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Nagoya Math. J.
Vol. 105 (1987), 153-167

ON THE CUBIC THETA FUNCTION

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Introduction

The generalized theta function of a totally imaginary field including n-th roots of unity, which was defined by T. Kubota [2], was introduced in his investigation of the reciprosity law of the n-th power residue. If $n = 2$, it reduces to the classical theta function. In the case $n = 3$ for the Eisenstein field, the Fourier coefficients of the cubic theta function, which were explicitly expressed by S.J. Patterson, are essentially cubic Gauss sums [3]. Furthermore in the case $n = 4$ for the Gaussian field those of the biquadratic theta functions, which have been investigated by T. Suzuki [4], haven't been obtained completely yet.

The main purpose of the present paper is to construct the cubic theta function based on Weil's idea [5]. In this procedure Davenport-Hasse's formula is used, which implies the multiplicative property of the Gauss sums and corresponds to Gauss's multiplicative formula of Gamma functions that is also used in this process. This fact may be of some importance in the study of the Gauss sums with respect to the character of a general n-th power residue. Our method is far simpler than that of [3], although the automorphic property of the cubic theta function is proved for a slightly smaller discontinuous group than in the latter.

This paper consists of five sections. Since some of the ideas and the technique used in this paper are based on those of T. Kubota [2], S.J. Patterson [3], T. Suzuki [4] and A. Weil [5], overlapping arguments will be described roughly.

§ 1. $\Gamma(N)$

We denote by $\boldsymbol{Q}$ and $\boldsymbol{C}$ the field of the rational numbers and the field of the complex numbers respectively. Let $\boldsymbol{Z}$ be the ring of the ra-

Received October 9, 1985.

tional integers, $\lambda=\sqrt{-3}$, $O=\boldsymbol{Z}(\omega)$ ($\omega=\exp(2\pi i/3)$) and $e(z)=\exp(2\pi i(z+\bar{z}))$ for $z\in\boldsymbol{C}$. We define the upper half space $H=\boldsymbol{C}\times\boldsymbol{R}_+^\times$, where $\boldsymbol{R}_+^\times$ is a multiplicative group of positive numbers. If we regard $w\ (\in H)$ as $\begin{bmatrix} z & -v \\ v & \bar{z}\end{bmatrix}$ ($z\in\boldsymbol{C}, v\in\boldsymbol{R}_+^\times$) and put $\tilde{z}=\begin{bmatrix} z & 0 \\ 0 & \bar{z}\end{bmatrix}$ ($z\in\boldsymbol{C}$), then $SL(2,\boldsymbol{C})$ acts on H by

$$\sigma(w)=(\tilde{a}w+\tilde{b})(\tilde{c}w+\tilde{d})^{-1}\left(\sigma=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right). \tag{1.1}$$

Defining that the operation on H of the diagonal matrix $\begin{bmatrix} z & 0 \\ 0 & z\end{bmatrix}$, ($z\in\boldsymbol{C}$), is trivial, we obtain the operation on H of $GL(2,\boldsymbol{C})$.

Furthermore if we put

$$\Gamma(N)=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL(2,O)\middle|\begin{bmatrix} a & b \\ c & d\end{bmatrix}\equiv\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\ (\mathrm{mod}\ N)\right\} \tag{1.2}$$

for any $N\in O$, then $\Gamma(N)$ acts on H discontinuously and has a fundamental domain with the finite volume with respect to the invariant measure $v^{-3}dxdydv$, where $z=x+iy\in\boldsymbol{C}$, $v\in\boldsymbol{R}_+^\times$.

We put through the paper

$$N=3 \quad \text{or} \quad 3r\,, \tag{1.3}$$

where r is a prime number of degree 1 such that $r\equiv 1$ (9). We define for σ $\left(=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\Gamma(N)\right)$

$$\begin{aligned}\chi(\sigma)&=\left(\frac{c}{a}\right)_3 \qquad (c\neq 0)\,,\\ &=1 \qquad\quad (c=0)\,,\end{aligned} \tag{1.4}$$

where $\left(\frac{c}{a}\right)_3$ is the cubic residue symbol in $\boldsymbol{Q}(\omega)$. Then χ is a character of $\Gamma(N)$.

Let κ be a cusp of $\Gamma(N)$ i.e. $\kappa\in\boldsymbol{Q}(\omega)$ of $\kappa=\infty=1/0$. We write $\kappa=\alpha/\gamma$ ($(\alpha,\gamma)=1$). If $(\alpha,\lambda)=1$, then we may assume that $\alpha\equiv 1$ (3). Also, if $(\alpha,\lambda)=1$, then we may assume that $\gamma\equiv 1$ (3). From the assumptions we easily have

LEMMA 1. *Two cusps $\kappa=\alpha/\gamma$, $\kappa'=\alpha'/\gamma'$ are equivalent to each other under $\Gamma(N)$ if and only if $\alpha\equiv\alpha'$ (N), $\gamma\equiv\gamma'$ (N).*

We put for any cusp $\kappa=\alpha/\gamma$ of $\Gamma(N)$

$$\Gamma_\kappa=\{\sigma\in\Gamma(N)\,|\,\sigma\kappa=\kappa\}\,.$$

We note that $\Gamma_\kappa=\sigma_\kappa\Gamma_\infty\sigma_\kappa^{-1}$ putting $\kappa=\sigma_\kappa(\infty)$ for $\sigma_\kappa(\in SL(2,O))$.

A cusp κ is called essential if the restriction to Γ_κ of χ is trivial. We can confirm that the essential cusps of $\Gamma(N)$ depend only on the equivalence of $\Gamma(3)$. We readily check that the set of the essential cusps of $\Gamma(3)$ is $\{0,1,-1,\infty\}$. We further classify those of $\Gamma(N)$ $(N=3r)$ in the following four cases after T. Suzuki. For $\kappa=\alpha/\gamma$

(1) A-type i.e. $\alpha\equiv 0\ (N)$,

(2) B-type i.e. $\alpha\equiv 0\ (3)$, $(\alpha, r)=1$.

(3) C-type i.e. $\alpha\equiv 1\ (3)$, $\alpha\equiv 0\ (r)$,

(4) D-type i.e. $\alpha\equiv 1\ (3)$, $(\alpha, r)=1$.

