Y.-H. Lee
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THE EXISTENCE OF PERIODIC SOLUTIONS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

YONG-HOON LEE[1]

Abstract. We prove a multiplicity result of 2π-periodic solutions for certain weakly coupled system of ordinary differential equations with real parameters. The proofs are based on differential inequalities and coincidence degree.

§1. Introduction

In 1986, Fabry, Mawhin and Nkashama [4] have considered periodic problems of the form

$$
\begin{aligned}
&x''(t) + f(t, x(t)) = s \\
&x^{(k)}(0) = x^{(k)}(2\pi), \ k = 0, 1
\end{aligned}
\tag{1.1}
$$

with f continuous, and have proved that if

$$
f(t, x) \longrightarrow \infty \ \text{ as } |x| \to \infty \text{ uniformly in } t \in [0, 2\pi], \tag{H}
$$

an Ambrosetti-Prodi type result holds, namely there exists a number s_o such that the problem has no, at least one or at least two solutions according to $s < s_o$, $s = s_o$ or $s > s_o$. Similar results ([7], [9]) hold for

$$
\begin{aligned}
x'(t) + f(t, x(t)) &= s \\
x(0) &= x(2\pi).
\end{aligned}
$$

Chiappinelli, Mawhin and Nugari [2] have considered the Dirichlet problem

$$
\begin{aligned}
x''(t) + x(t) + f(t, x(t)) &= s\left(\sqrt{\frac{2}{\pi}}\right)\sin t \\
x(0) = 0 &= x(\pi).
\end{aligned}
$$

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Under the assumption (H), they have proved a weakened Ambrosetti-Prodi type result, namely there exist s_o and $\bar{s}$ with $s_o \leq \bar{s}$ such that the problem has no, at least one or at least two solutions according to $s < s_o$, $s = \bar{s}$ or $s > \bar{s}$.

In [3], Ding and Mawhin have studied higher order ordinary differential equations of the form

$$\begin{gathered} x^{(2m)}(t) + f(x(t)) = s + h(t) \\ x^{(k)}(0) = x^{(k)}(2\pi), \ 0 \leq k \leq 2m-1, \ m > 1. \end{gathered}$$

Under the assumption (H) and a supplementary growth condition on f, they proved a weakened Ambrosetti-Prodi type result. Ramos and Sanchez [10] generalize the above result allowing joint dependence of (t, x) in the nonlinear term. With the same conditions found in [3], they have proved that the problem

$$\begin{gathered} x^{(2m)}(t) + f(t, x(t)) = s \\ x^{(k)}(0) = x^{(k)}(2\pi), \ 0 \leq k \leq 2m-1, \ m > 1 \end{gathered} \tag{1.2}$$

has an Ambrosetti-Prodi type result.

Lee [6] has studied periodic solutions for a weakly coupled system of ordinary differential equations

$$x_i''(t) + g_i(t, x_i(t)) + h_i(t, x(t)) = s_i. \tag{1_s}$$

Under the assumption (H) for each g_i and the boundedness on h_i, he has proved that system (1_s) has a weakended Ambrosetti-Prodi type result.

In this paper, we prove under some suitable conditions that there exists $s_o \in \mathbf{R}^n$ such that (1_s) has no 2π-periodic solution, at least one 2π-periodic solution or at least two 2π-periodic solutions according to $s < s_o$, $s = s_o$ or $s > s_o$. Ambrosetti-Prodi type results of 1-dimensional cases are not precisely parallel to those of n-dimensional systems, since the parameter s_o for problems (1.1) or (1.2) is uniquely determined, but not necessarily for system (1_s) (one may refer to Remark in Section 2). We leave a question about more properties of s_o caused by nonuniqueness.

NOTATION. We first introduce some definitions and notation. $I = [0, 2\pi]$. For $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, $\|x\| = (x_1^2 + \cdots, +x_n^2)^{\frac{1}{2}}$, and inequalities in $\mathbf{R}^n$ will be defined componentwise, thus, $x \leq y$ if and only if $x_i \leq y_i$, for

$i = 1, \cdots n$, etc. Mean value $\bar{x}$ of x and the function $\tilde{x}$ of mean value 0 will be respectively defined by $\bar{x} = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt$ and $\tilde{x}(t) = x(t) - \bar{x}$. $\overline{\Omega}$ will denote the closure of Ω, $C^k(I, \mathbf{R}^n)$ will denote the space of continuous functions defined on I into $\mathbf{R}^n$ whose derivative through order k are continuous, and $C^k_{2\pi}(I, \mathbf{R}^n)$ the space of 2π-periodic functions of C^k. Finally for $u \in C^k(I, \mathbf{R}^n)$, $\|u\|_\infty = \sup_{t\in I}\|u(t)\|$ and $\|u\|_2 = (\frac{1}{2\pi}\int_0^{2\pi}\|u(t)\|^2dt)^{\frac{1}{2}}$.

Let us consider the n-dimensional second order system

$$x_i''(t) + F_i(t, x(t), x_i'(t)) = s_i, \tag{2_s}$$

where s_i is a real parameter, $F_i : I \times \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}$ is 2π-periodic in the first variable and continuous. We recall some definitions.

