

A CONSTRUCTION OF ASYMPTOTIC SOLUTIONS AND THE EXISTENCE OF SMOOTH NULL-SOLUTIONS FOR A CLASS OF NON-FUCHSIAN PARTIAL DIFFERENTIAL OPERATORS

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§1. Introduction

Consider a partial differential operator

$$(1.1) \quad P = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) \partial_t^j \partial_x^\alpha, \quad a_{m,0}(t, x) \equiv t^\kappa,$$

where κ is a non-negative integer and $a_{j,\alpha}$ are real-analytic in a neighborhood of $(0,0) \in \mathbf{R}_t \times \mathbf{R}_x^n$.

M. S. Baouendi and C. Goulaouic [1] defined *Fuchsian partial differential operators*, and proved the unique solvability of the characteristic Cauchy problems in the category of real-analytic (or holomorphic) functions, which is a generalization of the classical Cauchy-Kowalevsky theorem. They also proved a generalization of the Holmgren uniqueness theorem. Especially, from their results it easily follows that if P is a Fuchsian operator with real-analytic coefficients, then there exist no sufficiently smooth null-solutions. Here, a Schwartz distribution u in a neighborhood of $(0,0)$ is called a *null-solution* for P at $(0,0)$, if $Pu = 0$ in a neighborhood of $(0,0)$ and $(0,0) \in \text{supp } u \subset \{t \geq 0\}$, where $\text{supp } u$ denotes the support of u .

The author considered the characteristic Cauchy problems for a class of operators wider than the Fuchsian operators in [3]. In that result, he showed the unique solvability of the characteristic Cauchy problems in the category of functions which are of class C^∞ with respect to t and real-analytic with respect to x . He also showed the non-existence of sufficiently smooth null-solutions. (As for

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distribution null-solutions, see [4]). This class of operators is defined in terms of four conditions. He gave a conjecture that if the third condition is violated, then there exists a C^∞ null-solution.

In this article, we construct an asymptotic solution of $Pu = 0$ in the form

$$(1.2) \quad u(t, x) := \exp\left(-\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \cdot t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x),$$

where

- (i) M is a non-negative integer, and q is a positive integer,
- (ii) $\mu[j] (j = 0, 1, \dots, M)$ are positive rational numbers such that $\mu[0] > \mu[1] > \dots > \mu[M] > 0$.
- (iii) $\lambda[j] (j = 0, 1, \dots, M+1)$ and $v_{l,p} (l \geq 0; 0 \leq p \leq lm)$ are real-analytic in a fixed open neighborhood of $0 \in \mathbf{R}^n$,

for a class of operators wider than that considered in [3].

Further, using these asymptotic solutions, we prove the conjecture in [3] mentioned above under an additional assumption. The C^∞ null-solution constructed here is one of the most fastly decaying nontrivial solutions of $Pu = 0$.

In Section 2, we give the statements of the main theorems. After giving some preliminaries in Section 3, we prove the main theorems in Sections 4 and 5.

NOTATIONS:

- (i) The set of all integers (resp. nonnegative integers) is denoted by $\mathbf{Z}$ (resp. $\mathbf{N}$). Put $\mathbf{N}/q := \{p/q : p \in \mathbf{N}\}$ for a positive integer q , and put $\mathbf{Z}/q$ similarly.
- (ii) Put $\mathcal{G} := t\partial_t$.
- (iii) For a bounded domain Ω in $\mathbf{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω .
- (iv) The space of the Schwartz distributions on U is denoted by $\mathcal{D}'(U)$.
- (v) For a complete locally convex topological vector space E , put

$$C_{flat}^N([0, T]; E) := \{f \in C^N([0, T]; E) : \frac{d^j f}{dt^j} \Big|_{t=0} = 0 \text{ for } 0 \leq j \leq N-1\}.$$

- (vi) Put $(\lambda)_j := \prod_{l=0}^{j-1} (\lambda - l)$ for $\lambda \in \mathbf{C}$ and $j \in \mathbf{N}$.
- (vii) For a commutative ring R , the ring of polynomials of λ with the coefficients belonging to R is denoted by $R[\lambda]$. The degree of $F \in R[\lambda]$ is denoted by $\deg_\lambda F$.

§2. Statement of the main result

Let q be a positive integer, Ω be a bounded domain in $\mathbf{C}^n$ that includes the origin 0, and T be a positive real number. Consider a linear partial differential operator of the form (1.1). We assume only the following weaker condition on the coefficients.

$$(A-0) \quad a_{j,\alpha} \in \widehat{\mathcal{F}}_q([0, T]; \mathcal{O}(\Omega)) \quad (j + |\alpha| \leq m),$$

where

$$\begin{aligned} \mathcal{F}_q([0, T]; \mathcal{O}(\Omega)) &:= \{\phi \in C^\infty((0, T]; \mathcal{O}(\Omega)) \\ &\quad : [s \mapsto \phi(s^q)] \in C^\infty([0, T^{1/q}]; \mathcal{O}(\Omega))\}, \\ \widehat{\mathcal{F}}_q([0, T]; \mathcal{O}(\Omega)) &:= \{\phi \in C^\infty((0, T]; \mathcal{O}(\Omega)) \\ &\quad : t^M \phi(t) \in \mathcal{F}_q([0, T]; \mathcal{O}(\Omega)) \text{ for some } M \in \mathbf{N}\}. \end{aligned}$$

Let $r(j, \alpha)$ be the *generalized vanishing order* of $a_{j,\alpha}$ on the hypersurface $\Sigma := \{(0, x) : x \in \Omega\}$, that is

$$(2.1) \quad r(j, \alpha) := \sup\{r \in \mathbf{Z}/q : t^{-r} a_{j,\alpha} \in \mathcal{F}_q([0, T]; \mathcal{O}(\Omega))\}.$$

If $r(j, \alpha) = \infty$, then we redefine $r(j, \alpha) := R$ for a sufficiently large R ($R := \max\{r(j, \alpha) : r(j, \alpha) < \infty\} + 1$ will suffice). Put

$$(2.2) \quad \tilde{a}_{j,\alpha}(t, x) := t^{-r(j,\alpha)} a_{j,\alpha}(t, x) \quad (\in \mathcal{F}_q([0, T]; \mathcal{O}(\Omega))).$$

