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GENERALIZED CENTERS AND CHARACTERIZATIONS OF INNER PRODUCT SPACES

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ABSTRACT. In this paper, we present new Garkavi-Klee type characterizations of inner product spaces using the notion of generalized centers of three points sets introduced by using absolute normalized norms.

1. Introduction

Throughout this paper, the term "normed space" always means a real normed space. It is well-known, as a result due to Jordan and von Neumann, that a normed space X satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for each $x, y \in X$ if and only if the norm of X is induced by an inner product (that is, X is an inner product space). Inspired by this simple characterization of inner product spaces, many mathematicians have been provided several hundreds of results on this research area, based on various geometric ideas such as norm inequalities, generalized orthogonality types in normed spaces, and so on. The celebrated book of Amir [1] contains many of classical and major results in such topics.

In 1960's, Garkavi [6] and Klee [8] gave a characterization of inner product spaces using the notion of Chebyshev centers. For each non-empty bounded subset A of a normed space X and each element $x \in X$, let

$$r(x, A) = \sup\{\|x-y\| : y \in A\},$$
$$r(A) = \inf\{r(x, A) : x \in X\},$$

respectively. Then the value $r(A)$ is called the *Chebyshev radius* of A, and an element $z \in X$ is called a *Chebyshev center* of A if $r(z, A) = r(A)$. Let $Z(A)$ denote the set of all Chebyshev centers of A. The result of Garkavi and Klee is as follows:

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A normed space X with $\dim X \geq 3$ is an inner product space if and only if the condition (GK) holds, that is, for each $u, v, w \in X$,

$$Z(\{u, v, w\}) \cap \operatorname{co}(\{u, v, w\}) \neq \emptyset.$$

Using the notion of ∞-direct sum of normed spaces, the notion of Chebyshev centers is interpreted as follows: Let X be a normed space, and let $u, v, w \in X$. Then an element $z \in X$ is in $Z(\{u, v, w\})$ if and only if

$$\|(z-u, z-v, z-w)\|_\infty = \inf\{\|(x-u, x-v, x-w)\|_\infty : x \in X\}.$$

From this, we can immediately generalize the notion of Chebyshev centers. Let $1 \leq p \leq \infty$, let X be a normed space, and let $z, u, v, w \in X$. Then z is called a *p-center* of the three points set $\{u, v, w\}$ if

$$\|(z-u, z-v, z-w)\|_p = \inf\{\|(x-u, x-v, x-w)\|_p : x \in X\}.$$

Let $Z_p(u, v, w)$ denote the set of all p-centers of u, v, w. It is apparent from the definition that the notions of Chebyshev centers and ∞-centers coincide for any three points set, that is, $Z(\{u, v, w\}) = Z_\infty(u, v, w)$ for each $u, v, w \in X$. The case of $p = 1$, the set $Z_1(u, v, w)$ is often called the *Felmat center* of $\{u, v, w\}$.

In terms of p-centers of three points sets, Benítez, Fernández and Soriano [2, 3] provided characterizations of inner product spaces analogous to that of Garkavi and Klee, which also gave an answer to the problem proposed by Durier [5] in 1997. Namely, they showed that for $1 \leq p < \infty$, a normed space X with $\dim X \geq 3$ is an inner product space if and only if it satisfies the condition (GK_p) which states that

$$Z_p(u, v, w) \cap \operatorname{co}(\{u, v, w\}) \neq \emptyset$$

for each $u, v, w \in X$. Furthermore, in this direction, Mendoza and Pakhrou [9] presented in 2005 the following improvement of results mentioned above: Let $1 \leq p \leq \infty$. Then a normed space X with $\dim X \geq 3$ is an inner product space if and only if $Z_p(u, v, w) \cap \operatorname{co}(\{u, v, w\}) \neq \emptyset$ for each $u, v, w \in S_X$, where S_X is the unit sphere of X. They called this condition (GK_p^s).