DEFINITION 1. $\alpha \in C^2(I, \mathbf{R}^n)$ is called a *lower solution* of (2_s) if

$$\begin{aligned} &\alpha_i''(t) + F_i(t, \alpha(t), \alpha_i'(t)) \geq s_i \\ &\alpha_i(0) = \alpha_i(2\pi),\ \alpha_i'(0) \geq \alpha_i'(2\pi),\ i = 1, \cdots, n. \end{aligned}$$

Similarly, $\beta \in C^2(I, \mathbf{R}^n)$ is called an *upper solution* of (2_s) if

$$\begin{aligned} &\beta_i''(t) + F_i(t, \beta(t), \beta_i'(t)) \leq s_i \\ &\beta_i(0) = \beta_i(2\pi),\ \beta_i'(0) \leq \beta_i'(2\pi),\ i = 1, \cdots, n. \end{aligned}$$

Denote a vector-valued function $F(t, x, y)$ on $I \times \mathbf{R}^n \times \mathbf{R}^n$ whose i^{th}-component is $F_i(t, x, y_i)$.

DEFINITION 2. $F : I \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is said to be *quasi-monotone nondecreasing* if for $1 \leq i \leq n$, $F_i(t, u_1, \cdots, u_n, y_i) \leq F_i(t, v_1, \cdots, v_n, y_i)$, whenever $u_j \leq v_j$, for $j \neq i$ and $u_i = v_i$.

DEFINITION 3. A function $F : I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies a *Nagumo condition* if for each $R > 0$, there exist $\phi_i \in C(\mathbf{R}_+, (0, \infty))$, $i = 1, \cdots, n$, increasing and $\lim_{s\to\infty}\frac{s^2}{\phi_i(s)} = \infty$ such that for all $\|x\| \leq R$, $y_i \in \mathbf{R}$ and $t \in I$,

$$|F_i(t, x, y_i)| \leq \phi_i(|y_i|).$$

It is well known that the Nagumo condition provides *a priori* bound of x' as shown in Lemma 1.

LEMMA 1. *Let x be any 2π-periodic solution of (2_s) satisfying $\|x\|_\infty \leq R$. If F satisfies a Nagumo condition, then there exists $M_1 > 0$ depending only on s, R and ϕ_i, $i = 1, \cdots, n$ such that*

$$\|x'\|_\infty \leq M_1.$$

We give a fundamental theorem on the existence for a 2π-periodic solution of (2_s).

THEOREM 1. (Hu and Lakshmikantham [5]) *Assume that:*

(1) *α and β are respectively lower and upper solution of (2_s) with $\alpha(t) \leq \beta(t)$ on I.*

(2) *F is quasimonotone nondecreasing.*

(3) *F satisfies a Nagumo condition.*

Then (2_s) has a 2π-periodic solution $x(t)$ such that $\alpha(t) \leq x(t) \leq \beta(t)$ on I and

$$\|x'\|_\infty \leq M_1.$$

§2. Existence

In this section, we give an existence theorem for a 2π-periodic solution of (2_s).

THEOREM 2. *Let $\bar{s} \in \mathbf{R}^n$ be given by $\bar{s}_i = \max_{t \in I} F_i(t, 0, 0)$. Suppose that*

(A_1) *For each s^*, there exists $R_1(s^*) \in \mathbf{R}^n$ such that for $i = 1, \cdots, n$,*

$$F_i(t, x, 0) \geq \max\{s_i^*,\ \bar{s}_i\}, \ \text{whenever } x \leq -R_1(s^*), \quad \text{for all } t. \tag{2.1}$$

(A_2) *$F(\cdot, \cdot, 0)$ is bounded below by ρ:*

$$F_i(t, x, 0) \geq \rho_i, \quad \text{for all } t \in I,\ x \in \mathbf{R}^n.$$

(A_3) *For each s^*, there exists a positive real number $M_2(s^*)$ such that for each $s \leq s^*$ and each possible 2π-periodic solution x of (2_s), one has*

$$\|x\|_\infty < M_2(s^*).$$

(A_4) *F is quasi-monotone nondecreasing.*

(A_5) *F satisfies a Nagumo condition.*

Then there exists $s_o \in \mathbf{R}^n$ such that (2_s) has no 2π-periodic solution for $s < s_o$ and at least one 2π-periodic solution for $s \geq s_o$.