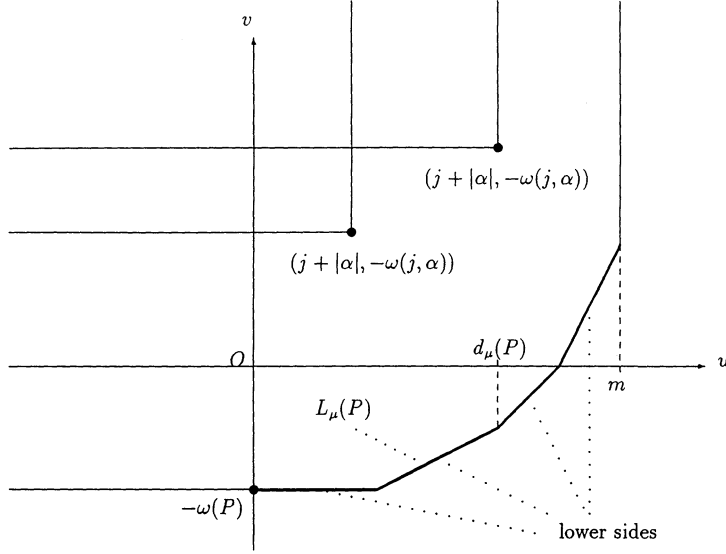
Note that if $r(j, \alpha) < R$, then $\tilde{a}_{j,\alpha}(0, x) \not\equiv 0$.

Associating a *weight* $\omega(j, \alpha) := j - r(j, \alpha)$ to each differential monomial $a_{j,\alpha}(t, x) \partial_t^j \partial_x^\alpha$, we draw a Newton polygon $\Delta(P)$ using the points $(j + |\alpha|, -\omega(j, \alpha))$ ($j + |\alpha| \leq m$) in (u, v) -plane as follows.

DEFINITION 2.1 ([3]). (1) Put

$$\Delta(P) := ch\left(\bigcup_{j+|\alpha| \leq m} \{(u, v) \in \mathbf{R}^2 : u \leq j + |\alpha|, v \geq -\omega(j, \alpha)\}\right),$$

where $ch(A)$ denotes the convex hull of A . This is called the *Newton polygon* of P .

FIGURE 1. Newton polygon of $P : \Delta(P)$

(2) Put

$$\hat{V} = \hat{V}(P) := \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : (j + |\alpha|, -\omega(j, \alpha)) \text{ is a vertex of } \Delta(P)\}.$$

(3) Put

$$\omega = \omega(P) := \max\{\omega(j, \alpha) \in \mathbf{R} : j + |\alpha| \leq m\},$$

which is the *maximum weight* of P .

(4) The boundary of $\Delta(P) \cap ([0, \infty) \times \mathbf{R})$ is the union of two vertical half-lines and a finite number of compact line segments with distinct slopes. Each of these compact line segments is called a *lower side* of $\Delta(P)$. The set of the slopes of the lower sides of $\Delta(P)$ is denoted by $S = S(P) (\subset \mathbf{Q})$. For $\mu \in S(P)$, the lower side of $\Delta(P)$ with slope μ is denoted by $L_\mu = L_\mu(P)$. Put

$$I_\mu = I_\mu(P) := \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : (j + |\alpha|, -\omega(j, \alpha)) \in L_\mu(P)\}.$$

Let the right end points of $L_\mu(P)$ be (u_1, v_1) . We put $d_\mu(P) := u_1$, and call it the *degree* of the slope μ .

If $0 \notin S$, we put $L_0(P) := \{(0, -\omega(0, 0))\} = \{(0, -\omega(P))\} \subset \mathbf{R}^2$, $I_0(P) := \{(0, 0)\} \subset \mathbf{N} \times \mathbf{N}^n$, and $d_0(P) := 0$.

By the use of these notions, Fuchsian operators in the sense of M. S. Baouendi and C. Goulaouic [1] are characterized as follows. (In fact, they assumed that the

coefficients belong to $C^m([0, T]; \mathcal{O}(\Omega))$. This difference is, however, not essential and hence we ignore the difference of the classes of coefficients.)

PROPOSITION 2.2. *The operator P is Fuchsian if and only if $\omega(P) \geq 0$, $S(P) = \{0\}$, and there exist no $(j, \alpha) \in I_0(P)$ such that $\alpha \neq 0$.*

We consider a class of operators wider than the class of Fuchsian operators. First, we assume the following condition.

(A-1) For all $\mu \in S(P)$, there exist no $(j, \alpha) \in I_\mu(P)$ such that $\alpha \neq 0$.

DEFINITION 2.3. For $\mu \in S(P)$ with $\mu > 0$, we put

$$\mathcal{G}_\mu[P](x; \lambda) := \sum_{(j,0) \in I_\mu(P)} \tilde{a}_{j,0}(0, x) \lambda^j \in \mathcal{O}(\Omega)[\lambda].$$

We also put

$$\mathcal{G}_0[P](x; \lambda) := \sum_{(j,0) \in I_0(P)} \tilde{a}_{j,0}(0, x) (\lambda)_j \in \mathcal{O}(\Omega)[\lambda].$$

The polynomial $\mathcal{G}_\mu[P]$ of λ is called the *indicial polynomial of P associated with the slope $\mu \in S(P) \cup \{0\}$* . Note that $d_\mu(P) = \deg_\lambda \mathcal{G}_\mu[P]$.

For $\mu \in S(P) \cup \{0\}$, we consider the following condition.

(A-2; μ) If $(j, 0) \in \hat{V}(P)$ and $j \geq d_\mu(P)$, then $\tilde{a}_{j,0}(0, 0) \neq 0$.

This is equivalent to the following.

(A-2; μ) For every $\nu \in S(P)$ with $\nu \geq \mu$, the coefficient of the top order term of $\mathcal{G}_\nu[P](x; \lambda) \in \mathcal{O}(\Omega)[\lambda]$ does not vanish at $x = 0$.

Remark 2.4. Note that if $(j, 0) \in \hat{V}(P)$, then $\tilde{a}_{j,0}(0, x) \not\equiv 0$. Thus, the condition (A-2; μ) is a kind of non-degeneracy at $x = 0$. Further, the condition (A-2; μ) for $\mu > 0$ is weaker than the condition (A-2; 0), and (A-2; 0) is equivalent to (A-2) in [3].

Now, the following is one of the three main theorems in this article.

THEOREM 2.5. *Assume that P satisfies (A-0) and (A-1). Let $\mu_0 \in S(P) \cap \mathbf{N}/q$, $\mu_0 > 0$, and assume the condition (A-2; μ_0). If λ_0 is a simple root of $\mathcal{G}_{\mu_0}[P](0; \lambda) = 0$, then there exist*

(i) $M \in \mathbf{N}$,

- (ii) $\mu[j] \in \mathbf{N}/q (j = 0, 1, \dots, M)$, where $\mu_0 = \mu[0] > \mu[1] > \dots > \mu[M] > 0$,
- (iii) a subdomain Ω_0 of Ω including 0,
- (iv) $\lambda[j] \in \mathcal{O}(\Omega_0) (j = 0, 1, \dots, M+1)$, where $\lambda0 = \lambda_0$,

such that the following holds.