§ 2. Eisenstein series

For any 2×2 matrix M we consider ℓ-fold tensor product M_ℓ, we put

$$M_\ell^*=\overline{M}_\ell^t\ (^-=\text{complex conjugate},\ ^t=\text{transposition})\,,$$

and for $g=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in SL(2,C)$ define

$$j(g,w)=(\tilde{c}w+\tilde{d})\det(\tilde{c}w+\tilde{d})^{-1/2}\,. \tag{2.1}$$

We also take M_0 to be the constant 1. If κ is an essential cusp of $\Gamma(N)$, then for $w\in H$, $\mathrm{Re}\,(s)>2$, we define

$$E_\ell(w,\kappa,\Gamma(N),s)=\sum_{g\in\Gamma_\kappa\backslash\Gamma(N)}\overline{\chi}(g)j_\ell^*(\sigma_\kappa^{-1}g,w)v(\sigma_\kappa^{-1}g(w))^s\,, \tag{2.2}$$

where $v(w)=v$ for $w=(z,v)$. From (2.2) we obtain for $\sigma\in\Gamma(N)$

$$E_\ell(\sigma(w),\kappa,\Gamma(N),s)=\chi(\sigma)j_\ell(\sigma,w)E_\ell(w,\kappa,\Gamma(N),s)\,. \tag{2.3}$$

We put

$$K_{s,\ell}(\alpha)=\int_C(|z|^2+1)^{-s-\ell/2}\begin{bmatrix}z & -1\\ 1 & \bar{z}\end{bmatrix}_\ell^* e(-\alpha z)dxdy\,. \tag{2.4}$$

If p is another essential cusp and g represents a coset in $\Gamma_\kappa\backslash\Gamma(N)/\Gamma_p$, then we write $\sigma_\kappa^{-1}g\sigma_p=\begin{bmatrix}* & *\\ c(g) & d(g)\end{bmatrix}$. For $\mu\in\lambda^{-3}O$, we further define

$$\psi_{\kappa p}^+(s,\mu,\Gamma(N),\ell)=V(N)^{-1}\sum_{g\in\Gamma_\kappa\backslash\Gamma(N)/\Gamma_p}{}'\,\overline{\chi(g)}e\Big(\frac{\mu d(g)}{c(g)r}\Big)(\widetilde{c(g)_\ell})^*|c(g)|^{-2s-\ell}\,, \tag{2.5}$$

where $'$ indicates that the coset with $c(g)=0$ is omitted and $V(N)$ means the volume of the fundamental domain of $\Gamma(N)$. Then by a standard argument $E_\ell(w, \kappa, \Gamma(N), s)$ has a Fourier expansion

$$(2.6)\qquad E_\ell(\sigma_p(w), \kappa, \Gamma(N), s) = \delta_{\kappa p} I_\ell v^s + \sum_{\mu\in\lambda^{-3}\mathcal{O}} v^{2-s} K_{s,\ell}(\mu v) \times \psi^+_{\kappa p}(s, \mu, \Gamma(N), \ell) e(\mu z)\,,$$

where $\delta_{\kappa p}$ is the Kronecker's symbol.

By a general theory of the Eisenstein series $E_\ell(w, \kappa, \Gamma(N), s)$ can be analytically continued to the whole plane and has a functional equation

$$(2.7)\qquad E_\ell(w, \kappa, \Gamma(N), s) = \sum_{p\in P(N)} E_\ell(w, p, \Gamma(N), 2-s) M_\ell(s) \psi^+_{\kappa p}(s, \Gamma(N), \ell)\,,$$

where $M_\ell(s) = K_{s,\ell}(0)$, $\psi^+_{\kappa p}(s, \Gamma(N), \ell) = \psi^+_{\kappa p}(s, 0, \Gamma(N), \ell)$, $P(N)$ is the set of all essential cusps of $\Gamma(N)$. Therefore by (2.6) and (2.7) we obtain

$$(2.8)\qquad K_{s,\ell}(\mu v)\psi^+_\kappa(s, \mu, \Gamma(N), \ell) = \sum_{p\in P(N)} v^{2s-2} K_{2-s,-\ell}(\mu v) \psi^+_p(2-s, \mu, \Gamma(N), \ell) M_\ell(s) \psi^+_{\kappa p}(s, \Gamma(N), \ell)\,,$$

where $\psi^+_p(s, \mu, \Gamma(N), \ell) = \psi^+_{p\infty}(s, \mu, \Gamma(N), \ell)$.

§ 3. The functional equations

Put $i=(i_1, \cdots, i_\ell)$, $j=(j_1, \cdots, j_\ell)$, $(i_k, j_k = 1, 2,\ 1\le k\le \ell)$,

$$m^{(\ell)}_{ij} = \prod_{r=1}^{\ell} m_{i_r, j_r}\,.$$

Then $m^{(\ell)}_{ij}$ form the entries of M_ℓ which is the ℓ-fold tensor product of 2×2 Matrix $M=(m_{ij})$, and are written as M_{ij}. For $\ell\in \boldsymbol{Z}$ we define

$$(3.1)\qquad \begin{aligned} \psi_{\kappa p}(s, \mu, \Gamma(N), \ell) &= \psi^+_{\kappa p}(s, \mu, \Gamma(N), \ell)_{11} \quad (\ell>0)\,,\\ &= \psi^+_{\kappa p}(s, \mu, \Gamma(N), -\ell)_{22} \quad (\ell<0)^{1)}\,,\\ &= \psi^+_{\kappa p}(s, \mu, \Gamma(N), 0) \quad (\ell=0)\,. \end{aligned}$$

We shall first consider the case $N=3$. Calculation in the same way as in [3] gives

$$(3.2)\qquad \psi_{00}(s, \Gamma(3), \ell) = V(3)^{-1}(1+(-1)^\ell)(3^{3s-3}i^\ell - 1)^{-1}\zeta(3s-3, \ell) \times \zeta(3s-2, \ell)^{-1}\,,$$

$$(3.3)\qquad \psi_{01}(s, \Gamma(3), \ell) = \psi_{0,-1}(s, \Gamma(3), \ell) = V(3)^{-1}(-1)^\ell\zeta(3s-3, \ell)\zeta(3s-2, \ell)^{-1}$$

[1] $11=(1,\cdots,1)(1,\cdots,1)$, $22=(2,\cdots,2)(2,\cdots,2)$.

$$\psi_{0\infty}(s, \Gamma(3), \ell) = V(3)^{-1}(-1)^{\ell}\zeta(3s-3, \ell)\zeta(3s-2, \ell)^{-1}\,{}^{2)}, \tag{3.4}$$

where $\zeta(s, \ell) = \sum_{c\equiv 1(3)} \left(\frac{\bar{c}}{|c|}\right)^{3\ell}|c|^{-2s}$. By (2.7), (3.2), (3.3), (3.4) and the results of (3) we have the following theorem.

