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FAMILIES OF SOLVABLE FROBENIUS SUBGROUPS IN FINITE GROUPS

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Abstract. We introduce the notion of abelian system on a finite group G, as a particular case of the recently defined notion of kernel system (see this Journal, September 2001). Using a famous result of Suzuki on *CN*-groups, we determine all finite groups with abelian systems. Except for some degenerate cases, they turn out to be special linear group of rank 2 over fields of characteristic 2 or Suzuki groups. Our ideas were heavily influenced by [1] and [8].

§0. Introduction

The purpose of this paper is to classify all abelian Frobenius systems on finite groups. By a Frobenius system on a finite group G, we mean Frobenius a kernel system in the sense of [4], *i.e.* a mapping $\mathcal{F}$ from the set $\mathcal{MS}(G)$ of maximal solvable subgroups of G to the power set $\mathcal{P}(G)$ of G, such that the following conditions are satisfied, for all $M \in \mathcal{MS}(G)$:

(FS1) $\mathcal{F}(M)$ is a normal subgroup of M;
(FS2) $\forall a \in M \setminus \mathcal{F}(M),\ \ C_{\mathcal{F}(M)}(a) = 1$;
(FS3) $\forall g \in G \setminus M,\ \ \mathcal{F}(M) \cap \mathcal{F}(M)^g = \{1\}$.

The Frobenius system is said to be abelian if one has in addition:

(FS4) $\forall M \in \mathcal{MS}(G),\ \ M/\mathcal{F}(M)$ is abelian.

As seen in [4, Lemma 1.2], if G is a nonidentity finite *CA*-group, then G possesses a canonical Frobenius system. The proof of the aforementioned lemma even yields that this Frobenius system is abelian. In particular $SL_2(K)$, for K a finite field of characteristic 2, does possess a canonical abelian Frobenius system $\mathcal{F}_K$.

Let $n \geq 1$ be an integer; then Theorem 9 (pp. 137–138) of [6] implies that the Suzuki group $\mathcal{S}z(2^{2n+1})$ (there denoted by $G(q)$, where $q = 2^{2n+1}$)

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possesses a Frobenius subgroup H of order $q^2(q-1)$, a dihedral subgroup B_0 of order $2(q-1)$, and cyclic subgroups A_1, A_2 of respective orders $q+r+1$ and $q-r+1$ ($r = \sqrt{2q} = 2^{r+1}$). By an easy application of the same Theorem, the elements of $\mathcal{MS}(G)$ are exactly the conjugates of H, B_0, B_1 and B_2, where $B_i = N_G(A_i)$ is a Frobenius group of order $4|A_i|$ (i = 1, 2). Let A_0 be the subgroup of B_0 of order $q-1$ and N the Frobenius kernel of H; it is easily seen that by setting, for each $g \in \mathcal{S}z(2^{2n+1})$:

$$\mathcal{F}_{(n)}(H^g) = N^g\,,$$
$$\mathcal{F}_{(n)}(B_0^g) = A_0^g\,, \quad \text{and}$$
$$\mathcal{F}_{(n)}(B_i^g) = A_i^g \quad (i = 1, 2)\,,$$

one defines an abelian Frobenius system $\mathcal{F}_{(n)}$ on $\mathcal{S}z(2^{2n+1})$.

There is an obvious notion of isomorphism for groups with Frobenius systems: if $\mathcal{F}_1$, $\mathcal{F}_2$ are Frobenius systems respectively on G_1, G_2, an *isomorphism* between $(G_1, \mathcal{F}_1)$ and $(G_2, \mathcal{F}_2)$ is by definition an isomorphism $\alpha : G_1 \to G_2$ such that:

$$\forall M \in \mathcal{MS}(G_1), \;\; \mathcal{F}_2(\alpha(M)) = \alpha(\mathcal{F}_1(M)).$$

The purpose of this work is to prove the following:

THEOREM 0.1. *Let $\mathcal{F}$ be an abelian Frobenius system on the finite group G; then one of the following holds*:

(1) *G is abelian and $\mathcal{F}(G) = \{1\}$.*
(2) *G is a nonidentity solvable group and $\mathcal{F}(G) = G$.*
(3) *G is a solvable Frobenius group with cyclic complement, and $\mathcal{F}(G)$ is the Frobenius kernel of G.*
(4) *$(G, \mathcal{F})$ is isomorphic to $(SL_2(\mathbf{F}_{2^n}), \mathcal{F}_{\mathbf{F}_{2^n}})$ for some $n \geq 2$.*
(5) *$(G, \mathcal{F})$ is isomorphic to $(\mathcal{S}z(2^{2n+1}), \mathcal{F}_{(n)})$ for some $n \geq 1$.*

Clearly these possibilities are mutually exclusive, and each of them yields an abelian Frobenius system.

The notations are mostly standard, and conform to those in [4].

§1. General preliminary lemmas

For the moment, let G denote an arbitrary finite group.

LEMMA 1.1. *If some element A of $\mathcal{MS}(G)$ is abelian, then $G = A$ is.*

Proof. Let us proceed by induction on $|G|$ (the result being trivial for $G = \{1\}$). We may assume that $A \neq G$; then, for each subgroup H with $A \subseteq H \subset G$, one has $A \in \mathcal{MS}(H)$, whence (by the induction hypothesis applied to H) $H = A$ – this means that A is a maximal subgroup of G. By a Theorem of Herstein ([2]), G is solvable; but then $\mathcal{MS}(G) = \{G\}$ and $A = G$, a contradiction. ∎

From now on, let $\mathcal{F}$ denote an abelian Frobenius system on the finite group G.