For an arbitrarily given $v_{0,0}(x) \in \mathcal{O}(\Omega_0)$, there exists $v_{l,p}(x) \in \mathcal{O}(\Omega_0) (l \geq 0; 0 \leq p \leq lm)$ such that a formal series

$$(2.3) \quad u(t, x) := \exp\left(-\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \cdot t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)$$

is an asymptotic solution of $Pu = 0$. That is, for every $N \in \mathbf{N}$ there holds

$$(2.4) \quad t^{-\lambda[M+1](x)} \cdot \exp\left(\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \cdot P\left(\exp\left(-\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \times t^{\lambda[M+1](x)} \cdot \sum_{l=0}^N t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)\right) = o(t^{N/q-r_0}),$$

with some $r_0 \in \mathbf{N}$.

This theorem shall be proved in Section 4. We shall also give a proposition which corresponds to the case of $\mu_0 = 0$ and $M = -1$.

Remark 2.6. Even if $\mu_0 \in S(P)$ but $\mu_0 \notin \mathbf{N}/q$, we can retake another q such that $\mu_0 \in \mathbf{N}/q$ and (A-0) is satisfied. Hence, we can always apply this theorem with this new q .

Next, we consider the following condition for $\mu \in S(P)$.

(A-6; μ) If $\nu \in S(P)$ and $\nu > \mu$, then all non-zero roots λ of $\mathcal{C}_\nu[P](0; \lambda) = 0$ satisfy $\operatorname{Re} \lambda < 0$. Further, there exists $\lambda_0 \in \mathbf{C}$ which satisfies the following.

- (i) $\operatorname{Re} \lambda_0 > 0$,
- (ii) λ_0 is a simple root of $\mathcal{C}_\mu[P](0; \lambda) = 0$ and the other roots λ satisfy $\operatorname{Re} \lambda < \operatorname{Re} \lambda_0$.

Remark 2.7. In this section, we define only the conditions (A-0), (A-1), (A-2; μ), and (A-6; μ). This apparently strange numbering is for the consistency with [3]. We shall introduce another condition (A-3) in Section 5.

Using the theorem above, we can show the existence theorem of smooth null-solutions, which is the second of the main theorems.

THEOREM 2.8. Assume the conditions (A-0), (A-1), (A-2; μ_0), and (A-6; μ_0) for some $\mu_0 \in S(P)$ with $\mu_0 > 0$. Then, P has a C^∞ null-solution at $(0,0)$.

The C^∞ null-solution given in this theorem is one of the most fastly decaying nontrivial solutions as $t \rightarrow +0$. In fact, we have the following theorem, which is the last of the main theorems.

THEOREM 2.9. Assume the conditions (A-0), (A-1), (A-2; μ_0), and (A-6; μ_0) for some $\mu_0 \in S(P)$ with $\mu_0 > 0$. Assume that u is a C^0 solution of $Pu = 0$ for $t > 0$. If there exist $\delta > \operatorname{Re} \lambda_0$ and $C_0 > 0$ such that the inequality

$$|u(t, x)| \leq C_0 \exp\left(-\frac{\delta}{\mu_0} t^{-\mu_0}\right)$$

holds for $t > 0$ in a neighborhood of $(0,0)$, then $u = 0$ for $t > 0$ in a neighborhood of $(0,0)$.

Theorems 2.8 and 2.9 shall be proved in Section 5.

Finally, let us consider a typical example.

EXAMPLE 2.10. First, we consider the following ordinary differential operator decomposed into first order operators.

$$P_0 := t^d (t^{k_1} \partial_t - \lambda_1(t, x)) \cdots (t^{k_r} \partial_t - \lambda_r(t, x)) (\partial_t - \tilde{\lambda}_{r+1}(t, x)) \cdots (\partial_t - \tilde{\lambda}_m(t, x)),$$

where $m, r, d \in \mathbf{N}$, $0 \leq r \leq m$, $k_j \in \mathbf{N}$ ($1 \leq j \leq r$) and $\lambda_j, \tilde{\lambda}_l \in C^\infty([0, T]; \mathcal{O}(\mathcal{Q}))$ ($1 \leq j \leq r$; $r+1 \leq l \leq m$). Assume that $\lambda_j(0, x) \neq 0$ ($1 \leq j \leq r$) and $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$. For this operator, $S(P_0) = \{k_1, \dots, k_r, 0\}$ if $r < m$, and $S(P_0) = \{k_1, \dots, k_m\}$ if $r = m$. The condition (A-1) is trivially satisfied, and the condition (A-2; μ) is “if $k_j > \mu$ then $\lambda_j(0, 0) \neq 0$ ”. We can also show that

$$\mathcal{C}_\mu[P_0](x; \lambda) = \prod_{j: k_j > \mu} (-\lambda_j(0, x)) \cdot \prod_{j: k_j = \mu} (\lambda - \lambda_j(0, x)) \cdot \lambda^{h(\mu) + m - r}$$

for $\mu \in S(P_0)$ with $\mu > 0$, where $h(\mu)$ is the number of k_j 's that satisfy $k_j < \mu$. Thus, the condition (A-6; μ_0) for $\mu_0 > 0$ is the following.

If $k_j > \mu_0$ then $\operatorname{Re} \lambda_j(0, 0) < 0$. Further, there exists j_0 such that

(i) $k_{j_0} = \mu_0$,

(ii) $\operatorname{Re} \lambda_{j_0}(0, 0) > 0$,

(iii) If $k_j = \mu_0$ and $j \neq j_0$, then $\operatorname{Re} \lambda_j(0, 0) < \operatorname{Re} \lambda_{j_0}(0, 0)$.

Next, we consider a partial differential operator. Put $\mu_j := 0$ ($1 \leq j \leq m - r$) and $\mu_{m-r+j} := k_{r+1-j}$ ($1 \leq j \leq r$). Also put $\omega_j := d + \sum_{l=1}^j \mu_l$ ($0 \leq j \leq m$). Consider an operator

$$P = P_0 + \sum_{j=0}^m t^{\omega_j+1} B_j(t, x; \vartheta, \partial_x),$$

where $B_j(t, x; \vartheta, \partial_x) = \sum_{|\alpha| \leq j} b_{j,\alpha}(t, x) \partial_x^\alpha \vartheta^{j-|\alpha|}$ and $b_{j,\alpha} \in C^\infty([0, T]; \mathcal{O}(\Omega))$. Then, P satisfies the condition (A-1), and there hold $\Delta(P) = \Delta(P_0)$, $S(P) = S(P_0)$, $\mathcal{C}_\mu[P] = \mathcal{C}_\mu[P_0]$. (See Lemma 3.1.) Hence, P satisfies the condition (A-2; μ_0) (resp. (A-6; μ_0)), if and only if P_0 satisfies (A-2; μ_0) (resp. (A-6; μ_0)).