The purpose of this paper is to generalize the above result of Mendoza and Pakhrou using absolute normalized norms on $\mathbb{R}^3$. Recall that a norm $\|\cdot\|$ on $\mathbb{R}^3$ is said to be *absolute* if $\|(a, b, c)\| = \|(|a|, |b|, |c|)\|$ holds for each $(a, b, c) \in \mathbb{R}^3$, and *normalized* if $\|(1, 0, 0)\| = \|(0, 1, 0)\| = \|(0, 0, 1)\| = 1$ holds. The ℓ_p-norm $\|\cdot\|_p$ satisfies these conditions for each $1 \leq p \leq \infty$. This allows us to consider natural generalizations of the notion of p-centers of three points sets using absolute norms. In terms of these generalized centers, we show the same characterization of inner product spaces as that of Mendoza and Pakhrou, for a certain class of absolute normalized norms on $\mathbb{R}^3$ containing symmetric, strictly convex and smooth ones as well as the ℓ_p-norms for $1 < p < \infty$.

2. Notation and preliminaries

Let AN_3 denote the set of all absolute normalized norms on $\mathbb{R}^3$, and let Ψ_3 be the set of all convex functions ψ on

$$\Delta_3 = \{(r, s, t) : r, s, t \geq 0,\ r + s + t = 1\}$$

satisfying $\psi(1, 0, 0) = \psi(0, 1, 0) = \psi(0, 0, 1) = 1$ and

$$\psi(r, s, t) \geq (1 - r)\psi\left(0, \frac{s}{1 - r}, \frac{t}{1 - r}\right),$$

$$\psi(r, s, t) \geq (1 - s)\psi\left(\frac{r}{1 - s}, 0, \frac{t}{1 - s}\right),$$

$$\psi(r, s, t) \geq (1 - t)\psi\left(\frac{r}{1 - t}, \frac{s}{1 - t}, 0\right)$$

for each $(r, s, t) \in \Delta_3$ (with $r < 1$, or $s < 1$, or $t < 1$ in the corresponding case). For each $\psi \in \Psi_3$, let $\|(a, b, c)\|_\psi$ be 0 or

$$(|a| + |b| + |c|)\psi\left(\frac{|a|}{|a| + |b| + |c|}, \frac{|b|}{|a| + |b| + |c|}, \frac{|c|}{|a| + |b| + |c|}\right)$$

according as $(a, b, c) = 0$ or $(a, b, c) \neq 0$. Then we have the following correspondence between AN_3 and Ψ_3.

Theorem 2.1 (Saito, Kato and Takahashi [11])**.** *The mapping $\psi \to \|\cdot\|_\psi : \Psi_3 \to AN_3$ is a bijection.*

We recall a simple but important characterization of absolute norms from [4, Proposition IV.1.1] (or [11, Lemma 4.1]). A norm $\|\cdot\|$ on $\mathbb{R}^3$ is said to be *monotone* if $\|(a_1, b_1, c_1)\| \leq \|(a_2, b_2, c_2)\|$ whenever $|a_1| \leq |a_2|$, $|b_1| \leq |b_2|$ and $|c_1| \leq |c_2|$.

Proposition 2.2. *A norm on $\mathbb{R}^3$ is absolute if and only if it is monotone.*

We remark, in particular, that every absolute normalized (hence monotone) norm $\|\cdot\|$ on $\mathbb{R}^3$ satisfies the inequality

$$\|(a, b, c)\|_\infty = \max\{|a|, |b|, |c|\} \leq \|(a, b, c)\| \leq |a| + |b| + |c| = \|(a, b, c)\|_1$$

for each (a, b, c).

Now let X be a normed space, and let $\psi \in \Psi_3$. Then the equation

$$\|(x, y, z)\|_\psi = \|(\|x\|, \|y\|, \|z\|)\|_\psi$$

defines a norm on the cartesian product $X \times X \times X$. Indeed, the triangle inequality follows from the preceding proposition, while the other properties required of norms are apparent. The space $X \times X \times X$ endowed with this norm is called the ψ-direct sum of X, and is denoted by $(X \oplus X \oplus X)_\psi$ ([7]).

Using the notion of ψ-direct sums, we can consider natural generalizations of the notion of p-centers of three points sets. It should be noted that the following definition is a special case of the corresponding notion considered in [5].

Definition 2.3 (Durier [5]). Let X be a normed space, and let $\psi \in \Psi_3$. Then an element $z \in X$ is called a *ψ-center* of a three points set $\{u, v, w\}$ in X if

$$\|(z-u, z-v, z-w)\|_\psi = \inf\{\|(x-u, x-v, x-w)\|_\psi : x \in X\}.$$

The set of all ψ-center of $\{u, v, w\}$ is denoted by $Z_\psi(u, v, w)$.

