqrt{d_1}(\varepsilon^2+1)}\right)\right. \\ &\times\sin\left(2\pi S(n\nu/\sqrt{-d_1})\frac{\varepsilon^2-1}{\varepsilon^2+1}\right) - 8F_{n\nu/\sqrt{-d_1}}(4\pi|n\nu|/\sqrt{d_1}) \\ &\left. + 8F_{n\nu/\sqrt{-d_1}}\left(\frac{8\varepsilon\pi|n\nu|}{\sqrt{d_1}(\varepsilon^2+1)}\right)\right\}\cos(\pi\,\mathrm{Tr}_{L/Q}(n\omega/\sqrt{-d_1})). \end{aligned}$$

If $\mathfrak{f} \neq (1)$, *then for any* $(m, n) \in \mathfrak{g} \oplus \mathfrak{n}$ *satisfying* $n(m - u_1) \neq 0$ *we have*

$$(5.26)\quad \begin{aligned} H(n(m-u_1), nu_2) &= \frac{1}{|n(m-u_1)|S((n\nu(m-u_1))^{-1})} \\ &\times\left\{\frac{\varepsilon^2+1}{\varepsilon\pi|n\nu(m-u_1)|}K_1\left(\frac{8\varepsilon\pi|n\nu(m-u_1)|}{\varepsilon^2+1}\right)\sin\left(2\pi S(n\nu(m-u_1))\frac{\varepsilon^2-1}{\varepsilon^2+1}\right)\right. \\ &\left. - 8F_{n\nu(m-u_1)}(4\pi|n\nu(m-u_1)|) + 8F_{n\nu(m-u_1)}\left(\frac{8\varepsilon\pi|n\nu(m-u_1)|}{\varepsilon^2+1}\right)\right\} \\ &\times\cos(\pi\,\mathrm{Tr}_{L/Q}(n(m-u_1)\omega + nu_2)). \end{aligned}$$

Finally, we obtained

THEOREM 5.6. *Let C be any absolute ideal class in L. For an integral ideal $\mathfrak{c} \in C$, we write $\mathfrak{a} = \mathfrak{c}/\vartheta_L = \mathfrak{g}\omega_1 + \mathfrak{n}\omega_2$ (as $\mathfrak{g}$-module), where $\mathfrak{n}$ is an ideal in k. We put $\omega = \omega_1^{-1}\omega_2$ and $\nu = \frac{1}{2}(\omega - \bar{\omega})$. Let $\Psi(C, s)$ be the function defined by (3.2) with $\mathfrak{f} = (1)$. Then we have*

$$(5.27)\quad \Psi(C, s) = \frac{4\pi^2 R_L(1)}{w_L(1)d_1 d_2}\Big\{\frac{1}{s-1} + \frac{w_k\sqrt{d_1}}{\pi}\alpha_0(\mathfrak{n}^{-1}) - \log d_2$$
$$- \log N_{k/Q}(\mathfrak{n}) + \frac{1}{\log \varepsilon}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}(1 - \varepsilon^{-2n}) + \log \varepsilon$$
$$- \log 4 - \log|\nu|^2 + \frac{w_k d_1}{2\pi^2 \log \varepsilon}\,\frac{\varepsilon^2 - 1}{\varepsilon^2 + 1}N_{k/Q}(\mathfrak{n})\zeta_k((\mathfrak{g}), 2)|\nu|^2$$
$$+ \frac{2}{\log \varepsilon}\sum_{0 \neq n \in \mathfrak{n}/\{\pm 1\}}{}' \tau_{1/2}(\mathfrak{g}, \mathfrak{n}, n)H(n/\sqrt{-d_1}, 0)\Big\} + O(|s-1|)$$

where $H(n/\sqrt{-d_1}, 0)$ are given by (5.25).

THEOREM 5.7. *Let $\mathfrak{f} \neq (1)$ be any integral ideal in L and let C be any ray class modulo $\mathfrak{f}$ in L. Suppose $\mathfrak{c} \in C$ is an integral ideal which is prime to $\mathfrak{f}$. Put $\mathfrak{a} = \mathfrak{c}/(\vartheta_L\mathfrak{f}) = \mathfrak{g}\omega_1 + \mathfrak{n}\omega_2$ (as $\mathfrak{g}$-module), where $\mathfrak{n}$ is an ideal in k. Further we put $u_j = \mathrm{Tr}_{L/k}(\omega_j)$ $(j = 1, 2)$, $\omega = \omega_1^{-1}\omega_2$ and $\nu = \frac{1}{2}(\omega - \bar{\omega})$. Let $\Psi(C, s)$ be the function defined by (3.2). Then the function $\Psi(C, s)$ is holomorphic at $s = 1$ and we have*