§3. Preliminaries

In this section, we give some preliminaries for the proofs of the main theorems.

Let P be an operator (1.1) satisfying (A-0). By $t^j \partial_t^j = \vartheta(\vartheta - 1) \dots (\vartheta - j + 1) = (\vartheta)_j$, we can easily show the following lemma, which is useful in our arguments.

LEMMA 3.1. *We can rewrite P as*

$$(3.1) \quad P = \sum_{j+|\alpha| \leq m} b_{j,\alpha}(t, x) \vartheta^j \partial_x^\alpha,$$

with $b_{j,\alpha} \in \widehat{\mathcal{F}}_q([0, T]; \mathcal{O}(\Omega))$. For this $b_{j,\alpha}$, we define the generalized vanishing order

$$r'(j, \alpha) := \sup\{r \in \mathbf{Z}/q : t^{-r} b_{j,\alpha} \in \mathcal{F}_q([0, T]; \mathcal{O}(\Omega))\}.$$

For $\mu \geq 0$, we put $\omega_\mu(P) := \max\{-r'(j, \alpha) + \mu(j + |\alpha|) : j + |\alpha| \leq m\}$. Then, we have

$$\begin{aligned} \Delta(P) &= ch\left(\bigcup_{j+|\alpha| \leq m} \{(u, v) \in \mathbf{R}^2 : u \leq j + |\alpha|, v \geq r'(j, \alpha)\}\right), \\ \hat{V}(P) &= \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : (j + |\alpha|, r'(j, \alpha)) \text{ is a vertex of } \Delta(P)\}, \\ \omega(P) &= \max\{-r'(j, \alpha) \in \mathbf{R} : j + |\alpha| \leq m\} = \omega_0(P), \\ I_\mu(P) &= \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}^n : -r'(j, \alpha) + \mu(j + |\alpha|) = \omega_\mu(P)\}. \end{aligned}$$

Further, the condition (A-1) is stated as follows:

(A-1) For every $\mu \in S(P)$, if $-r'(j, \alpha) + \mu(j + |\alpha|) = \omega_\mu(P)$, then $\alpha = 0$.

Under (A-1), there holds

$$(3.2) \quad \begin{aligned} \mathcal{C}_\mu[P](x; \lambda) &= \sum_{j=0}^m \{b_{j,0}(t, x) t^{\omega_\mu(P) - \mu j}\} \Big|_{t=0} \lambda^j \\ &= \begin{cases} [t^{\omega_\mu(P)} e^{\lambda t^{-\mu/\mu}} P(e^{-\lambda t^{-\mu/\mu}})] \Big|_{t=0} & (\mu > 0), \\ [t^{\omega(P)} t^{-\lambda} P(t^\lambda)] \Big|_{t=0} & (\mu = 0), \end{cases} \end{aligned}$$

and the condition (A-2; μ) is stated as follows:

(A-2; μ) If $(j, 0) \in \hat{V}(P)$ and $j \geq d_\mu(P)$, then $\{b_{j,0}(t, 0)t^{-r'(j,0)}\} |_{t=0} \neq 0$.

It is convenient to consider the operator in the form (3.1) rather than the form (1.1).

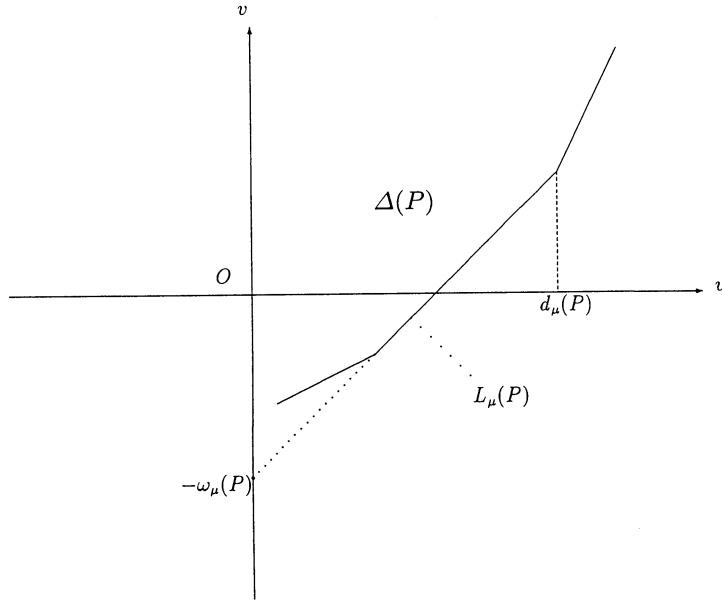


FIGURE 2. $\omega_\mu(P)$

Remark 3.2. For $\mu \geq 0$, we can define $\mathcal{C}_\mu[P]$ by (3.2), even if $\mu \notin S$. If $\mu \in S$ and $\mu > 0$, then $\mathcal{C}_\mu[P]$ has more than one term as a polynomial of λ . If $\mu \notin S$ and $\mu > 0$, then $\mathcal{C}_\mu[P]$ has only one term.

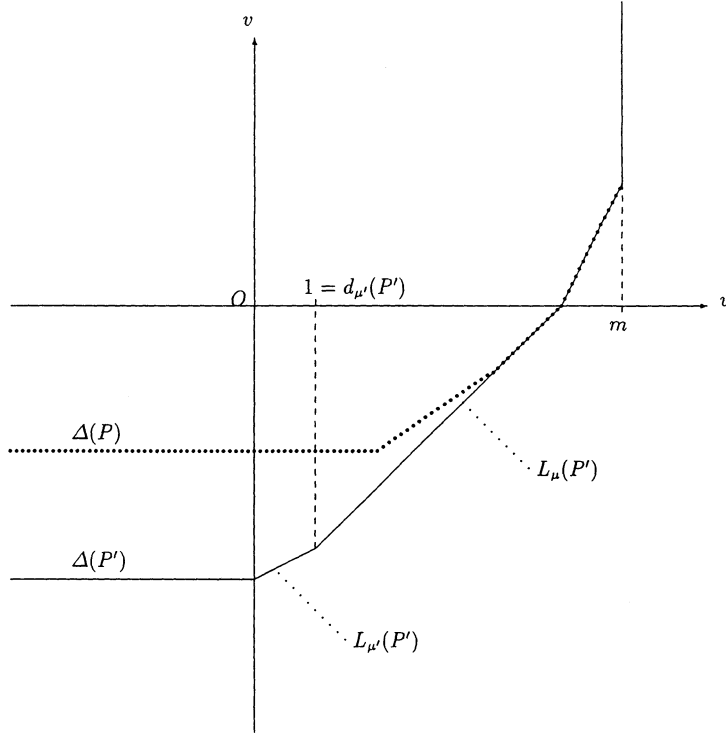
The key tool for the proofs of main theorems is the following type of transformation of operators.