Needless to say, the ℓ_p-norm is in AN_3 for each $1 \leq p \leq \infty$ with the corresponding function $\psi_p \in \Psi_3$ given by

$$\psi_p(r, s, t) = \begin{cases} (r^p + s^p + t^p)^{1/p} & (1 \leq p < \infty), \\ \max\{r, s, t\} & (p = \infty), \end{cases}$$

for each $(r, s, t) \in \Delta_3$. Hence we have $Z_{\psi_p}(u, v, w) = Z_p(u, v, w)$ for each u, v, w. This shows that Definition 2.3 provides natural generalizations of the notion of p-centers of three points sets.

We close this section with some auxiliary results on absolute norms and their direct sums. For each $\psi \in \Psi_3$, let ψ^* be the function given by

$$\psi(s) = \max_{t \in \Delta_3} \frac{\langle s, t \rangle}{\psi(t)}$$

for each $s \in \Delta_3$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^3$. Then the dual space of $(\mathbb{R}^3, \|\cdot\|_\psi)$ can be associated with ψ^* defined above.

Proposition 2.4 (Bhatia [4]; Mitani, Oshiro and Saito [10]). *Let $\psi \in \Psi_3$. Then $\psi^* \in \Psi_3$, and generalized Hölder's inequality*

$$|\langle x, y \rangle| \leq \|x\|_\psi \|y\|_{\psi^*}$$

holds for each $x, y \in \mathbb{R}^3$. Moreover, the dual space $(\mathbb{R}^3, \|\cdot\|_\psi)^*$ of the normed space $(\mathbb{R}^3, \|\cdot\|_\psi)$ is isometrically isomorphic to the normed space $(\mathbb{R}^3, \|\cdot\|_{\psi^*})$. In particular, $\psi^{**} = \psi$ holds.

We recall that if X is a normed space, then the dual space of the p-direct sum $(X \oplus X \oplus X)_p$ of X is isometrically isomorphic to the q-direct sum $(X^* \oplus X^* \oplus X^*)_q$ of the dual space X^* of X, where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. The following result is a natural generalization of this fact to absolute direct sums.

Proposition 2.5 (Mitani, Oshiro and Saito [10]). *Let X be a normd space, and let $\psi \in \Psi_3$. Then the dual space of the ψ-direct sum $(X \oplus X \oplus X)_\psi$ of X is isometrically isomorphic to the ψ^*-direct sum $(X^* \oplus X^* \oplus X^*)_{\psi^*}$ of the dual space X^* of X. Moreover, if $F \in (X \oplus X \oplus X)_\psi^*$ is identified with $(f, g, h) \in (X^* \oplus X^* \oplus X^*)_{\psi^*}$, then $F(x, y, z) = f(x) + g(y) + h(z)$ for each $x, y, z \in X$.*

3. Characterizations of inner product spaces

We start this section with the following lemma.

Lemma 3.1. *Let $\psi \in \Psi_3$. Then ψ takes the minimum at $(1/3, 1/3, 1/3)$ if and only if ψ^* does, and in which case, the equation*

$$\psi\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\psi^*\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$$

holds.

Proof. Let $c = (1/3, 1/3, 1/3)$. We first note that the inequality

$$\frac{1}{3} = \langle c, (r, s, t)\rangle \leq \|c\|_\psi \|(r, s, t)\|_{\psi^*} = \psi(c)\psi^*(r, s, t)$$

holds for each $(r, s, t) \in \Delta_3$ by generalized Hölder's inequality, and hence

$$\frac{1}{3\psi(c)} \leq \min_{(r,s,t)\in\Delta_3} \psi^*(r, s, t)$$

Now suppose that ψ takes the minimum at c, then we have

$$\psi^*(c) = \max_{(r,s,t)\in\Delta_3} \frac{\langle c, (r, s, t)\rangle}{\psi(r, s, t)} = \max_{(r,s,t)\in\Delta_3} \frac{1}{3\psi(r, s, t)} = \frac{1}{3\psi(c)},$$

which shows the equation stated in the lemma; and by the preceding paragraph,

$$\frac{1}{3\psi(c)} = \min_{(r,s,t)\in\Delta_3} \psi^*(r, s, t)$$

Thus ψ^* also takes the minimum at c. Since $\psi^{**} = \psi$, the converse immediately follows from the fact just proved above. □

We next present two auxiliary results as variations of [3, Lemma 1]. The first one is an easy consequence of the Hahn-Banach theorem. We include its proof for the sake of completeness.