Proof. We first show that $(2_{\bar{s}})$ has a 2π-periodic solution. By Theorem 1, it is sufficient to find upper and lower solutions β and α of $(2_{\bar{s}})$ with $\alpha(t) \leq \beta(t)$ on I. It is obvious that $\beta \equiv 0$ is an upper solution. On the other hand, by (2.1), $-R_1(\bar{s})$ is a lower solution. Let $\mathcal{S} = \{ s \in \mathbf{R}^n : (2_s)$ has a 2π-periodic solution $\}$, then $\mathcal{S} \neq \emptyset$ and $(\mathcal{S}, \leq)$ is a partially ordered set. We notice that ρ is a lower bound for $\mathcal{S}$ in $\mathbf{R}^n$, otherwise, there is $s \in \mathcal{S}$ with $s_k < \rho_k$ for some $k = 1, \cdots, n$. Let x be a 2π-periodic solution of (2_s) and let $x_k(t_o) = \min_t x_k(t)$, then $x_k'(t_o) = 0$, $x_k''(t_o) \geq 0$ and

$$\rho_k \leq F_k(t_o, x(t_o), 0) \leq s_k < \rho_k.$$

This is a contradiction. We show that $\mathcal{S}$ has at least one minimal element. Let $\mathcal{C}$ be a chain in $\mathcal{S}$, we claim that $\mathcal{C}$ has a lower bound in $\mathcal{S}$. Let $s_{(1)} \in \mathcal{C}$ and without loss of generality, we suppose that it is not a lower bound for $\mathcal{C}$, then we may choose $s_{(2)} \in \mathcal{C}$ such that $s_{(2)} \neq s_{(1)}$ and $s_{(2)} \leq s_{(1)}$. Suppose similarly that $s_{(2)}$ is not a lower bound for $\mathcal{C}$, then we also choose $s_{(3)} \in \mathcal{C}$ such that $s_{(3)} \neq s_{(2)}$ and $s_{(3)} \leq s_{(2)}$. Continuing this process, we obtain a distinct sequence $(s_{(k)}) \subset \mathcal{S}$ such that $s_{(k+1)} \leq s_{(k)}$, $k = 1, 2, \cdots$. We notice that for each $i = 1, \cdots, n$, sequence of i^{th}-components $(s_{(k)_i})_{k=1}^{\infty}$ is decreasing in $\mathbf{R}$ bounded below by ρ_i. Thus the sequence converges to a number s_{c_i}, $i = 1, \cdots, n$. Let $s_c = (s_{c_i})_{i=1}^n$, then $s_c \in \mathcal{S}$. Indeed, let $(x_{(k)})$ be a sequence of 2π-periodic solutions of $(2_{s_{(k)}})$. We notice that the bound M_1 in Lemma 1 can be taken independent of s if s belongs to some bounded set. Therefore, by this fact and (A_3), the sequence $(x_{(k)})$ is bounded in $C^1_{2\pi}(I, \mathbf{R}^n)$. Thus it is also bounded in $C^2_{2\pi}(I, \mathbf{R}^n)$, since $x_{(k)}$ is a solution of $(2_{s_{(k)}})$. By compact imbedding property of $C^2_{2\pi}$ in $C^1_{2\pi}$, the sequence contains a subsequence converging to some $x \in C^1_{2\pi}$. It is easy to check by the integrated form that x is a 2π-periodic solution of (2_{s_c}), thus $s_c \in \mathcal{S}$. It is not hard to see that s_c is a lower bound for $\mathcal{C}$. Therefore, by Zorn's lemma, $\mathcal{S}$ has a minimal element, say s_o. To complete the proof, we show that for each $s^* > s_o$, $[s_o, s^*] \subset \mathcal{S}$. Let u be a 2π-periodic solution of (2_{s_o}), then for $s \in [s_o, s^*]$,

$$u_i''(t) + F_i(t, u(t), u_i'(t)) = s_{oi} \leq s_i.$$

This implies that $u(t)$ is an upper solution of (2_s). On the other hand, taking $R_1(s^*)$ in (2.1) large enough to satisfy $-R_1(s^*) < u(t)$, for all $t \in I$, we get $-R_1(s^*)$ a lower solution of (2_s). Therefore (2_s) has at least one 2π-periodic solution for all $s \in [s_o, s^*]$, and the proof is complete.

We consider the following differential system:

$$(3_s) \qquad x_i''(t) + g_i(t, x_i(t), x_i'(t)) + h_i(t, x(t)) = s_i,$$

where $g_i : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $h_i : I \times \mathbf{R}^n \to \mathbf{R}$ are 2π-periodic in the first variable and continuous. We first give *a priori* estimate for possible 2π-periodic solutions of (3_s).

LEMMA 2. *Suppose for* $i = 1, \cdots, n$,

(1) $\lim_{|x|\to\infty} g_i(t, x, y) = \infty$ *uniformly in* t *and* y.

(2) g_i *is bounded below.*

(3) h_i *is bounded below.*

Then for each $s^* \in \mathbf{R}^n$, *there exist* $r(s^*) \in \mathbf{R}^n$ *and* $R(s^*) \in \mathbf{R}$ *such that for each* $s \leq s^*$ *and each possible* 2π*-periodic solution* x *of* (3_s), *one has*

$$-r_i(s^*) < x_i(t) < r_i(s^*) + R(s^*),$$

for all $t \in I$ *and* $i = 1, \cdots, n$.