LEMMA 3.3. Assume that an operator P of the form (1.1) (or (3.1)) satisfies the conditions (A-0) and (A-1). Let $\mu \in S(P) \cap \mathbf{N}/q$, $\mu > 0$, and assume (A-2; μ). Let λ_1 be a simple root of $\mathcal{C}_\mu[P](0; \lambda) = 0$. Take a subdomain Ω' of Ω including 0 and $\lambda(x) \in \mathcal{O}(\Omega')$ so that they satisfy $\lambda(0) = \lambda_1$ and $\mathcal{C}_\mu[P](x; \lambda(x)) \equiv 0$ on Ω' . If we put

$$P' := \exp\left(\frac{\lambda(x)}{\mu} t^{-\mu}\right) \circ P \circ \exp\left(-\frac{\lambda(x)}{\mu} t^{-\mu}\right),$$

then P' is an operator on $[0, T] \times \Omega'$ of the form (1.1) and satisfies the following:

- (a) The operator P' satisfies (A-0) and (A-1).
- (b) $S(P') \cap (\mu, \infty) = S(P) \cap (\mu, \infty)$.
- (c) $\mathcal{C}_\nu[P'](x; \cdot) = \mathcal{C}_\nu[P](x; \cdot)$ for every $\nu > \mu$ and $x \in \Omega'$.
- (d) There holds $\mathcal{C}_\mu[P'](x; \lambda) = \mathcal{C}_\mu[P](x; \lambda + \lambda(x))$. Further, if $d_\mu(P) > 1$, then $\mu \in S(P')$; if $d_\mu(P) = 1$, then $\mu \notin S(P')$.
- (e) There exists $\mu' < \mu$ such that $\mu' \in \mathbf{N}/q$ and $S(P') \cap [0, \mu) = \{\mu'\}$.
- (f) $d_{\mu'}(P') = 1$ and P' satisfies (A-2; μ').



The upper part of the dotted line is $\Delta(P)$.

The upper part of the real line is $\Delta(P')$.

FIGURE 3. $\Delta(P')$ and $\Delta(P)$

Proof. First, note that

$$\begin{aligned}
(3.3) \quad & \exp\left(\frac{\lambda(x)}{\mu} t^{-\mu}\right) \circ \mathcal{G} \circ \exp\left(-\frac{\lambda(x)}{\mu} t^{-\mu}\right) = \mathcal{G} + \lambda(x)t^{-\mu}, \\
& \exp\left(\frac{\lambda(x)}{\mu} t^{-\mu}\right) \circ \partial_x \circ \exp\left(-\frac{\lambda(x)}{\mu} t^{-\mu}\right) = \partial_x + \frac{-\lambda_x(x)}{\mu} t^{-\mu}.
\end{aligned}$$

From these, it is easy to see that P' is an operator of the form (3.1) and satisfies the conditions (A-0), (A-1), and (A-2; μ). It is also easy to see that there hold the conclusions (b), (c). Further, we have $\mathcal{C}_\mu[P'](x; \lambda) = \mathcal{C}_\mu[P](x; \lambda + \lambda(x))$. Since $\mathcal{C}_\mu[P'](x; 0) \equiv 0$ and since $(\partial_\lambda \mathcal{C}_\mu[P'])(0; 0) = (\partial_\lambda \mathcal{C}_\mu[P])(0; \lambda_1) \neq 0$, we have $(1, 0, \dots, 0) \in \hat{V}(P') (\subset \mathbf{N} \times \mathbf{N}^n)$. Hence, if $d_\mu(P) > 1$, then $\mu \in S(P')$; if $d_\mu(P) = 1$, then $\mu \notin S(P')$. Further, there exists $\mu' \in \mathbf{N}/q$ such that $\mu' < \mu$, $S(P') \cap [0, \mu] = \{\mu'\}$, and $d_{\mu'}(P') = 1$. The condition (A-2; μ) and the fact that $(\partial_\lambda \mathcal{C}_\mu[P'])(0; 0) \neq 0$ imply (A-2; μ'). $\square$

By an iterative use of this lemma, we have the following.

PROPOSITION 3.4. *Assume that P satisfies (A-0) and (A-1). Let $\mu_0 \in S(P) \cap \mathbf{N}/q$, $\mu_0 > 0$, and assume (A-2; μ_0). Let λ_0 be a simple root of $\mathcal{C}_{\mu_0}[P](0; \lambda) = 0$. Then, there exist*

- (i) $M \in \mathbf{N}$,
- (ii) $\mu[j] \in \mathbf{N}/q$ ($j = 0, 1, \dots, M$), where $\mu_0 = \mu[0] > \mu[1] > \dots > \mu[M] > 0$,
- (iii) a subdomain Ω_{M+1} of Ω including 0,
- (iv) $\lambda[j] \in \mathcal{O}(\Omega_{M+1})$ ($j = 0, 1, \dots, M$), where $\lambda0 = \lambda_0$,

such that the operator

$$P^{(M+1)} := \exp\left(\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \circ P \circ \exp\left(-\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right)$$

is an operator on $[0, T] \times \Omega_{M+1}$ of the form (1.1) and satisfies the following:

- (a) The operator $P^{(M+1)}$ satisfies (A-0) and (A-1).
- (b) $S(P^{(M+1)}) \cap (\mu_0, \infty) = S(P) \cap (\mu_0, \infty)$.
- (c) $\mathcal{C}_\nu[P^{(M+1)}](x; \cdot) = \mathcal{C}_\nu[P](x; \cdot)$ for every $\nu > \mu_0$ and $x \in \Omega_{M+1}$.
- (d) There holds $\mathcal{C}_{\mu_0}[P^{(M+1)}](x; \lambda) = \mathcal{C}_{\mu_0}[P](x; \lambda + \lambda[0](x))$. If $d_{\mu_0}(P) > 1$, then $\mu_0 \in S(P^{(M+1)})$; if $d_{\mu_0}(P) = 1$, then $\mu_0 \notin S(P^{(M+1)})$.
- (e) $S(P^{(M+1)}) \cap [0, \mu_0] = \{0\}$.
- (f) $d_0(P^{(M+1)}) = 1$ and $P^{(M+1)}$ satisfies (A-2; 0).