Lemma 3.2. *Let X be a normed space, and let $u, v, w \in S_X$. Suppose that $\psi \in \Psi_3$. Then $0 \in Z_\psi(u, v, w)$ if and only if there exists an element (f, g, h) in $(X^* \oplus X^* \oplus X^*)_{\psi^*}$ such that $\|(f, g, h)\|_{\psi^*} = 1$, $f(u)+g(v)+h(w) = \|(u, v, w)\|_\psi$ and $f+g+h = 0$.*

Proof. Let M be the closed subspace $\{(x, x, x) : x \in X\}$ of $(X \oplus X \oplus X)_\psi$. From the definition, $0 \in Z_\psi(u, v, w)$ if and only if

$$\begin{aligned}\|(u, v, w)\|_\psi &= \inf\{\|(x - u, x - v, x - w)\|_\psi : x \in X\} \\ &= \inf\{\|m - (u, v, w)\|_\psi : m \in M\} = d((u, v, w), M).\end{aligned}$$

Now suppose that $0 \in Z_\psi(u,v,w)$. We note that $d((u,v,w),M) = \|(u,v,w)\|_\psi > 0$; so $(u,v,w) \notin M$. Then, by the Hahn-Banach theorem, there exists an element $F = (f,g,h)$ in $(X \oplus X \oplus X)^*_\psi (= (X^* \oplus X^* \oplus X^*)_{\psi^*})$ such that $\|(f,g,h)\|_{\psi^*} = 1$,

$$f(u) + g(v) + h(w) = F(u,v,w) = \|(u,v,w)\|_\psi$$

and $M \subset \ker F$. This last inclusion shows that $f + g + h = 0$.

Conversely, if we have an element $(f,g,h)(= F)$ of $(X^* \oplus X^* \oplus X^*)_{\psi^*}$ with the stated properties, it follows that

$$\begin{aligned}
\|(u,v,w)\|_\psi = F(u,v,w) &= |F(u,v,w) - F(x,x,x)| \\
&\le \|F\|\|(u-x, v-x, w-x)\|_\psi \\
&= \|(f,g,h)\|_{\psi^*}\|(u-x, v-x, w-x)\|_\psi \\
&= \|(u-x, v-x, w-x)\|_\psi .
\end{aligned}$$

for each $x \in X$. Thus $\|(u,v,w)\|_\psi = \inf\{\|(x-u, x-v, x-w)\|_\psi : x \in X\}$, that is, $0 \in Z_\psi(u,v,w)$. □

For each $x \in X$, let $Jx = \{f \in X^* : f(x) = \|f\|^2 = \|x\|^2\}$. The mapping $J : X \to X^*$ is called the (normalized) duality mapping. The second lemma provides a rather algebraic interpretation of the statement that $0 \in Z_\psi(u,v,w)$.

Lemma 3.3. *Let X be a normed space, and let $u,v,w \in S_X$. Suppose that $\psi \in \Psi_3$, and that ψ^* takes the minimum only at $(1/3,1/3,1/3)$. Then $0 \in Z_\psi(u,v,w)$ if and only if there exist $f \in Ju$, $g \in Jv$ and $h \in Jw$ such that $f + g + h = 0$.*

Proof. Suppose that $0 \in Z_\psi(u,v,w)$. Then, by the preceding lemma, there exists an element (f,g,h) in $(X^* \oplus X^* \oplus X^*)_{\psi^*}$ such that $\|(f,g,h)\|_{\psi^*} = 1$, $f(u)+g(v)+h(w) = \|(u,v,w)\|_\psi$ and $f + g + h = 0$. Put

$$k = 3\psi\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{\psi^*(1/3,1/3,1/3)}.$$