THEOREM 1. (Patterson). *Let ℓ be an even integer, $A(s, \ell) = 3^{-1}i^{-|\ell|}(1-3^{3-3s}i^{\ell})(1-3^{3s-4}i^{\ell})^{-1}$,*

$$F(s, \mu, \Gamma(3), \ell) = (2\pi)^{-2s}\Gamma(s+|\ell|/2-1/3)\Gamma(s+|\ell|/2-2/3)\psi_{0\infty}(s, \mu, \Gamma(3), \ell),$$

where $\Gamma(s)$ is the Gamma function and

$$F_{\infty}(s, \mu, \Gamma(3), \ell) = \sum_{\varepsilon^6=1}\sum_{b=2}^{\infty}\Gamma(\mu, \varepsilon\lambda^b)3^{-bs}(\varepsilon i^b)^{-\ell}F(s, \varepsilon\lambda^b, \mu, \ell),$$

where $\Gamma(\mu, \varepsilon\lambda^b) = \sum_{\substack{\delta(\operatorname{mod}\varepsilon\lambda^{b+2})\\ \delta\equiv 1(3)}} (\varepsilon\lambda^b/\delta)_3 e(\mu\delta/\varepsilon\lambda^b)$.

Then the function $F(s, \mu, \Gamma(3), \ell)$ can be analytically continued to the whole plane as an entire function if $\ell \neq 0$ and to a meromorphic function at most simple poles at $s = 2/3, 4/3$ if $\ell = 0$. Furthermore $F(s, \mu, \Gamma(3), \ell)$ is bounded when $|\mathrm{Im}(s)|$ is large in every vertical strip of finite width and satisfies the functional equation

$$F(s, \mu, \Gamma(3), \ell) = A(s, \ell)3^{9(1-s)}|\mu|^{2(1-s)}\left(\frac{\bar{\mu}}{|\mu|}\right)^{\ell}\Big(F_{\infty}(2-s, \mu, \Gamma(3), -\ell)$$
$$+ F(2-s, \mu, \Gamma(3), -\ell)\Big(e(\mu)+(-1)^{\ell}e(-\mu)+\frac{1+(-1)^{\ell}}{3^{3s-3}i^{\ell}-1}\Big)\Big).$$

Next, we consider the case $N = 3r$ such that r is a prime number of degree 1 satisfying $r \equiv 1$ (9). We define $g(c, \mu) = \sum_{a \bmod c}\left(\frac{a}{c}\right)_3 e(\mu a/c)$ for $\mu \in (1/\lambda)O$. Let $p = (\alpha'/\gamma') \in P(N)$ and α be a character of O defined modulo r. Then we obtain

$$\psi_p(s, \mu, \Gamma(N), \ell) = V(N)^{-1}\left(\frac{\alpha'}{\gamma'}\right)_3^{-1} e\left(\frac{-\hat{\gamma}'\alpha'\mu}{r}\right)$$
$$\times \sum_{c\equiv -\gamma(3r)}\left(\frac{c}{r}\right)_3 g(c, \mu)\left(\frac{\bar{c}}{|c|}\right)^{3\ell}|c|^{-6s}, \tag{3.5}$$

if $p = \alpha'/\gamma'$ such that $(\gamma', \lambda r) = 1$, where $\hat{\gamma}' \equiv \gamma'^{-1}(r)$,

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2) $\psi_{\kappa p}(s, \Gamma(3), \ell) = \psi_{\kappa p}(s, 0, \Gamma(3), \ell)$.

$$\psi_p(s, \mu, \Gamma(N), \ell) = V(N)^{-1}\left(\frac{-\gamma'}{\alpha'}\right)_3^{-1} \sum_{\substack{\varepsilon^6=1\\ b\geq 2}} e\left(\frac{-\hat{\gamma}'\alpha'\mu}{r}\right)\Gamma(\mu, \varepsilon\lambda^b)$$
$$\times |\varepsilon\lambda^b|^{-2s}\left(\frac{\overline{\varepsilon\lambda^b}}{|\varepsilon\lambda^b|}\right)^{\ell} \sum_{c\equiv -\gamma'(3r)} \left(\frac{c}{r}\right)_3 g(c, \varepsilon\lambda^b\mu)\left(\frac{\bar{c}}{|c|}\right)^{3\ell}|c|^{-6s} \tag{3.6}$$ [3)],

if $p = \alpha'/\gamma'$ such that $(\gamma', \lambda) \neq 1$ and $(\gamma', r) = 1$, where $\Gamma(\mu, \varepsilon\lambda^b)$ is the same as one defined above.

We now use an idea of T. Suzuki i.e. for $\kappa = \alpha/\gamma$ which is an essential cusp of A-type we consider

$$\sum_{\substack{\gamma \bmod 3r\\ \gamma\equiv 1(3)}} \mathfrak{a}(\gamma)\left(\frac{\alpha}{\gamma}\right)_3 \psi_\kappa(s, \mu, \Gamma(N), \ell)\,. \tag{3.7}$$

By (3.5) we obtain

$$(3.7) = V(N)^{-1}(-1)^{\ell} \sum_{\substack{c\equiv 1(3)\\ (c,r)=1}} \mathfrak{a}(c)\left(\frac{c}{r}\right)_3 g(c, \mu)\left(\frac{\bar{c}}{|c|}\right)^{3\ell}|c|^{-2s}\,. \tag{3.8}$$

We put

$$\psi(s, \mu, \Gamma(N), \mathfrak{a}, \ell) = \sum_{\substack{c\equiv 1(3)\\ (c,r)=1}} \mathfrak{a}(c)\left(\frac{c}{r}\right)_3 g(c, \mu)\left(\frac{\bar{c}}{|c|}\right)^{3\ell}|c|^{-2s}\,,$$

so that we have the following lemmas like ones in [4].