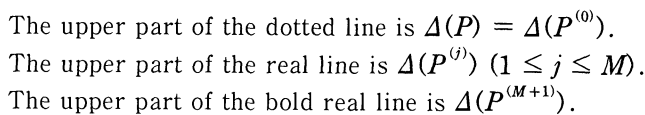


FIGURE 4. $\Delta(P) = \Delta(P^{(0)})$ and $\Delta(P^{(M+1)}) \subset \dots \subset \Delta(P^{(1)})$

Proof. Since λ_0 is a simple root, we can take a subdomain Ω_1 of Ω including 0 and $\lambda[0](x) \in \mathcal{O}(\Omega_1)$ such that they satisfy $\lambda0 = \lambda_0$ and $\mathcal{E}_{\mu_0}[P](x; \lambda[0](x)) \equiv 0$ on Ω_1 .

Put $P^{(0)} := P$ and $\mu[0] := \mu_0$. If we put

$$P^{(1)} := \exp\left(\frac{\lambda[0](x)}{\mu[0]} t^{-\mu[0]}\right) \circ P^{(0)} \circ \exp\left(-\frac{\lambda[0](x)}{\mu[0]} t^{-\mu[0]}\right),$$

then by Lemma 3.3, the operator $P^{(1)}$ is also an operator of the form (1.1) on $[0, T] \times \Omega_1$ and satisfies the following:

- (a) The operator $P^{(1)}$ satisfies (A-0) and (A-1).

- (b) $S(P^{(1)}) \cap (\mu[0], \infty) = S(P^{(0)}) \cap (\mu[0], \infty)$.
- (c) $\mathcal{C}_\nu[P^{(1)}](x; \cdot) = \mathcal{C}_\nu[P^{(0)}](x; \cdot)$ for every $\nu > \mu[0]$ and $x \in \Omega_1$.
- (d) There holds $\mathcal{C}_{\mu[0]}[P^{(1)}](x; \lambda) = \mathcal{C}_{\mu[0]}[P^{(0)}](x; \lambda + \lambda[0](x))$. If $d_{\mu[0]}(P^{(0)}) > 1$, then $\mu[0] \in S(P^{(1)})$; if $d_{\mu[0]}(P^{(0)}) = 1$, then $\mu[0] \notin S(P^{(1)})$.
- (e) There exists $\mu[1] < \mu[0]$ such that $\mu[1] \in \mathbf{N}/q$ and $S(P^{(1)}) \cap [0, \mu[0]) = \{\mu[1]\}$.
- (f) $d_{\mu[1]}(P^{(1)}) = 1$ and $P^{(1)}$ satisfies (A-2; $\mu[1]$).

By (f), we have $\mathcal{C}_{\mu[1]}[P^{(1)}](x; \lambda) = a[1](x)\lambda - b[1](x)$ for some $a[1], b[1] \in \mathcal{O}(\Omega_1)$ with $a[1](0) \neq 0$.

If $\mu[1] = 0$, then put $M = 0$. Consider the case when $\mu[1] > 0$. We can take a subdomain Ω_2 of Ω_1 including 0 such that $a[1](x) \neq 0$ on Ω_2 , and hence we can take $\lambda[1] \in \mathcal{O}(\Omega_2)$ such that $\mathcal{C}_{\mu[1]}[P^{(1)}](x; \lambda[1](x)) \equiv 0$ on Ω_2 .

If we put

$$P^{(2)} := \exp\left(\frac{\lambda[1](x)}{\mu[1]} t^{-\mu[1]}\right) \circ P^{(1)} \circ \exp\left(-\frac{\lambda[1](x)}{\mu[1]} t^{-\mu[1]}\right),$$

then by Lemma 3.3 and by $d_{\mu[1]}(P^{(1)}) = 1$, the operator $P^{(2)}$ is also an operator of the form (1.1) and satisfies the following:

- (a) The operator $P^{(2)}$ satisfies (A-0) and (A-1).
- (b) $S(P^{(2)}) \cap (\mu[1], \infty) = S(P^{(1)}) \cap (\mu[1], \infty)$.
- (c) $\mathcal{C}_\nu[P^{(2)}](x; \cdot) = \mathcal{C}_\nu[P^{(1)}](x; \cdot)$ for every $\nu > \mu[1]$ and $x \in \Omega_2$.
- (d) There holds $\mathcal{C}_{\mu[1]}[P^{(2)}](x; \lambda) = \mathcal{C}_{\mu[1]}[P^{(1)}](x; \lambda + \lambda[1](x)) = a[1](x)\lambda$ and $\mu[1] \notin S(P^{(2)})$.
- (e) There exists $\mu[2] < \mu[1]$ such that $\mu[2] \in \mathbf{N}/q$ and $S(P^{(2)}) \cap [0, \mu[1]) = \{\mu[2]\}$.
- (f) $d_{\mu[2]}(P^{(2)}) = 1$ and $P^{(2)}$ satisfies (A-2; $\mu[2]$).

We can continue this procedure unless $\mu[j] = 0$. Since $\mu[j] \in \mathbf{N}/q$ and $\mu[0] > \mu[1] > \dots \geq 0$, we necessarily reach $\mu[M+1] = 0$. $\square$

The following lemma is used to construct each term of infinite series in asymptotic solutions.

LEMMA 3.5. *Let $Q(x; \lambda) \in \mathcal{O}(\Omega)[\lambda]$ and $\Lambda \in \mathcal{O}(\Omega)$. Assume that $Q(x; \Lambda(x)) \neq 0$ on Ω . Then, we can solve the equation*

$$(3.4) \quad Q(x; \mathcal{D})v = t^{\Lambda(x)} \sum_{p=0}^L g_p(x) (\log t)^p, \quad g_p \in \mathcal{O}(\Omega) \quad (0 \leq p \leq L)$$

as $v = t^{\Lambda(x)} \sum_{p=0}^L v_p(x) (\log t)^p$, $v_p \in \mathcal{O}(\Omega)$ ($0 \leq p \leq L$).