(This last equality follows from Lemma 3.1.) From generalized Hölder's inequality, we have

$$\begin{aligned}
\|(u,v,w)\|_\psi = \|(1,1,1)\|_\psi &= k \\
&= f(u) + g(v) + h(w) \\
&\le \|f\| + \|g\| + \|h\| \\
&\le \|(1,1,1)\|_\psi \|(f,g,h)\|_{\psi^*} = k.
\end{aligned}$$

It follows that $f(u) = \|f\|$, $g(v) = \|g\|$, $h(w) = \|h\|$ and $\|f\| + \|g\| + \|h\| = k$. Since

$$1 = \|(f,g,h)\|_{\psi^*} = k\psi^*\left(\frac{\|f\|}{k}, \frac{\|g\|}{k}, \frac{\|h\|}{k}\right),$$

we obtain

$$\psi^*\left(\frac{\|f\|}{k},\frac{\|g\|}{k},\frac{\|h\|}{k}\right)=\frac{1}{k}=\psi^*\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right).$$

However, the function ψ^* takes the minimum only at $(1/3,1/3,1/3)$, the last equality implies that $\|f\|=\|g\|=\|h\|=k/3$. Thus putting $f'=(3/k)f$, $g'=(3/k)g$, and $h'=(3/k)h$ yields $f'\in Ju$, $g'\in Jv$, $h'\in Jw$ and

$$f'+g'+h'=\frac{3}{k}(f+g+h)=0.$$

Conversely, let f,g,h be the functionals with the stated properties. We note that $\|f\|=\|g\|=\|h\|=1$. Put $f_0=(k/3)f$, $g_0=(k/3)g$, and $h_0=(k/3)h$, respectively. Then it follows that

$$\|(f_0,g_0,h_0)\|_{\psi^*}=k\|(1/3,1/3,1/3)\|_{\psi^*}=k\psi^*\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=1,$$

that

$$f_0(u)+g_0(v)+h_0(w)=k=3\psi\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\|(u,v,w)\|_\psi,$$

and that $f_0+g_0+h_0=(k/3)(f+g+h)=0$. This, together with the preceding lemma, implies that $0\in Z_\psi(u,v,w)$. $\square$

Using Lemmas 3.1 and 3.3, we have the following lemma corresponding to [3, Lemma 2].

Lemma 3.4. *Let X be a normed space, and let $u,v,w\in S_X$. Suppose that both $\psi\in\Psi_3$ and its dual function ψ^* take the minima only at $(1/3,1/3,1/3)$, and that $0\in Z_\psi(u,v,w)$. If $z\in Z_\psi(u,v,w)$, then*

$$\|u-tz\|=\|v-tz\|=\|w-tz\|=1$$

for each $t\in[0,1]$.

Proof. By the preceding lemma, there exist $f\in Ju$, $g\in Jv$ and $h\in Jw$ such that $f+g+h=0$. Put $k=\psi(1/3,1/3,1/3)$, $f'=kf$, $g'=kg$ and $h'=kh$, respectively. Then we have $\|f'\|=\|g'\|=\|h'\|=f'(u)=g'(v)=h'(w)=k$ and

$$\|(f',g',h')\|_{\psi^*}=3k\|(1/3,1/3,1/3)\|_{\psi^*}=1$$

by Lemma 3.1. Let Φ be the convex function on $\mathbb{R}$ given by

$$\Phi(t)=\|(u-tz,v-tz,w-tz)\|_\psi.$$

Then $\Phi(0)=\Phi(1)=\|(u,v,w)\|_\psi=3k$ since $0,z\in Z_\psi(u,v,w)$, which and the convexity of Φ imply that $\Phi(t)\le 3k$ for each $t\in[0,1]$. However, since

$$\Phi(t)\ge\inf\{\|(u-x,v-x,w-x)\|_\psi : x\in X\}=\|(u,v,w)\|_\psi=3k$$

for each $t \in \mathbb{R}$, it follows that $\Phi(t) = 3k$ for each $t \in [0,1]$. From the fact that $f' + g' + h' = 0$, generalized Hölder's inequality guarantees that

$$\begin{aligned} 3k = f'(u) + g'(v) + h'(w) &= f'(u - tz) + g'(v - tz) + h'(w - tz) \\ &\leq \|f'\|\|u - tz\| + \|g'\|\|v - tz\| + \|h'\|\|w - tz\| \\ &\leq \|(f', g', h')\|_{\psi^*}\Phi(t) = 3k \end{aligned}$$