$$(5.28)\quad \Psi(C, 1) = \frac{2R_L(\mathfrak{f})}{w_L(\mathfrak{f})d_2}\Big\{b(u_1, u_2) + \frac{1}{\log \varepsilon}\,\frac{\varepsilon^2 - 1}{\varepsilon^2 + 1}N_{k/Q}(\mathfrak{n})G_{\mathfrak{g}}(2, u_1)|\nu|^2$$
$$+ \frac{4\pi^2}{\sqrt{d_1}\log \varepsilon}N_{k/Q}(\mathfrak{n})\sum_{0 \neq n \in \mathfrak{n}/\{\pm 1\}}{}'\ \sum_{u_1 \neq m \in \mathfrak{g}}{}' \left|\frac{m - u_1}{n}\right| H(n(m - u_1), nu_2)\Big\}$$

where $H(n(m - u_1), nu_2)$ are given in (5.26).

Remark. In the case of imaginary quadratic field $\boldsymbol{Q}(\sqrt{-d})$ ($-d$; the discriminant), the Kronecker limit formula was given by

$$\zeta(s, A) = \frac{2\pi}{w\sqrt{d}}\Big\{\frac{1}{s-1} + 2\gamma - \log\sqrt{d} - \log 2 - \log y - 2\log|\eta(z)|^2\Big\}$$
$$+ O(|s-1|) \qquad (\gamma;\ \text{Euler constant})$$

Here A is an absolute ideal class; $\mathfrak{b} \in A$ is an ideal with $\boldsymbol{Z}$-basis $[1, z]$, $z = x + yi$ $(y > 0)$; w is the number of roots of unity in $\boldsymbol{Q}(\sqrt{-d})$ and

$$- \log|\eta(z)|^2 = \frac{\pi}{6}y + 2\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n y}\cos(2\pi n x)\,.$$

The formula (5.27) may be regarded as a generalization of this. In fact, $\mu = \frac{1}{2}(\omega + \tilde{\omega})$ and $\nu = \frac{1}{2}(\omega - \tilde{\omega})$ corresponds to x and yi, respectively. The function

$$\Phi(\omega, \tilde{\omega}) = \frac{\varepsilon^2 - 1}{\varepsilon^2 + 1} \cdot \frac{w_k d_1}{\pi^2} N_{k/Q}(\mathfrak{n})\zeta_k((\mathfrak{g}), 2)|\nu|^2 + 4 \sum_{0 \neq n \in \mathfrak{n}/\{\pm 1\}}' \tau_{1/2}(\mathfrak{g}, \mathfrak{n}, n) H(n/\sqrt{-d_1}, 0)$$

(the Fourier cosine series in μ whose Fourier coefficients are the functions of ν), can be considered to be an analogy of $-\log|\eta(z)|^2$.

REFERENCES

[1] T. Asai, On a certain function analogous to $\log|\eta(z)|$, Nagoya Math. J., **40** (1970), 193–211.

[2] A. F. Beardon, The geometry of discrete groups; Discrete groups and automorphic functions, Proc. Conf. Univ. Cambridge (1975).

[3] J. Elstrodt, F. Grunewald, J. Mennicke, Eisenstein series on three-dimensional hyperbolic space and imaginary quadratic fields, J. reine angew. Math., **360** (1985).

[4] G. Fujisaki, Algebraic number theory (in Japanese), Shōkabō (1974).

[5] E. Hecke, Mathematische Werke, Göttingen (1959).

[6] S. Konno, On Kronecker limit formula for certain biquadratic fields, Advanced Studies in Pure Math., **13** (1988), 297–311.

[7] T. Kubota, Über diskontinuierliche Gruppen vom Picardschen Typus und zugehörige Eisensteinsche Reihen, Nagoya Math. J., **32** (1968), 259–271.

[8] W. Magnus, F. Oberhettinger, R. P. Soni, Formulas and theorems for the special functions of mathematical physics, Grundl. math. Wiss. 52, Berlin-Heidelberg-New York (1966).

[9] C. L. Siegel, Lecture notes on advanced analytic number theory, Bombay (1961).

[10] ——, Bernoullische Polynome und quadratische Zahlkörper, Gesammelte Abhandlungen Bd IV.

[11] A. Weil, Dirichlet series and automorphic forms, Lecture notes in Math. 189, Springer-Verlag (1971).

[12] ——, Elliptic functions according to Eisenstein and Kronecker, Berlin-Heidelberg-New York (1976).

Kobe Women's College of Pharmacy
Motoyama Kitamachi 4-19-1
Higashinada-ku, Kobe 657, Japan