Proof. By an easy calculation, we have

$$Q(x; \mathcal{D})(t^{\Lambda(x)} (\log t)^p) = \sum_{j=0}^p \binom{p}{j} (\partial_\lambda^j Q)(x; \Lambda(x)) \cdot t^{\Lambda(x)} (\log t)^{p-j}.$$

Hence, (3.4) is equivalent to

$$\begin{aligned} & Q(x; \Lambda(x)) \cdot v_p(x) \\ & + \sum_{j=1}^{L-p} \binom{p+j}{j} (\partial_\lambda^j Q)(x; \Lambda(x)) \cdot v_{p+j}(x) = g_p(x) \quad (p = 0, 1, \dots, L). \end{aligned}$$

Thus, by $Q(x; \Lambda(x)) \neq 0$, we can uniquely determine $v_L, v_{L-1}, \dots, v_0$. $\square$

§4. Proof of Theorem 2.5

In this section, we prove Theorem 2.5. First, we give the existence of an asymptotic solution with no exponential factor, which corresponds to the case $\mu_0 = 0$ and $M = -1$ in Theorem 2.5. Although we use only the case when $\deg_\lambda \mathcal{C}_0[P] = 1$ in the proof of main theorems, this proposition has its own value.

PROPOSITION 4.1. *Assume that P satisfies (A-0), (A-1), and (A-2; 0). Let $\lambda(x) \in \mathcal{O}(\Omega_0)$ satisfy*

- (i) $\mathcal{C}_0[P](x; \lambda(x)) \equiv 0$ on Ω_0 ,
- (ii) $\mathcal{C}_0[P](x; \lambda(x) + l/q) \neq 0$ on Ω_0 for $l \in \mathbb{N} \setminus \{0\}$,

for some subdomain Ω_0 of Ω including 0. Then, for an arbitrarily given $v_{0,0}(x) \in \mathcal{O}(\Omega_0)$, there exist $v_{l,p}(x) \in \mathcal{O}(\Omega_0)$ ($l \geq 0; 0 \leq p \leq lm$) such that

$$(4.1) \quad u(t, x) := t^{\lambda(x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)$$

is an asymptotic solution of $Pu = 0$. That is

$$t^{-\lambda(x)} P \left(t^{\lambda(x)} \cdot \sum_{l=0}^N t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x) \right) = o(t^{N/q - \omega(P)}),$$

for every $N \in \mathbb{N}$.

Proof. We can formally expand P with respect to t as

$$P = t^{-\omega} \left(\mathcal{C}_0[P](x; \vartheta) + \sum_{h=1}^{\infty} B_h(x, \partial_x; \vartheta) t^{h/q} \right),$$

where $B_h(x, \partial_x; \vartheta) = \sum_{j+|\alpha| \leq m} b_{h,j,\alpha}(x) \partial_x^\alpha \vartheta^j$ with $b_{h,j,\alpha} \in \mathcal{O}(\Omega)$ and $\omega := \omega(P)$. Hence, we have only to find $v_{l,p}$ that satisfy

$$\begin{aligned} & \mathcal{C}_0[P](x; \vartheta) \left(t^{\lambda(x)+l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x) \right) \\ &= - \sum_{h=0}^{l-1} B_{l-h}(x, \partial_x; \vartheta) \left(t^{\lambda(x)+l/q} \sum_{p=0}^{hm} (\log t)^p v_{h,p}(x) \right) \quad (l \in \mathbf{N}). \end{aligned}$$

Since

$$\begin{aligned} & \mathcal{C}_0[P](x; \vartheta) (t^{\lambda(x)} v_{0,0}(x)) = \mathcal{C}_0[P](x; \lambda(x)) \cdot t^{\lambda(x)} v_{0,0}(x) \equiv 0, \\ & \partial_x (t^{\lambda(x)+l/q} (\log t)^p v(x)) \\ &= t^{\lambda(x)+l/q} (\log t)^p (\partial_x v)(x) + t^{\lambda(x)+l/q} (\log t)^{p+1} (\partial_x \lambda)(x) v(x), \end{aligned}$$

and since

$$\mathcal{C}_0[P](x; \lambda(x) + l/q) \neq 0 \quad (l \geq 1) \quad \text{on } \Omega_0,$$

we can get $v_{l,p}$ with an arbitrarily given $v_{0,0}$ by applying Lemma 3.5. $\square$

Proof of Theorem 2.5. We can apply Proposition 3.4 to P . By (f) of the proposition, we have

$$\mathcal{C}_0[P^{(M+1)}](x; \lambda) = a[M+1](x)\lambda - b[M+1](x)$$

for some $a[M+1], b[M+1] \in \mathcal{O}(\Omega_{M+1})$ with $a[M+1](0) \neq 0$. Hence, we can take a subdomain Ω_0 of Ω_{M+1} including 0 such that $a[M+1](x) \neq 0$ on Ω_0 . We can take $\lambda[M+1] \in \mathcal{O}(\Omega_0)$ such that $\mathcal{C}_0[P^{(M+1)}](x; \lambda[M+1](x)) \equiv 0$ and $\mathcal{C}_0[P^{(M+1)}](x; \lambda[M+1](x) + l/q) \neq 0$ on Ω_0 for $l \in \mathbf{N} \setminus \{0\}$.

By applying Proposition 4.1 to $P^{(M+1)}$, we can construct an asymptotic solution

$$(4.2) \quad v = t^{\lambda[M+1](x)} \cdot \sum_{l=0}^{\infty} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)$$

of $P^{(M+1)}v = 0$ for an arbitrarily given $v_{0,0} \in \mathcal{O}(\Omega_0)$.

Thus, the proof of Theorem 2.5 is completed. $\square$

§5. Proof of Theorems 2.8 and 2.9

In this section, we prove Theorems 2.8 and 2.9.

First, we introduce another condition (A-3).

(A-3) If $\mu \in S(P)$ and $\mu > 0$, then all the non-zero roots λ of $\mathcal{C}_\mu[P](0; \lambda) = 0$ satisfy $\operatorname{Re} \lambda < 0$.

From the results in [3], we easily get the following theorem, which shall be used later.

THEOREM 5.1. *Assume the conditions (A-0), (A-1), (A-2; 0), and (A-3). Then, there exist $N_0 \in \mathbf{N}$, $T_0 > 0$, and a domain Ω_0 including 0 for which the following holds:*

- (1) *For every $N \geq N_0$ and every $f \in C_{\text{flat}}^{N-\omega(P)}([0, T]; \mathcal{O}(\Omega))$, there exists a unique $u \in C_{\text{flat}}^N([0, T_0]; \mathcal{O}(\Omega_0))$ such that $Pu = f$ on $[0, T_0] \times \Omega_0$.*
- (2) *If $u \in t^{N_0} \times C^0([0, T]; \mathcal{D}'(\Omega \cap \mathbf{R}^n))$ and $Pu = 0$ for $t > 0$ in a neighborhood of $(0, 0)$, then $u = 0$ for $t > 0$ in a neighborhood of $(0, 0)$. Especially, there exists no sufficiently smooth null-solution for P at $(0, 0)$.*

In (2) of this theorem, the domain where $u = 0$ may depend not only on the domain where $Pu = 0$ but also on u itself. As for solutions in $C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n))$, however, we can show the existence of a common domain of uniqueness, by a standard argument as follows.