Proof. Let s^* be given and let x be a 2π-periodic solution of (3_s) for $s \leq s^*$. Then

$$(2.2) \qquad \frac{1}{2\pi} \int_0^{2\pi} g_i(t, x_i(t), x_i'(t)) + h_i(t, x(t)) dt = s_i\,.$$

We may assume by hypotheses that there exist σ, $\nu \in \mathbf{R}^n$ such that $g_i(t, x, y) \geq \sigma_i$ and $h_i(t, u) \geq \nu_i$ for $i = 1, \cdots, n$ and for all $t \in I$, $x, y \in \mathbf{R}$ and $u \in \mathbf{R}^n$. Without loss of generality, we may assume $s \geq \sigma + \nu$, otherwise (3_s) has no 2π-periodic solution. Indeed; Assume $s_k < \sigma_k + \nu_k$, for some $k = 1, \cdots, n$. Let x be a 2π-periodic solution of (3_s) and let $x_k(t_o) = \min_{t\in I} x_k(t)$, then $x_k''(t_o) \geq 0$ and

$$\sigma_k + \nu_k \leq g_k(t_o, x_k(t_o), x_k'(t_o)) + h_k(t_o, x(t_o)) \leq s_k < \sigma_k + \nu_k.$$

The contradiction shows that $s \geq \sigma + \nu$ if (3_s) has a 2π-periodic solution. Taking the Euclidean inner product of (3_s) with $\tilde{x}(t)$ and integrating over the period, by (2.2), we get

$$\begin{aligned}
\|x'\|_2^2 &= \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi} \tilde{x}_i(t)\ \{g_i(t,x_i(t),x_i'(t)) - \sigma_i + h_i(t,x(t)) - \nu_i\}dt \\
&\leq \frac{1}{2\pi}\sum_{i=1}^{n}\|\tilde{x}_i\|_\infty\{\int_0^{2\pi} g_i(t,x_i(t),x_i'(t)) + h_i(t,x(t))dt - 2\pi(\sigma_i+\nu_i)\} \\
&= \sum_{i=1}^{n}\|\tilde{x}_i\|_\infty\{s_i - (\sigma_i+\nu_i)\} \\
&\leq \frac{\pi}{\sqrt{3}}\sum_{i=1}^{n}\|x_i'\|_2\{s_i^* - (\sigma_i+\nu_i)\} \quad \text{by Sobolev inequality and } s^* \geq s \\
&\leq \frac{\pi}{\sqrt{3}}\|x'\|_2\|s^* - (\sigma+\nu)\|.
\end{aligned}$$

Thus

$$\|x'\|_2 \leq \frac{\pi}{\sqrt{3}}\|s^* - (\sigma+\nu)\|.$$

We may assume by hypotheses that there exists $r(s^*) \geq 0$ in $\mathbf{R}^n$ such that

$$g_i(t,x,y) + h_i(t,u) > s_i^*, \tag{2.3}$$

whenever $|x| \geq r_i(s^*)$, for all $t \in I$, $y \in \mathbf{R}$, $u \in \mathbf{R}^n$. We claim that for $i = 1,\cdots,n$, there exists $\tau_i \in I$ such that $|x_i(\tau_i)| < r_i(s^*)$ for all possible solutions x of (3_s). Suppose that the claim is not true, then there exist a solution x and an index k such that

$$|x_k(t)| \geq r_k(s^*),$$

for all $t \in I$. Thus by (2.3)

$$g_k(t,x_k(t),x_k'(t)) + h_k(t,x(t)) > s_k^* \geq s_k.$$

This contradicts to (2.2). Now

$$\begin{aligned}
|x_i(t)| &\leq |x_i(\tau_i)| + |\int_{\tau_i}^{t} x_i'(\tau)d\tau| \\
&\leq |x_i(\tau_i)| + 2\pi\|x'\|_2 \\
&< r_i(s^*) + \frac{2\pi^2}{\sqrt{3}}\|s^* - (\sigma+\nu)\| \\
&\leq r_i(s^*) + R(s^*),
\end{aligned}$$

where $R(s^*)$ is a constant greater than $\frac{2\pi^2}{\sqrt{3}}\|s^* - (\sigma + \nu)\|$. We claim that $-r_i(s^*) < x_i(t) < r_i(s^*) + R(s^*)$, for all $t \in I$. If it is not true, then there exist a solution x, an index k and $t_o \in I$ such that

$$x_k(t_o) \leq -r_k(s^*).$$

Let $x_k(\tau) = \min x_k(t)$, then $x_k(\tau) \leq -r_k(s^*)$ and by (2.3), we get

$$g_k(\tau, x_k(\tau), x_k'(\tau)) + h_k(\tau, x(\tau)) > s_k^*.$$

Since $x_k''(\tau) \geq 0$,

$$s_k^* < x_k''(\tau) + g_k(\tau, x_k(\tau), x_k'(\tau)) + h_k(\tau, x(\tau)) = s_k.$$

This contradicts $s_k \leq s_k^*$ and the proof is complete.

Conditions (1) $\sim$ (3) in Lemma 2 implies (A_1), (A_2) in Theorem 2 and Lemma 2 itself implies (A_3) in Theorem 2. Therefore we have the following corollary for the existence result of (3_s).