for each $t \in [0,1]$. Hence one obtains, for each $t \in [0,1]$,

$$\|u - tz\| + \|v - tz\| + \|w - tz\| = 3$$

and

$$3k = \Phi(t) = 3\psi\left(\frac{\|u - tz\|}{3}, \frac{\|v - tz\|}{3}, \frac{\|w - tz\|}{3}\right);$$

and therefore

$$\psi\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \psi\left(\frac{\|u - tz\|}{3}, \frac{\|v - tz\|}{3}, \frac{\|w - tz\|}{3}\right).$$

Since the function ψ takes the minimum only at $(1/3, 1/3, 1/3)$, we have

$$\|u - tz\| = \|v - tz\| = \|w - tz\| = 1$$

for each $t \in [0,1]$, as desired. $\square$

Our next result generalizes [9, Proposition 2], and is a key ingredient for the proof of the main theorem.

Lemma 3.5. *Let X be a normed space, and let $u, v, w \in S_X$.*

(i) *If the dual function ψ^* of $\psi \in \Psi_3$ takes the minimum only at $(1/3, 1/3, 1/3)$, then $0 \in Z_\psi(u, v, w)$ if and only if $0 \in Z_1(u, v, w)$.*

(ii) *If the functions ψ and ψ^* both take the minima only at $(1/3, 1/3, 1/3)$ and $0 \in Z_1(u, v, w)$, then $Z_\psi(u, v, w) \subset Z_1(u, v, w)$,*

Proof. We note that $(\psi_1)^* = \psi_\infty$ takes the minimum $1/3$ only at $(1/3, 1/3, 1/3)$. This and Lemma 3.3 together show that $0 \in Z_\psi(u, v, w)$ if and only if $0 \in Z_1(u, v, w)$ provided that the dual function ψ^* of $\psi \in \Psi_3$ takes the minimum only at $(1/3, 1/3, 1/3)$. This proves (i).

Next we assume that the functions ψ and ψ^* both take the minima only at $(1/3, 1/3, 1/3)$ and $0 \in Z_1(u, v, w)$. Let $z \in Z_\psi(u, v, w)$. Since $0 \in Z_\psi(u, v, w)$ by (i), the preceding lemma assures that $\|u - z\| = \|v - z\| = \|w - z\| = 1$, which implies that

$$\|(u - z, v - z, w - z)\|_1 = 3 = \|(u, v, w)\|_1.$$

Since $0 \in Z_1(u, v, w)$, it follows that $z \in Z_1(u, v, w)$; and therefore $Z_\psi(u, v, w) \subset Z_1(u, v, w)$. Hence (ii) holds. $\square$

Another key ingredient for our characterizations of inner product spaces is the following result of Mendoza and Pakhrou [9, Lemma 3].

Lemma 3.6 (Mendoza and Pakhrou [9])**.** *Let X be a normed space with $\dim X \geq 3$. Suppose that X is not an inner product space. Then there exists a subspace Y of X and $u, v, w \in S_Y$ such that $0 \in Z_{1,Y}(u, v, w)$ and $Z_{1,Y}(u, v, w) \cap \mathrm{co}(\{u, v, w\}) = \emptyset$, where $Z_{1,Y}(u, v, w)$ is the set of Felmat centers of u, v, w considered with respect to Y.*

We now introduce new conditions that characterize inner product spaces. For $\psi \in \Psi_3$, we say that a normed space X satisfies the condition (GK_ψ^s) if $Z_\psi(u, v, w) \cap \mathrm{co}(\{u, v, w\}) \neq \emptyset$ for each $u, v, w \in S_X$. As the following result of Durier [5, Proposition 3.2] shows, inner product spaces satisfy (GK_ψ^s) for each ψ in a stronger sense. The proof is given for the sake of completeness.