COROLLARY 5.2. *Assume the same assumptions as in the theorem above. Then there exists $N_0 \in \mathbf{N}$ such that for every $T' \in (0, T)$ and every open neighborhood U' of $0 \in \mathbf{R}^n$, there exist $T'' \in (0, T')$ and an open neighborhood U'' of 0 for which the following holds. If $u \in t^{N_0} \times C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n))$ and $Pu = 0$ on $(0, T') \times U'$, then $u = 0$ on $(0, T'') \times U''$.*

Proof. Put $K := \{u \in t^{N_0} \times C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n)) : Pu = 0 \text{ on } (0, T') \times U'\}$. This is a closed subspace of a Fréchet space $t^{N_0} \times C^0([0, T]; C^0(\Omega \cap \mathbf{R}^n))$, and hence it is also a Fréchet space. Let $\{T_n\}_{n \in \mathbf{N}}$ be a decreasing sequence of positive real numbers converging to 0 and let $\{U_n\}_{n \in \mathbf{N}}$ be a fundamental system of open neighborhoods of 0. Put $L_n := \{u \in K : u = 0 \text{ on } (0, T_n) \times U_n\}$, which are closed subspaces of K . By Theorem 5.1-(2), there holds $K = \bigcup_{n=0}^{\infty} L_n$. Since a Fréchet space is a Baire space, there exists an n such that L_n has an inner point, that is $L_n = K$. $\square$

Now, we give a proof of Theorem 2.8.

Proof of Theorem 2.8. We may assume that $\mu_0 \in \mathbf{N}/q$ without loss of generality, and we can apply Proposition 3.4 to P . The operator $P^{(M+1)}$ satisfies (A-0), (A-1) and (A-2; 0). By the assumption (A-6; μ_0) for P and by the conditions (c), (d), (e) in Proposition 3.4, the operator $P^{(M+1)}$ satisfies (A-3). Further, as we have shown in the proof of Theorem 2.5, the operator $P^{(M+1)}$ has a formal solution (4.2) with $v_{0,0} \equiv 1$.

If we put

$$v_N := t^{\lambda(M+1)(x)} \cdot \sum_{l=0}^{qN} t^{l/q} \sum_{p=0}^{lm} (\log t)^p v_{l,p}(x)$$

and $g_N := P^{(M+1)}(v_N)$ for sufficiently large $N \in \mathbf{N}$, then we have

$$g_N \in C_{flat}^{N-r_0}([0, T]; \mathcal{O}(\Omega_0)),$$

where Ω_0 is a subdomain of Ω including 0 and $r_0 \in \mathbf{N}$, both independent of N . By Theorem 5.1, we get $w_N \in C_{flat}^{N+\omega(P^{(M+1)})-r_0}([0, T_0]; \mathcal{O}(\Omega'_0))$ such that $P^{(M+1)}(w_N) = -g_N$, where $T_0 > 0$ and Ω'_0 is a subdomain of Ω_0 including 0. Thus, $v := v_N + w_N$ satisfies $P^{(M+1)}(v) = 0$ and $t^{-\lambda(M+1)(x)} v(t, x) \rightarrow 1 (t \rightarrow +0)$. Note that Corollary 5.2 implies that v is independent of N for sufficiently large N in a neighborhood of $(0,0)$.

Since $\operatorname{Re} \lambda0 > 0$ by the assumption, we can easily show that

$$u(t, x) := \exp\left(-\sum_{j=0}^M \frac{\lambda[j](x)}{\mu[j]} t^{-\mu[j]}\right) \cdot v(t, x)$$

belongs to $C_{flat}^\infty([0, T_0]; \mathcal{O}(\Omega'_0))$. Thus, u is a C^∞ null-solution for P . $\square$

Next, we give a proof of Theorem 2.9.

Proof of Theorem 2.9. If we take δ' as $\delta > \delta' > \operatorname{Re} \lambda_0$, and if we put $v := \exp(\delta' t^{-\mu_0}/\mu_0) u$, then we have $v \in t^N \times C^0([0, T]; \mathcal{D}'(\Omega_0 \cap \mathbf{R}^n))$ for every $N \in \mathbf{N}$ with some domain Ω_0 and $T > 0$. We also have

$$0 = P\left(\exp\left(-\frac{\delta'}{\mu_0} t^{-\mu_0}\right) v\right) = \exp\left(-\frac{\delta'}{\mu_0} t^{-\mu_0}\right) \bar{P}v,$$

that is, $\bar{P}v = 0$, where $\bar{P} := \exp(\delta' t^{-\mu_0}/\mu_0) \circ P \circ \exp(-\delta' t^{-\mu_0}/\mu_0)$. We have only to show that $v = 0$ for $t > 0$ in a neighborhood of $(0,0)$.

By an argument similar to and easier than that in the proof of Lemma 3.3, the

operator $\tilde{P}$ is an operator of the form (1.1) and satisfies the following:

- (a) The operator $\tilde{P}$ satisfies (A-0), (A-1), and (A-2; μ_0).
- (b) $S(\tilde{P}) \cap (\mu_0, \infty) = S(P) \cap (\mu_0, \infty)$.
- (c) $\mathcal{C}_\nu[\tilde{P}](x; \cdot) = \mathcal{C}_\nu[P](x; \cdot)$ for every $\nu > \mu_0$ and $x \in \Omega_0$.
- (d) $\mathcal{C}_{\mu_0}[\tilde{P}](x; \lambda) = \mathcal{C}_{\mu_0}[P](x; \lambda + \delta)$.
- (e) $S(\tilde{P}) \cap [0, \mu_0] = \{\mu_0\}$.

By (d) and the condition (A-6; μ_0) for P , all the roots λ of $\mathcal{C}_{\mu_0}[\tilde{P}](0; \lambda) = 0$ satisfy $\operatorname{Re} \lambda < 0$. This and the conditions (c), (e) imply that the operator $\tilde{P}$ satisfies (A-3). Further, also by (d), we have $\mathcal{C}_{\mu_0}[\tilde{P}](0; 0) \neq 0$. This and the assumption (A-2; μ_0) imply (A-2; 0). Thus, we can apply Theorem 5.1 to $\tilde{P}$, and hence, we have $v = 0$ for $t > 0$ in a neighborhood of $(0, 0)$. $\square$

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