COROLLARY 1. *Suppose for* $i = 1, \cdots, n$,

(1) $\lim_{|x| \to \infty} g_i(t, x, y) = \infty$ *uniformly in* t *and* y.

(2) g_i *is bounded below.*

(3) h_i *is bounded below.*

(4) $h(t, x)$ *is quasi-monotone nondecreasing in* x.

(5) g *satisfies a Nagumo condition.*

then there exists $s_o \in \mathbf{R}^n$ *such that* (2_s) *has no* 2π*-periodic solution for* $s < s_o$ *and at least one* 2π*-periodic solution for* $s \geq s_o$.

Now let us consider equation (1_s), i.e.

$$(1_s) \qquad x_i''(t) + g_i(t, x_i(t)) + h_i(t, x(t)) = s_i,$$

where $g_i : I \times \mathbf{R} \to \mathbf{R}$ and $h_i : I \times \mathbf{R}^n \to \mathbf{R}$ are 2π-periodic in the first variable and continuous, for $i = 1, \cdots, n$. We have the following lemma for *a priori* estimate, and the proof follows on the lines of the proof of Lemma 2.

LEMMA 3. *Suppose for* $i = 1, \cdots, n$,

(1) $\lim_{|x| \to \infty} g_i(t, x) = \infty$ *uniformly in* t.

(2) h_i *is bounded below.*

Then for each $s^* \in \mathbf{R}^n$, *there exist* $r(s^*) \in \mathbf{R}^n$ *and* $R(s^*) \in \mathbf{R}$ *such that for each* $s \leq s^*$ *and each possible* 2π*-periodic solution of* (1_s), *one has*

$$-r_i(s^*) < x_i(t) < r_i(s^*) + R(s^*),$$

for all $t \in I$ *and* $i = 1, \cdots, n$.

The conditions (2) and (5) in Corollary 1 are redundant and we have the existence result for (1_s) as follows.

COROLLARY 2. *Suppose for* $i = 1, \cdots, n$,

$(H1)$ $\lim_{|x| \to \infty} g_i(t, x) = \infty$, *uniformly in* t.

$(H2)$ h_i *is bounded below.*

$(H3)$ $h(t, x)$ *is quasi-monotone nondecreasing in* x.

Then there exists $s_o \in \mathbf{R}^n$ *such that* (1_s) *has no* 2π*-periodic solution for* $s < s_o$ *and at least one* 2π*-periodic solution for* $s \geq s_o$.

Remark. We notice that the parameter s_o in Theorem 2 is not necessarily unique if $n \geq 2$. Let us consider the following system

$$\begin{aligned} x'' + cx' + x^2 &= s_1 \\ y'' + dy' + y^2 + \tan^{-1} x &= s_2, \end{aligned}$$

where c and d are both nonzero real constants. We can see by integrating on a period after multiplication by x' that the equation $x'' + cx' + x^2 = s_1$ admits only constant solutions $x \equiv \pm\sqrt{s_1}$, $s_1 \geq 0$. Therefore the system has exactly four couples of solutions as follows;

$$\begin{pmatrix} \sqrt{s_1} \\ \sqrt{s_2 - \tan^{-1}\sqrt{s_1}} \end{pmatrix}, \begin{pmatrix} \sqrt{s_1} \\ -\sqrt{s_2 - \tan^{-1}\sqrt{s_1}} \end{pmatrix},$$
$$\begin{pmatrix} \sqrt{s_1} \\ \sqrt{s_2 + \tan^{-1}\sqrt{s_1}} \end{pmatrix}, \begin{pmatrix} \sqrt{s_1} \\ -\sqrt{s_2 + \tan^{-1}\sqrt{s_1}} \end{pmatrix}.$$

We also easily see that this system satisfies assumptions in Theorem 2 and the set of minimal elements of $\mathcal{S}$ becomes $\{(s_1, -\tan^{-1}\sqrt{s_1}) : s_1 \geq 0\}$. Therefore the curve $s_1 = 0, s_2 \geq 0$ and $s_2 = -\tan^{-1}\sqrt{s_1}, s_1 \geq 0$ in (s_1, s_2) space separates the parametric points with 2π-periodic solutions from those without 2π-periodic solutions. We notice that the parameter s_o in Theorem 2 is $(0, 0)$ in this example.

§3. Degree computations and multiplicity

Our main concern is the multiplicity of 2π-periodic solution for equation (1_s) and we state the main theorem.