Lemma 2. *Let $\gamma \equiv 0\,(3)$, $\alpha \equiv 1\,(3)$. Then*

$$\sum_{c\equiv -r(3r)} \sum_{\substack{d \,(\mathrm{mod}\, 3rc)\\ d\equiv \alpha(3r)}} \left(\frac{c}{d}\right)_3 |c|^{-2s}\left(\frac{\bar{c}}{|c|}\right)^{\ell}$$

is given by:

(1) *in case $\gamma \equiv 0\,(3r)$, $\alpha \equiv 1\,(3)$*

$$\frac{1+(-1)^{\ell}}{3^{3s-3}i^{\ell} - 1} \sum_{r|M|r^{\infty}} \left(\frac{\overline{M}}{|M|}\right)^{\ell}\left(\frac{\alpha}{M}\right)_3 |M|^{2-2s} \sum_{\substack{c\equiv 1(3)\\ (c,r)=1}} \Phi(c)\left(\frac{\bar{c}}{|c|}\right)^{3\ell}|c|^{4-6s}\,.$$

where Φ is the Euler function on O and $r|M|r^{\infty}$ means that M runs over all powers of r.

(2) *in case $\gamma \equiv 0\,(3)$, $(\gamma, r) = 1$, $\alpha \equiv 1\,(3)$*

$$(-1)^{\ell} \sum_{\substack{\pm\lambda^{3b}c^3\equiv\gamma(r)\\ b\geq 1, c\equiv 1(3)\\ (c,r)=1}} 3^{3b(1-s)}\Phi(c)|c|^{4-6s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell}.$$

Lemma 3. *Let $\gamma \equiv 1(3)$.* Then

3) $\psi_p(s, \mu, \Gamma(N), \ell) = \psi_{p^\infty}(s, \mu, \Gamma(N), \ell)$.

$$\sum_{c\equiv -r(3r)} \sum_{\substack{d \bmod 3rc \\ d\equiv\alpha(3r)}} \left(\frac{d}{c}\right)_3 |c|^{-2s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell}$$

is given by:

(1) *in case* $\gamma \equiv 0\,(r)$

$$(-1)^\ell \sum_{r\mid M\mid r^\infty} \left(\frac{\overline{M}}{|M|}\right)^\ell \left(\frac{\alpha}{M}\right)_3 |c|^{-2(1-s)} \sum_{\substack{c\equiv 1(3)\\(c,r)=1}} \Phi(c)|c|^{4-6s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell},$$

(2) *in case* $(\gamma, r) = 1$

$$(-1)^\ell \sum_{\substack{c^3\equiv\gamma(r)\\ c\equiv 1(3),(c,r)=1}} \Phi(c)|c|^{4-6s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell}.$$

Let $\kappa = \alpha/\gamma$ be of A-type. Then $\psi_{\kappa p}(s, \Gamma(N), \ell)$[4] for $p = \alpha'/\gamma'$ is given as follows:

(1) $p = \alpha'/\gamma'$ is of A-type.

By Lemma 2 we have

$$\left(\frac{\alpha}{\gamma}\right)_3 V(N)\psi_{\kappa p}(s, \Gamma(N), \ell)$$

$$(3.9)\qquad = \left(\frac{\alpha'}{\gamma'}\right)_3 (3^{3s-3}i^\ell - 1)^{-1}(1+(-1)^\ell) \sum_{r\mid M\mid r^\infty} \left(\frac{\gamma}{M}\right)_3 \left(\frac{\gamma'}{M}\right)_3 \left(\frac{\overline{M}}{|M|}\right)^\ell |M|^{2-2s}$$

$$\times \sum_{\substack{c\equiv 1(3)\\(c,r)=1}} \Phi(c)|c|^{4-6s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell}.$$

(2) $p = \alpha'/\gamma'$ is of B-type.

By Lemma 2 we have

$$\left(\frac{\alpha}{\gamma}\right)_3 V(N)\psi_{\kappa p}(s, \Gamma(N), \ell)$$

$$(3.10)\qquad = (-1)^\ell \left(\frac{\alpha'}{\gamma'}\right)_3 \sum_{\substack{\pm\lambda^{3b}c^3\equiv -\gamma\alpha'(r)\\ b\geq 1, c\equiv 1(3)\\ (c,r)=1}} 3^{3b(1-s)}\Phi(c)|c|^{4-6s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell}.$$

(3) $p = \alpha'/\gamma'$ is of C-type.

By Lemma 3 we have

$$\left(\frac{\alpha}{\gamma}\right)_3 V(N)\psi_{\kappa p}(s, \Gamma(N), \ell)$$

$$(3.11)\qquad = (-1)^\ell \left(\frac{\gamma'}{\alpha'}\right)_3 \sum_{r\mid M\mid r^\infty} \left(\frac{\gamma}{M}\right)_3 \left(\frac{\gamma'}{M}\right)_3^{-1} |M|^{2-2s} \left(\frac{\overline{M}}{|M|}\right)^\ell$$

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[4] $\psi_{\kappa p}(s, \Gamma(N), \ell) = \psi_{\kappa p}(s, 0, \Gamma(N), \ell)$.

$$\times \sum_{\substack{c\equiv 1(3)\\(c,r)=1}} \Phi(c)|c|^{4-6s}\left(\frac{\bar{c}}{|c|}\right)^{3\ell}.$$

(4) $p = \alpha'/\gamma'$ is of D-type.

By Lemma 3 we have

$$\left(\frac{\alpha}{\gamma}\right)_3 V(N)\psi_{\kappa p}(s, \Gamma(N), \ell) = (-1)^{\ell}\left(\frac{\gamma'}{\alpha'}\right)_3 \sum_{\substack{c^3\equiv\gamma\alpha'(r)\\ c\equiv 1(3)\\(c,r)=1}} \Phi(c)|c|^{4-6s}. \tag{3.12}$$

Let

$$\zeta(s, \mathfrak{a}, \ell) = \sum_{\substack{c\equiv 1(3)\\(c,r)=1}} \left(\frac{\bar{c}}{|c|}\right)^{3\ell} \mathfrak{a}(c)|c|^{-2s},$$

and note $\sum_{\gamma \pmod r} \mathfrak{a}(\gamma)(\gamma/M)_3 = 0$ for any $M\in O$ such that $r|M|r^{\infty}$ if $\mathfrak{a}^3$ is not the trivial character 1_r. Then by (3.9), (3.10), (3.11) and (3.12) we have