Lemma 1.2. *Let us suppose that*:

$$\forall M \in \mathcal{MS}(G), \quad \mathcal{F}(M) \neq M.$$

Then every nonabelian Sylow subgroup of G is a TI-set.

Proof. Let us assume that the Sylow q-subgroup Q of G is not abelian. There is an $M \in \mathcal{MS}(G)$ with

$$Q \subseteq M\,;$$

as

$$Q/Q \cap \mathcal{F}(M) \simeq Q\mathcal{F}(M)/\mathcal{F}(M) \subseteq M/\mathcal{F}(M)\,,$$

$Q/Q \cap \mathcal{F}(M)$ is abelian according to (FS4). Therefore $Q \cap \mathcal{F}(M) \neq \{1\}$, so q divides $|\mathcal{F}(M)|$. As $\mathcal{F}(M)$ is a Hall subgroup of M (see [4, Corollary 1.4]), it follows that q does not divide the order of $M/\mathcal{F}(M)$. But $Q\mathcal{F}(M)/\mathcal{F}(M)$ is a q-subgroup of $M/\mathcal{F}(M)$, hence $Q\mathcal{F}(M)/\mathcal{F}(M) = \{\bar{1}\}$ and:

$$Q \subseteq \mathcal{F}(M)\,.$$

But $\mathcal{F}(M)$ is nilpotent ([4, Proposition 1.5]), therefore $Q = O_q(\mathcal{F}(M)) \lhd M$ and $M \subseteq N_G(Q)$. If $Q \cap Q^x \neq \{1\}$, then $\mathcal{F}(M) \cap \mathcal{F}(M)^x \neq \{1\}$ (because $Q \subseteq \mathcal{F}(M)$), hence $x \in M$ (FS3), whence $x \in N_G(Q)$ and $Q = Q^x$: Q is a TI-set. ∎

§2. The proof of Theorem 0.1

Let $\mathcal{F}$ be an abelian Frobenius system on the finite group G. If $\mathcal{F}(M) = \{1\}$ for some $M \in \mathcal{MS}(G)$, then $M \simeq M/\mathcal{F}(M)$ is abelian (by (FS4)). But Lemma 1.1 now yields that $G = M$, and we are in case (1). If $\mathcal{F}(M) = M \neq \{1\}$ for some $M \in \mathcal{MS}(G)$, then either $G = M$ (hence we are in case (2)), or (according to (FS3)) M is a Frobenius complement in G. By Frobenius' Theorem, G possesses a Frobenius kernel N, and, by [5, 12.6.13, p. 354], N is nilpotent. Therefore N and

$$G/N = MN/N \simeq M/M \cap N \simeq M$$

are solvable, hence so is G; but then $\mathcal{MS}(G) = \{G\}$ and $M = G$, a contradiction. Therefore we may assume that:

$$\forall M \in \mathcal{MS}(G), \quad \{1\} \neq \mathcal{F}(M) \neq M.$$

It now follows from (FS1) and (FS2) that $\mathcal{F}(M)$ is a Frobenius kernel in M for all $M \in \mathcal{MS}(G)$.

Lemma 2.1. *G is a CN-group.*

Proof. Let $x \in G^\sharp$, and let $S \in \mathcal{MS}(C_G(x))$; there is $M \in \mathcal{MS}(G)$ with $S \subseteq M$. If S is abelian, then, by Lemma 1.1, $C_G(x) = S$ is abelian, hence nilpotent. Else one has $S \cap \mathcal{F}(M) \neq \{1\}$ (because of (FS4)); let $u \in (S \cap \mathcal{F}(M))^\sharp$. One has $u \in S \subseteq C_G(x)$, whence $x \in C_G(u)$; but $C_G(u) \subseteq \mathcal{F}(M)$ by Lemma 1.3 of [4], whence $x \in \mathcal{F}(M)$. A second application of the same Lemma now yields $C_G(x) \subseteq \mathcal{F}(M)$; but $\mathcal{F}(M)$ is nilpotent according to Proposition 1.5 of [4], hence so is $C_G(x)$. ∎

If G is solvable, then $\mathcal{MS}(G) = \{G\}$ and $\mathcal{F}(G)$ is a Frobenius kernel in G, with abelian complement by (FS4); it is well-known that such a complement is necessarily cyclic (this follows from [5, 12.6.15, p. 356]), and we are in case (3). Otherwise, G is a nonsolvable *CN*-group; by the main results of [6] and [7], G is therefore isomorphic to $SL_2(\mathbf{F}_{2^n})$ for some $n \geq 2$, $\mathcal{S}z(2^{2n+1})$ for some $n \geq 1$, or M_9 (a nonsimple, nonsolvable group of order 1440). But the Sylow 2-subgroups of M_9 are not abelian, otherwise so would be those of the alternating group A_6 which is a section of M_9 – a contradiction, as these last are dihedral of order 8; but they are not TI-sets either ([6]), whence, by Lemma 1.2, G is not isomorphic to M_9. Therefore we may assume that $G = SL_2(\mathbf{F}_{2^n})$ $(n \geq 2)$ or $G = \mathcal{S}z(2^{2n+1})$ $(n \geq 1)$. But we have seen that, for $M \in \mathcal{MS}(G)$, $\mathcal{F}(M)$ was a Frobenius kernel for M, and a finite group does possess *at most one* Frobenius kernel ([5, (12.6.12), p. 354]), therefore $\mathcal{F}$ is uniquely determined, and so has to be $\mathcal{F}_{\mathbf{F}_{2^n}}, \xi$ (resp. $\mathcal{F}_{(n)}, \xi$). ∎

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