THEOREM 3. *Suppose that $(H1)$, $(H2)$ and $(H3)$ in Corollary 2 are satisfied. Then there exists $s_o \in \mathbf{R}^n$ such that*

(1) *(1_s) has no 2π-periodic solution for $s < s_o$.*

(2) *(1_s) has at least one 2π-periodic solution for $s = s_o$.*

(3) *(1_s) has at least two 2π-periodic solutions for $s > s_o$.*

Operator set-up. We reduce problem (1_s) to an equivalent operator form. Let us define $L : D(L) \subset C^0_{2\pi}(I, \mathbf{R}^n) \longrightarrow C^0(I, \mathbf{R}^n)$ by $(x_1, \cdots, x_n) \mapsto (x_1'', \cdots, x_n'')$, where $D(L) = C^2_{2\pi}(I, \mathbf{R}^n)$, and $N_s : C^0_{2\pi}(I, \mathbf{R}^n) \longrightarrow C^0(I, \mathbf{R}^n)$ by

$$(N_s)_i x(\cdot) = g_i(\cdot, x_i(\cdot)) + h_i(\cdot, x(\cdot)) - s_i$$

so that (1_s) can be written as

$$Lx + N_s x = 0. \tag{3.1}$$

It is easy to see that L is a Fredholm operator of index 0 and N_s is $L-$compact on $\overline{\Omega}$ for any bounded open Ω in $C^0_{2\pi}(I, \mathbf{R}^n)$ (see [8]). The coincidence degree $D_L(L + N_s, \Omega)$ is well-defined if $Lx + N_s x \neq 0$ for $x \in D(L) \cap \partial\Omega$.

In what follows, without any further comments, s_o means the one given in Corollary 2. The following lemma is a common result for Ambrosetti-Prodi type problems.

LEMMA 4. *If $(H1)$, $(H2)$ and $(H3)$ are satisfied, then for each $s^* \in \mathbf{R}^n$ and each open bounded set Ω such that $\Omega \supset \{x \in C^0_{2\pi} : -r_i(s^*) < x_i(t) < r_i(s^*) + R(s^*),\ t \in I,\ i = 1, \cdots, n\}$, one has*

$$D_L(L + N_s, \Omega) = 0 \text{ whenever } s \leq s^*.$$

Proof. By Lemma 3, $D_L(L + N_s, \Omega)$ is well-defined and by Corollary 2, (1_s) has no solution for $s < s_o$. Therefore

$$D_L(L + N_s, \Omega) = 0 \text{ whenever } s < s_o.$$

For any fixed $\tilde{s}$ ($< s_o$), by the homotopy invariance of degree, we have

$$D_L(L + N_s, \Omega) = D_L(L + N_{\tilde{s}}, \Omega) = 0,$$

whenever $s \leq s^*$ and this completes the proof.

Modified function. For convenience, we define $f_s : I \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ by $f_{si}(t,x) = g_i(t,x_i) + h_i(t,x) - s_i$. Let $u,\ v \in C^0(I, \mathbf{R}^n)$ satisfy $v(t) \leq u(t)$. We define a *modified function F_s of f_s with respect to u and v* as

$$F_{si}(t,x) = \begin{cases} f_{si} \quad (t,\bar{x}) - \frac{x_i - u_i(t)}{1+x_i^2}, & \text{if } x_i > u_i(t) \\ f_{si} \quad (t,x), \quad \text{if } v_i(t) \leq x_i \leq u_i(t) \\ f_{si} \quad (t,\bar{x}) - \frac{x_i - v_i(t)}{1+x_i^2}, & \text{if } x_i < v_i(t) \end{cases}$$

and

$$\bar{x}_i = \begin{cases} u_i(t), & \text{if} \quad x_i > u_i(t) \\ x_i, & \text{if} \quad v_i(t) \leq x_i \leq u_i(t) \\ v_i(t), & \text{if} \quad x_i < v_i(t) \end{cases}.$$

We sometimes call F_s the *modification of f_s with respect to u and v.* Notice that F_s is continuous and bounded on $I \times \mathbf{R}^n$ and the bound of F_{si} is given by

$$\max\{|f_{si}(t,x)| : t \in I,\ v(t) \leq x \leq u(t)\} + \max_{t \in I} |u_i(t)| + \max_{t \in I} |v_i(t)| + 1.$$

Let $s^* > s_o$ and let $u(t)$ be a 2π-periodic solution of (1_{s_o}), then for $s \in (s_o, s^*]$, $u(t)$ is an upper solution and $-r(s^*)$, given in Lemma 3, is a lower solution of (1_s), respectively, and $-r(s^*) \leq u(t)$ for all $t \in I$. Therefore we may define the modification F_s of f_s with respect to u and $-r(s^*)$, and easily see that the bound of F_s does not depend of s, since $s \in (s_o, s^*]$. For fixed $s^*(> s_o)$, let us consider a homotopy

$$(4_s^\mu) \qquad x''(t) - (1-\mu)x(t) + \mu F_s(t, x(t)) = 0$$

where $\mu \in [0,1]$, F_s is the modification of f_s with respect to u and $-r(s^*)$. We now give *a priori* estimate for possible 2π-periodic solutions of the homotopy.

LEMMA 5. *Assume $(H1)$, $(H2)$ and $(H3)$ are satisfied. Let s^* $(> s_o)$ be given. For $s \in (s_o, s^*]$, let x be a possible 2π-periodic solution of (4_s^μ). Then there exists a real number $M(s^*) > 0$, such that*

$$\|x_i\|_\infty < r_i(s^*) + M(s^*),$$

where $r_i(s^*)$ is given in Lemma 3.