Proposition 1. *If $\mathfrak{a}^3$ is not 1_r,*

$$\begin{aligned}
&\sum_{\substack{\gamma \pmod{3r}\\ \gamma\equiv 1(3)}} \mathfrak{a}(\gamma)\left(\frac{\alpha}{\gamma}\right)_3 \psi_{\kappa p}(s, \Gamma(N), \ell)\\
&= V(N)^{-1}\left(\frac{\alpha'}{\gamma'}\right)_3 (1 + \mathfrak{a}(-1)(-1)^{\ell})\bar{\mathfrak{a}}(\alpha')(\bar{\mathfrak{a}}(\lambda^3)i^{\ell}3^{3s-2} - 1)^{-1}\\
&\qquad \times \zeta(3s-2, \mathfrak{a}^3, \ell)\zeta(3s-2, \mathfrak{a}^3, \ell)^{-1} \quad (\textit{if } p \textit{ is of } B\textit{-type}),\\
&= V(N)^{-1}\mathfrak{a}(-1)(-1)^{\ell}\left(\frac{\gamma'}{\alpha'}\right)_3 \bar{\mathfrak{a}}(\alpha')\zeta(3s-2, \mathfrak{a}^3, \ell)\zeta(3s-2, \mathfrak{a}^3, \ell)^{-1}\\
&\qquad (\textit{if } p \textit{ is of } D\textit{-type}),\\
&= 0 \qquad (\textit{otherwise}).
\end{aligned} \tag{3.13}$$

Now we can prove

Theorem 2. *Let ℓ be an integer, $N = 3r$ (r is a prime number of degree 1 such that $r\equiv 1$ (9)) and $\mathfrak{a}$ be a character defined by modulo r such that $\mathfrak{a}^3 \neq 1_r$ and $\mathfrak{a}(-1)(-1)^{\ell} = 1$. We put*

$$F(s, \mu, \Gamma(N), \mathfrak{a}, \ell) = |r|^{2s}(2\pi)^{-2s}\Gamma(s + |\ell|/2 - 1/3)\Gamma(s + |\ell|/2 - 2/3) \times \psi(s, \mu, \Gamma(N), \mathfrak{a}, \ell),$$

$$A(s, \mathfrak{a}, \ell) = 3^{-1}i^{-|\ell|}(1 - \bar{\mathfrak{a}}(\lambda^3)3^{3-2s}i^{\ell})(1 - \bar{\mathfrak{a}}(\lambda^3)3^{3s-4}i^{\ell})^{-1},$$

$$F_{\infty}(s, \mu, \Gamma(N), \mathfrak{a}, \ell) = \sum_{\varepsilon^6=1}\sum_{b=2}^{\infty} \Gamma(\mu, \varepsilon\lambda^b)3^{-bs}(\varepsilon i^b)^{-\ell}F(s, \varepsilon\lambda^b, \mu, \mathfrak{a}, \ell),$$

$$B(\mathfrak{a}, \mu) = G(\mathfrak{a}^3)G(\mathfrak{a}, \mu)|r|^{-2},$$

Here $G(\mathfrak{a}) = \sum_{a \bmod r} \mathfrak{a}(a)e(a/r)$, $G(\mathfrak{a}, \mu) = \sum_{a \bmod r} \mathfrak{a}(a)e(\mu a/r)$.

Then $F(s, \mu, \Gamma(N), \mathfrak{a}, \ell)$ can be analytically continued to the whole plane as an entire function. Furthermore $F(s, \mu, \Gamma(N), \mathfrak{a}, \ell)$ is bounded when $|\mathrm{Im}(s)|$ is large in every vertical strip of finite width, and satisfies the functional equation

$$\begin{aligned} &F(s, \mu, \Gamma(N), \mathfrak{a}, \ell) \\ &\quad = A(s, \mathfrak{a}, \ell)B(\mathfrak{a}, \mu)3^{9(1-s)}|\mu|^{2(1-s)}\left(\frac{\bar{\mu}}{|\mu|}\right)^{\ell}\left(\frac{\bar{r}}{|r|}\right)^{2\ell} \\ &\quad \times \Big(F_{\infty}(2-s, \mu, \Gamma(N), \bar{\mathfrak{a}}, -\ell) + F(2-s, \mu, \Gamma(N), \bar{\mathfrak{a}}, -\ell) \\ &\quad \times \left(1 + \mathfrak{a}(-1)(-1)^{\ell} + \frac{1+\mathfrak{a}(-1)(-1)^{\ell}}{3^{3s-3}\bar{\mathfrak{a}}(\lambda^3)i^{\ell} - 1}\right)\Big). \end{aligned} \tag{3.14}$$

Proof. If κ is of A-type, by (2.7) we readily obtain

$$\begin{aligned} &\psi_{\kappa}(s, \mu, \Gamma(N), \ell) \\ &\quad = \pi i^{-\ell}|\mu|^{2-2s}|r|^{4-4s}\left(\frac{\bar{\mu}}{|\mu|}\right)^{\ell}\left(\frac{\bar{r}}{|r|}\right)^{-\ell}\Gamma(s+|\ell|/2)\Gamma(2-s+|\ell|/2)^{-1} \\ &\quad \times \Big(\sum_{p \in P(N)} \psi_p(2-s, \mu, \Gamma(N), -\ell)\psi_{\kappa p}(s, \Gamma(N), \ell)\Big). \end{aligned} \tag{3.15}$$

By (3.8) we further obtain

$$\sum_{\substack{\gamma \bmod 3r \\ \gamma \equiv 1(3)}} \mathfrak{a}(\gamma)\left(\frac{\alpha}{\gamma}\right)_3 \psi_{\kappa}(s, \mu, \Gamma(N), \ell) = V(N)^{-1}(-1)^{\ell}\psi(s, \mu, \Gamma(N), \mathfrak{a}, \ell). \tag{3.16}$$

We also consider the same sum as in (3.16) related to the right hand side of (3.15) using (3.5), (3.6) and (3.13). We note that if we put

$$\zeta^*(s, \mathfrak{a}, \ell) = \zeta(s, \mathfrak{a}, \ell)(1 - 3^{-s}\bar{\mathfrak{a}}(\lambda^3)i^{\ell})^{-1}$$

and

$$\xi(s, \mathfrak{a}, \ell) = |r|^s(2\pi)^{-s}\Gamma(s + 3|\ell|/2)\zeta^*(s, \mathfrak{a}, \ell),$$

then

$$\xi(1-s, \bar{\mathfrak{a}}^3, -\ell)3^{(1-2s)/2}\frac{G'(\mathfrak{a}^3)}{|r|}\left(\frac{\bar{r}}{|r|}\right)^{3\ell} = \xi(s, \mathfrak{a}^3, \ell).$$

Moreover by using $\sum_{\alpha' \bmod r} \mathfrak{a}(\alpha')e(-\hat{\gamma}'\alpha'\mu/r) = \bar{\mathfrak{a}}(\hat{\gamma}')G(\bar{\mathfrak{a}}, \mu)$ we have (3.14).

The analytical property of $F(s, \mu, \Gamma(N), \mathfrak{a}, \ell)$ can be proved like the case $N = 3$.

§ 4. An analogue of Weil's theorem on the upper half space

We shall omit some proofs of the lemmas in this section because those are like ones in Weil [5].

Let

$$K_{1/3}(4\pi v) = (4\pi i)^{-1}\int_{\mathrm{Re}\,(s)=\sigma} \Gamma(t + |\ell|/2 - 1/3)\Gamma(t + |\ell|/2 - 2/3) \times (2\pi v)^{-2t-|\ell|-1}dt\,.$$

LEMMA 4. *Let* α *be a primitive character defined modulo* r *and assume that both*

$$\psi(s, \ell) = \sum_{m\in\lambda^{-3}\mathcal{O}-\{0\}} a_m|m|^{1-2s}\left(\frac{\bar{m}}{|m|}\right)^{\ell}$$

and

$$\psi(s, \alpha, \ell) = \sum_{m\in\lambda^{-3}\mathcal{O}-\{0\}} a_m\alpha(m)|m|^{1-2s}\left(\frac{\bar{m}}{|m|}\right)^{\ell}$$

converge absolutely in some half plane satisfying $a_m = a_{-m}$. *We further put*

$$\Phi(s, \ell) = (2\pi)^{-2s}\Gamma(s + |\ell|/2 - 1/3)\Gamma(s + |\ell|/2 - 2/3)\psi(s, \ell)\,,$$

$$\Phi(s, \alpha, \ell) = |r|^{2s}(2\pi)^{-2s}\Gamma(s + |\ell|/2 - 1/3)\Gamma(s + |\ell|/2 - 2/3)\psi(s, \alpha, \ell)\,.$$

Suppose that $\Phi(s, \ell)$ $(\ell \neq 0)$ *and* $\Phi(s, \alpha, \ell)$ *can be analytically continued to the whole plane as entire functions and* $\Phi(s, 0)$ *as a meromorphic function at most simple poles at* $s = 2/3,\ 4/3$. *Suppose that* $\Phi(s, \ell)$ *and* $\Phi(s, \alpha, \ell)$ *are bounded when* $|\mathrm{Im}\,(s)|$ *is large in every vertical strip of finite width.*

Put now $F(w) = \sum_{m\in\lambda^{-3}\mathcal{O}-\{0\}} a_m v K_{1/3}(4\pi|m|v)e(mz)$, $A = \operatorname*{Res}_{s=4/3} \Phi(s, 0)$ *and suppose that*

$$\Phi(s, \ell) = (-1)^{\ell}\Phi(2 - s, -\ell)\,. \tag{4.1}$$

Then

$$F(\omega_1(w)) + \pi A v(\omega_1(w))^{2/3} = F(w) + \pi A v(w)^{2/3}\,,$$

where $\omega_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Put, on the other hand, $F(w, \alpha) = \sum_{m\in\lambda^{-3}\mathcal{O}-\{0\}} a_m\alpha(m)vK_{1/3}(4\pi|m|v)e(mz)$ *and suppose that*

$$\Phi(s, \alpha, \ell) = (-1)^{\ell}\left(\frac{\bar{r}}{|r|}\right)^{2\ell} C\cdot\Phi\left(2 - s, \bar{\alpha}\cdot\left(\frac{\cdot}{r}\right)_3, -\ell\right), \tag{4.2}$$

where C *is a constant.*

Then

$$F(\omega_{r^2}(w), \mathfrak{a}) = C\cdot F\Big(w, \bar{\mathfrak{a}}\cdot\Big(\frac{\cdot}{r}\Big)_3\Big),$$

where $\omega_{r^2} = \begin{bmatrix} 0 & -1 \\ r^2 & 0 \end{bmatrix}$.

Proof. The first half of this lemma is Patterson's lemma (Lemma 7.2 in (3))[5] and the latter half of it can be proved similarly.

If we put $\alpha(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ for $a \in \boldsymbol{C}$, then we obtain

LEMMA 5.

$$F(w, \mathfrak{a}) = G(\mathfrak{a})^{-1} \sum_{u \bmod r} \mathfrak{a}(u) F_1(\alpha(u/r)(w), \mathfrak{a})\,, \tag{4.3}$$

where $F_1(w, \mathfrak{a}) = \pi A v^{2/3} + F(w, \mathfrak{a})$.

Let $\begin{bmatrix} r & -v \\ -u & r' \end{bmatrix}$ be an element of $\Gamma(9)$ and put $\gamma(r, v) = \begin{bmatrix} r & -v \\ -u & r' \end{bmatrix}$. Then we easily obtain

$$\alpha(u/r)\omega_{r^2} = r\cdot \omega_1 \gamma(r, v)\alpha(v/r)\,. \tag{4.4}$$

Using (4.3) and (4.4), we have

LEMMA 6. *Let* $C_{\mathfrak{a}} = G\Big(\mathfrak{a}\Big(\overline{\frac{\cdot}{r}}\Big)_3\Big)/G(\bar{\mathfrak{a}})$ *and suppose that*

$$F(\omega_{r^2}(w), \mathfrak{a}) = C_{\mathfrak{a}} F\Big(w, \bar{\mathfrak{a}}\cdot\Big(\frac{\cdot}{r}\Big)_3\Big)$$

for any $\mathfrak{a}$ *such that* $\mathfrak{a}^3 \neq 1_r$, *and* $F_1(\omega_1(w)) = F_1(w)$ *with* $F_1(w) = \pi A v^{2/3} + F(w)$. *Then for any integers* u *satisfying* $(r, u) = 1$

$$\begin{aligned}\Big(\frac{u}{r}\Big)_3^{-1} & F_1(w) - F_1(\gamma(r, u)\alpha(u/r)(w)) \\ &= \Big(\frac{-u}{r}\Big)_3^{-1} F_1(w) - F_1(\gamma(r, -u)\alpha(-u/r)(w))\,.\end{aligned}$$

By Lemma 6 and the fact $\Big(\dfrac{v}{av+1}\Big)_3 = 1$ for $a \equiv 0$ (3), $v \equiv 0$ (3), we have

LEMMA 7. *Let* r, r' *be prime numbers of degree* 1 *such that* r, $r' \equiv 1$

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5) Here the Laplace operator $v^2\Big(4\dfrac{\partial^2}{\partial z \partial \bar{z}} + \dfrac{\partial^2}{\partial v^2}\Big) - v\Big(\dfrac{\partial}{\partial v}\Big)$ is used.