Proof. Let x be a 2π-periodic solution of (4_s^μ) and let

$$|F_{s_i}(t,x)| \leq C_i(s^*)$$

for all $(t,x) \in I \times \mathbf{R}^n$. Following similar steps in the proof of Lemma 2, we get

$$\|x'\|_2 \leq \frac{\pi}{\sqrt{3}} C(s^*),$$

where $C(s^*) = (\sum_{i=1}^n C_i(s^*)^2)^{\frac{1}{2}}$. We show that there exist $\tau_i \in I$ for $i = 1, \cdots, n$ such that

$$|x_i(\tau_i)| \leq r_i(s^*).$$

The inequality is obvious for $\mu = 0$. If the inequality is not true for $\mu \in (0,1]$, then there exist a 2π-periodic solution x and an index k such that either $x_k(t) > r_k(s^*)$ or $x_k(t) < -r_k(s^*)$, for all $t \in I$. If $x_k(t) > r_k(s^*)$ then we may modify $r_k(s^*) \geq \|u_k\|_\infty$ if necessary, so that

$$\frac{x_k(t) - u_k(t)}{1 + x_k(t)^2} \geq 0.$$

Integrating (4_s^μ) over the period,

$$\begin{aligned}(1-\mu)\int_0^{2\pi} x_k(t)dt &\leq \mu \int_0^{2\pi} \{g_k(t,u_k(t)) + h_k(t,u(t)) - s_k\}dt \\ &\leq 2\pi\mu(s_{o_k} - s_k) < 0.\end{aligned}$$

This is a contradiction, since x_k is nonnegative on I. We can show a contradiction in a similar fashion for the case $x_k(t) < -r_k(s^*)$. Therefore we get

$$|x_i(\tau_i)| \leq r_i(s^*),$$

for some $\tau_i \in I$. Now

$$\begin{aligned}|x_i(t)| &\leq |x_i(\tau_i)| + \int_{\tau_i}^t |x_i'(\tau)|d\tau \\ &\leq r_i(s^*) + 2\pi\|x'\|_2 \\ &\leq r_i(s^*) + \frac{2\pi^2}{\sqrt{3}} C(s^*),\end{aligned}$$

for all $t \in I$. Taking $M(s^*) > \frac{2\pi^2}{\sqrt{3}} C(s^*)$, we get the conclusion.

We now set up the operator form of (4_s^μ) as follows. Define $T: C^0_{2\pi}(I,\mathbf{R}^n) \times [0,1] \longrightarrow C^0(I,\mathbf{R}^n)$ by

$$T(x,\mu)(\cdot) = -(1-\mu)x(\cdot) + \mu F_s(\cdot, x(\cdot))$$

then (4_s^μ) is equivalent to

$$Lx + T(x,\mu) = 0$$

and by the standard argument ([8]), T is L-compact on $\overline{\Omega}$ for any open bounded Ω, and $D_L(L + T(\cdot,\mu),\Omega)$ is well-defined and constant in μ if $Lx + T(x,\mu) \neq 0$ for $\mu \in [0,1]$ and $x \in D(L) \cap \partial\Omega$. To get multiple solutions, we need the following lemma.

LEMMA 6. *If $(H1)$, $(H2)$ and $(H3)$ are satisfied, then for each $s^* > s_o$, one can find an open bounded subset $\Omega_1 \subset C^0_{2\pi}(I,\mathbf{R}^n)$ such that for each $s \in (s_o, s^*]$,*

$$D_L(L + N_s, \Omega_1) = \pm 1.$$

Proof. Let $s^* > s_o$, $s \in (s_o, s^*]$ and let $u(t)$ be a 2π-periodic solution of (1_{s_o}). For the modification F_s of f_s with respect to u and $-r(s^*)$, consider

$$(4_s^1) \qquad x''(t) + F_s(t, x(t)) = 0.$$

If x is a 2π-periodic solution of (4_s^1), then we get

$$-r(s^*) \leq x(t) \leq u(t).$$

Indeed; Suppose that the inequalities are not true, so assume that there exist $k \in \{1,\cdots,n\}$ and $\tau \in I$ such that

$$x_k(\tau) > u_k(\tau).$$

Then $x_k - u_k$ has a positive maximum at $t_o \in I$ so that $x_k(t_o) > u_k(t_o)$ and $x_k''(t_o) \leq u_k''(t_o)$. And we get

$$\begin{aligned}
0 = \; & x_k''(t_o) + F_{sk}(t_o, x(t_o)) \\
= \; & x_k''(t_o) + f_{sk}(t_o, \overline{x(t_o)}) - \frac{x_k(t_o) - u_k(t_o)}{1 + x_k(t_o)^2} \\
< \; & x_k''(t_o) + g_k(t_o, u_k(t_o)) + h_k(t_o, \overline{x(t_o)}) - s_k \\
\leq \; & x_k''(t_o) + g_k(t_o, u_k(t_o)) + h_k(t_o, u(t_o)) - s_k, \text{ by quasi-monotonicity of } h \\
\leq \; & u_k''(t_o) + g_k(t_o, u_k(t_o)) + h_k(t_o, u(t_o)) - s_k = s_{ok} - s_k.
\end{aligned}$$

This contradicts $s > s_o$ and, thus, $x(t) \leq u(t)$ for all $t \in I$. Similarly, by (2.3) and quasi-monotonicity of h, we can show that $-r(s^*) \leq x(t)$ for all $t \in I$. Thus the inequalities imply, by the definition of modified function, that x is a 2π-periodic solution of (1_s). Since $s \in (s_o, s^*]$, $x(t) < u(t)$ and also by Lemma 3, $-r(s^*) < x(t)$. Therefore we have

$$-r(s^*) < x(t) < u(t). \tag{3.2}$$

Let

$$\Omega_1 = \{x \in C^0_{2\pi}(I, \mathbf{R}^n) : -r(s^*) < x(t) < u(t),\ t \in I\}.$$

Then $\Omega_1 \subset \Omega$, where Ω is given in Lemma 4, and (4^1_s) is equivalent to (1_s) on Ω_1. Therefore

$$D_L(L + T(\cdot, 1), \Omega_1) = D_L(L + N_s, \Omega_1).$$

Now it is enough to compute $D_L(L+T(\cdot,1),\Omega_1)$. Let Ω_o be an open bounded set in $C^0_{2\pi}(I, \mathbf{R}^n)$ such that

$$\Omega_o \supset \{x \in C^0_{2\pi}(I, \mathbf{R}^n) : \|x_i\|_\infty < r_i(s^*) + M(s^*),\ i = 1, \cdots, n\}.$$

Then $\Omega_1 \subset \Omega_o$ and by Lemma 5, $D_L(L + T(\cdot, \mu), \Omega_o)$ is well-defined for $\mu \in [0, 1]$. We know by (3.2) that every possible 2π-solution of (4^1_s) is contained in Ω_1. Thus by the excision property of degree,

$$D_L(L + T(\cdot, 1), \Omega_o) = D_L(L + T(\cdot, 1), \Omega_1).$$

Furthermore, by Proposition II. 16 in [8] and the homotopy invariance of degree, we get

$$\begin{aligned}\pm 1 &= D_L(L - I, \Omega_o)\\ &= D_L(L + T(\cdot, 0), \Omega_o)\\ &= D_L(L + T(\cdot, 1), \Omega_o)\\ &= D_L(L + T(\cdot, 1), \Omega_1).\end{aligned}$$

This completes the proof.

We now prove our main result.

Proof of Theorem 3. It is enough to show, by Corollary 2, that (1_s) has at least two solutions for $s > s_o$. Let us fix s with $s > s_o$. Then we may choose Ω and Ω_1 as in Lemma 4 and Lemma 6 respectively. By the additivity of degree, we have

$$0 = D_L(L + N_s, \Omega) = D_L(L + N_s, \Omega_1) + D_L(L + N_s, \Omega \setminus \overline{\Omega}_1).$$

Since $D_L(L + N_s, \Omega_1) = \pm 1$ by Lemma 6, we get

$$D_L(L + N_s, \Omega \setminus \overline{\Omega}_1) = \mp 1.$$

This implies that (1_s) has a 2π-periodic solution in Ω_1 and another in $\Omega \setminus \overline{\Omega}_1$. And the proof is done.

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References

[1] A. Ambrosetti and G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. pura Appl., **4** (93) (1972), 231–247.

[2] R. Chiappinelli, J. Mawhin and R. Nugari, *Generalized Ambrosetti-Prodi conditions for nonlinear two-point boundary value problems*, J. Diff. Equations, **69** (3) (1988), 422–434.

[3] S. H. Ding and J. Mawhin, *A multiplicity result for periodic solutions of higher order ordinary differential equations*, Differential and Integral Equations, **1** (1) (1988), 31–39.

[4] C. Fabry, J. Mawhin and M. N. Nkashama, *A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations*, Bull. London Math. Soc, **18** (1986), 173–180.

[5] S. Hu and V. Lakshmikantham, *Periodic boundary value problems for second order impulsive differential systems*, Nonlinear Analysis, T.M.A., **13** (1989), 75–85.

[6] Y.H. Lee, *An Ambrosetti-Prodi type result for second order weakly coupled systems*, Nonlinear Analysis T.M.A., **18** (8) (1992), 793–799.

[7] J. Mawhin, *First order ordinary differential equations with several periodic solutions*, J. Appl. Math. Physics, **38** (1987), 257–265.

[8] J. Mawhin, Topological degree methods in nonlinear boundary value problems, 40, CBMS Conference in Math, A.M.S. Providence R.I., 1979.

[9] M. N. Nkashama, *A generalized upper and lower solution method and multiplicity results for nonlinear first order ordinary differential equations*, J. Math. Anal. Appl., **140** (1989), 381–395.

[10] M. Ramos and L. Sanchez, *Multiple periodic solutions for some nonlinear ordinary differential equations of higher order*, Differential and Integral Equations, **1** (2) (1989), 81–90.

Department of Mathematics
Pusan National University
Pusan 609-735
Korea
yhlee@hyowon.cc.pusan.ac.kr