(9). *If for any character* α *defined modulo r satisfying* $\alpha^3 \neq 1_r$

$$F(\omega_{r^2}(w), \alpha) = C_\alpha F\left(w, \bar{\alpha}\cdot\left(\frac{\cdot}{r}\right)_3\right),$$

then for any $\gamma = \begin{bmatrix} r & -v \\ -u & r' \end{bmatrix} \in \Gamma(9)$

$$F_1(\gamma(w)) = \chi(\gamma)F_1(w)$$

holds.

Finally we can prove

THEOREM 3. *Let* $\ell \in \boldsymbol{Z}$, r *be a prime number of degree* 1 *such that* $r \equiv 1$ (9), *and* α *be a character defined modulo r such that* $\alpha^3 \neq 1_r$. *For*

$$F_1(w) = \pi A v^{2/3} + \sum_{m \in \lambda^{-3}O - \{0\}} a_m v K_{1/3}(4\pi|m|v) e(mz) \text{ with } a_m = a_{-m},$$

suppose that $F_1(\omega_1(w)) = F_1(w)$, *and* $\Phi(s, \alpha, \ell)$ *satisfies the assumptions in Lemma* 4 *and* (4.2) *for* $C = C_\alpha$ *for any* ℓ *and every* α *such that* $\alpha(-1)(-1)^\ell = 1$. *Then for* $\sigma \in \Gamma(9)$

$$F_1(\sigma(w)) = \chi(\sigma)F_1(w)\,. \tag{4.5}$$

Proof. Let $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(9)$. If $c = 0$, then we easily obtain (4.5). If $c \neq 0$, then $(a, 9c) = 1$. Hence there exist integers s and t such that $a + 9tc$ and $d + 9sc$ are prime numbers of degree 1. Put $p = a + 9tc$, $q = d + 9sc$, $u = -c$ and $v = -(b + 9sp + b9^2st + 9qt)$. Then we obtain

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -9t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p & -v \\ -u & q \end{bmatrix}\begin{bmatrix} 1 & -9s \\ 0 & 1 \end{bmatrix}.$$

If we put $\gamma = \begin{bmatrix} p & -v \\ -u & q \end{bmatrix}$, then by Lemma 7 we have $F_1(\gamma(w)) = \chi(\gamma)F_1(w)$. This completes the proof.

§ 5. The construction of $\theta(w)$

Let Γ_0 be the group consisting of, ω_1, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\Gamma(9)$. If we put $\chi\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \chi(\omega_1) = 1$, then χ can be extended to a character of Γ_0. By using the results of § 2–§ 4 we can prove

THEOREM 4 (Patterson). *For* $m \in \lambda^{-3}O$ *we define*

$$\tau_0(m) = \overline{g(\lambda^2, c)}\left|\frac{d}{c}\right| 3^{n/2+2}, \quad \textit{if } m = \pm\lambda^{3n-4}cd^3, n \geq 1,$$
$$= \zeta^{-1}\overline{g(\omega\lambda^2, c)}\left|\frac{d}{c}\right| 3^{n/2+2}, \quad \textit{if } m = \pm\omega\lambda^{3n-4}cd^3, n \geq 1,$$
$$= \zeta\overline{g(\omega^2\lambda^2, c)}\left|\frac{d}{c}\right| 3^{n/2+2}, \quad \textit{if } m = \pm\omega^2\lambda^{3n-4}cd^3, n \geq 1,$$
$$= \overline{g(1, c)}\left|\frac{d}{c}\right| 3^{n/2+5/2}, \quad \textit{if } m = \pm\lambda^{3n-3}cd^3, n \geq 0,$$
$$= 0, \quad \textit{otherwise},$$

where $\zeta = \exp(2\pi i/9)$, $c, d \in O$ *such that* $c, d \equiv 1$ (3) *and* c *is square free. There is then a constant* σ_0 *so that*

$$\theta(w) = \sigma_0 v^{2/3} + \sum_{m \in \lambda^{-3}O} \tau_0(m) v K_{1/3}(4\pi|m|v)e(mz)$$

is automorphic under Γ_0 *with* χ.

Proof. First if $\sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we evidently obtain $\theta(\sigma(w)) = \theta(w)$. Secondly if $\sigma = \omega_1$ then we have $\theta(\sigma(w)) = \theta(w)$ using Patterson's method by Theorem 1 and Lemma 4.

We now prove the case $\sigma \in \Gamma(9)$. We put

$$F_*(s, 1, \Gamma(N), \mathfrak{a}, \ell) = 3^{3s+1} i^{-\ell} \bar{\mathfrak{a}}(\lambda^3) F(s, 1, \Gamma(N), \mathfrak{a}, \ell),$$
$$F_*(s, \lambda^2, \Gamma(N), \mathfrak{a}, \ell) = 3^{4s} \bar{\mathfrak{a}}(\lambda^4) F(s, \lambda^2, \Gamma(N), \mathfrak{a}, \ell),$$
$$F_*(s, \omega\lambda^2, \Gamma(N), \mathfrak{a}, \ell) = \zeta 3^{4s} \bar{\mathfrak{a}}(\lambda^4) \omega^{-\ell} \mathfrak{a}(\omega) F(s, \omega\lambda^2, \Gamma(N), \mathfrak{a}, \ell),$$
$$F_*(s, \omega^2\lambda^2, \Gamma(N), \mathfrak{a}, \ell) = \zeta^{-1} 3^{4s} \bar{\mathfrak{a}}(\lambda^4) \omega^{\ell} \bar{\mathfrak{a}}(\omega) F(s, \omega^2\lambda^2, \Gamma(N), \mathfrak{a}, \ell),$$

$$\begin{aligned} \xi_0(s, \mathfrak{a}, \ell) = {} & (1 + \mathfrak{a}(-1)(-1)^\ell)(3^{3s-2}\bar{\mathfrak{a}}(\lambda^3) i^\ell - 1)^{-1}(F_*(s, \lambda^2, \Gamma(N), \mathfrak{a}, \ell) \\ & + F_*(s, \omega\lambda^2, \Gamma(N), \mathfrak{a}, \ell) + F_*(s, \omega^2\lambda^2, \Gamma(N), \mathfrak{a}, \ell)) \\ & + F_*(s, 1, \Gamma(N), \mathfrak{a}, \ell) + (1 + \mathfrak{a}(-1)(-1)^\ell) F_*(s, 1, \Gamma(N), \mathfrak{a}, \ell). \end{aligned}$$

If $\mathfrak{a}(-1)(-1)^\ell = 1$, the calculation by using Theorem 2 and Corollary 5.2 in (3) gives

$$\xi_0(s, \mathfrak{a}, \ell) = \mathfrak{a}(-1)(-1)^\ell \left(\frac{\bar{r}}{|r|}\right)^{2\ell} \bar{\mathfrak{a}}(\lambda^8) G(\mathfrak{a}^3) G''(\bar{\mathfrak{a}}) |r|^{-2} \xi_0(2 - s, \bar{\mathfrak{a}}, -\ell) \tag{5.1}$$

where $G''(\mathfrak{a}) = \sum_{x \bmod r} \mathfrak{a}(x) e(\lambda^2 x/r)$. We note that we use $\left(\frac{\lambda}{r}\right)_3 = \left(\frac{\omega}{r}\right)_3 = 1$ for obtaining (5.1) and in the case that $\mathfrak{a}(-1)(-1)^\ell = -1$ (5.1) is trivial.

Therefore if we put

$$L(s, \mathfrak{a}, \ell) = \sum_{m \in \lambda^{-3}\mathcal{O}} \tau_0(m)\mathfrak{a}(m)\left(\frac{\bar{m}}{|m|}\right)^{\ell} |m|^{1-2s}$$

and

$$\xi_1(s, \mathfrak{a}, \ell) = |r|^{2s}(2\pi)^{-2s}\Gamma(s + |\ell|/2 - 1/3)\Gamma(s + |\ell|/2 - 2/3)L(s, \mathfrak{a}, \ell)\,,$$

then by (5.1) we obtain

$$\begin{aligned} \xi_1(s, \mathfrak{a}, \ell) &= \mathfrak{a}(-1)(-1)^{\ell}\left(\frac{\bar{r}}{|r|}\right)^{2\ell} \bar{\mathfrak{a}}(\lambda^2)G''(\mathfrak{a}^3)\,G''\left(\bar{\mathfrak{a}}\cdot\left(\frac{\cdot}{r}\right)_3\right)|r|^{-2} \\ &\quad \times \xi_1\left(2 - s,\, \bar{\mathfrak{a}}\cdot\left(\frac{\cdot}{r}\right)_3,\, \ell\right). \end{aligned} \tag{5.2}$$

We now consider the Gauss sum in $\boldsymbol{Q}$ by putting $r\bar{r} = p$. If we define $\tau(\mathfrak{a}) = \sum_{x \bmod p} \mathfrak{a}(x) \exp(2\pi i x/p)$ for a prime number p $(\in \boldsymbol{Z})$, then we have

$$G(\mathfrak{a}) = \bar{\mathfrak{a}}(r)\tau(\mathfrak{a})\,. \tag{5.3}$$

By (5.3) we see

$$\begin{aligned} &\bar{\mathfrak{a}}(\lambda^2)G''(\mathfrak{a}^3)G''\left(\bar{\mathfrak{a}}\cdot\left(\frac{\cdot}{r}\right)_3\right)|r|^{-2} \\ &= \bar{\mathfrak{a}}(\lambda^2)\left(\frac{\bar{r}}{r}\right)_3 \mathfrak{a}^3(\lambda^2)\tau(\mathfrak{a}^3)\tau\left(\bar{\mathfrak{a}}\cdot\left(\frac{\cdot}{r}\right)_3\right)|r|^{-2}\,. \end{aligned} \tag{5.4}$$

Here we recall Davenport-Hasse's formula (in (1))

$$\tau(\mathfrak{a})\tau\left(\mathfrak{a}\cdot\left(\frac{\cdot}{r}\right)_3\right)\tau\left(\mathfrak{a}\cdot\left(\frac{\cdot}{r}\right)_3\right) = \mathfrak{a}^3(3)\tau(\mathfrak{a}^3)p\,. \tag{5.5}$$

By using (5.5), we obtain from (5.4)

$$\mathfrak{a}(-1)\mathfrak{a}(\lambda^2)G''(\mathfrak{a}^3)G''\left(\bar{\mathfrak{a}}\cdot\left(\frac{\cdot}{r}\right)_3\right)|r|^{-2} = G\left(\mathfrak{a}\cdot\overline{\left(\frac{\cdot}{r}\right)}_3\right)/G(\bar{\mathfrak{a}})\,. \tag{5.6}$$

Hence from (5.2) and (5.6) we finally have

$$\xi_1(s, \mathfrak{a}, \ell) = (-1)^{\ell}\left(\frac{\bar{r}}{|r|}\right)^{2\ell} C_{\mathfrak{a}}\xi_1\left(2 - s,\, \bar{\mathfrak{a}}\cdot\left(\frac{\cdot}{r}\right)_3,\, -\ell\right). \tag{5.7}$$

Therefore by applying Theorem 3 to (5.7) we have $\theta(\sigma(w)) = \chi(\sigma)\theta(w)$, for $\sigma \in \Gamma(9)$ putting $\sigma_0 = \pi A$. This completes the proof of the theorem.

Remarks 1. Theorem 4 is proved in [3] for $\Gamma(3)$ instead of $\Gamma(9)$, but in this paper we investigated the latter simplifying the calculation by using $\left(\frac{\lambda}{r}\right)_3 = \left(\frac{\omega}{r}\right)_3 = 1$.

2. If we apply the argument in the present paper to the biquadratic Gauss sum in $\boldsymbol{Q}(i)$, then roughly speaking the number corresponding to C_a in Lemma 6 is $G(a^4)G(\bar{a})|r|^{-2}$. Hence by using Davenport-Hasse's formula we see that we can't obtain an automorphic function whose coefficients are biquadratic Gauss sums in a simple analogue of the cubic theta function.

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