Lemma 3.7 (Durier [5])**.** *Let X be an inner product space, and let $u, v, w \in X$. Suppose that $\psi \in \Psi_3$. Then $\emptyset \neq Z_\psi(u, v, w) \subset \mathrm{co}(\{u, v, w\})$.*

Proof. We first note that the set $\mathrm{co}(\{u, v, w\})(= K)$ is norm compact since it is the image of the compact subset Δ_3 of the Euclidean space under the continuous mapping $(r, s, t) \to ru + sv + tw$. Since X is an inner product space, we have the metric projection P from X onto K, that is, $\|x - Px\| = \min\{\|x - y\| : y \in K\}$ for each $x \in X$. (The uniqueness of such a point easily follows from the parallelogram law.) We recall that P satisfies the inequality $\langle x - Px, Py - Px\rangle \leq 0$ for each $x, y \in X$. In particular, for each $x \in X$ and each $y = Py \in K$, we obtain

$$\begin{aligned}\|x - y\|^2 &= \|x - Px + Px - y\|^2 \\ &= \|x - Px\|^2 + 2\langle x - Px, Px - Py\rangle + \|Px - y\|^2 \\ &\geq \|x - Px\|^2 + \|Px - y\|^2.\end{aligned}$$

Now take an arbitrary $x \in X \backslash K$. Then $x \neq Px$, and the above inequality guarantees that

$$k = \max_{y \in K} \frac{\|Px - y\|}{\|x - y\|} < 1.$$

In particular, since $u, v, w \in K$, it follows that

$$\begin{aligned}\|(x - u, x - v, x - w)\|_\psi &> k\|(x - u, x - v, x - w)\|_\psi \\ &\geq \|(Px - u, Px - v, Px - w)\|_\psi \\ &\geq \min\{\|(y - u, y - v, y - w)\|_\psi : y \in K\} \\ &\geq \inf\{\|(x - u, x - v, x - w)\|_\psi : x \in X\}.\end{aligned}$$

This shows that $x \notin Z_\psi(u,v,w)$; hence $Z_\psi(u,v,w) \subset K$. Furthermore, taking the infimum over X in the last inequalities yields

$$\begin{aligned}&\inf\{\|(x-u,x-v,x-w)\|_\psi : x \in X\}\\ &= \min\{\|(y-u,y-v,y-w)\|_\psi : y \in K\}.\end{aligned}$$

This proves the existence of $y_0 \in K \cap Z_\psi(u,v,w)$. □

To prove the converse, we need some additional assumptions on ψ. The main result in this paper is the following. In view of Lemma 3.5, the proof is essentially the same as that of [9, Theorem 5].

Theorem 3.8. *Let X be a normed space with $\dim X \geq 3$. Suppose that $\psi \in \Psi_3$ and its dual function ψ^* both take the minima only at $(1/3,1/3,1/3)$. Then X is an inner product space if and only if it satisfies* (GK^s_ψ).

Proof. By the preceding lemma, it is enough to show the if part. Suppose that X is not an inner product space. Then, by Lemma 3.6, there exists a subspace Y of X and $u,v,w \in S_Y$ such that $0 \in Z_{1,Y}(u,v,w)$ and $Z_{1,Y}(u,v,w) \cap \mathrm{co}(\{u,v,w\}) = \emptyset$. Since $Z_{\psi,Y}(u,v,w) \subset Z_{1,Y}(u,v,w)$ by Lemma 3.5 (ii), it follows that

$$Z_{\psi,Y}(u,v,w) \cap \mathrm{co}(\{u,v,w\}) = \emptyset.$$

However, this means that

$$\begin{aligned}&\inf\{\|(x-u,x-v,x-w)\|_\psi : x \in X\}\\ &\leq \inf\{\|(y-u,y-v,y-w)\|_\psi : y \in Y\}\\ &< \min\{\|(y-u,y-v,y-w)\|_\psi : y \in \mathrm{co}(\{u,v,w\})\},\end{aligned}$$

that is,

$$Z_\psi(u,v,w) \cap \mathrm{co}(\{u,v,w\}) = \emptyset.$$

Hence X does not satisfy (GK^s_ψ). The proof is complete. □

A norm $\|\cdot\|$ on $\mathbb{R}^3$ is said to be *symmetric* if $\|(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)})\| = \|(x_1,x_2,x_3)\|$ for every permutation π of $\{1,2,3\}$. We also recall that a norm $\|\cdot\|$ on a Banach space is said to be *strictly convex* if $\|x+y\| = \|x\| + \|y\|$ implies that $x = ky$ for some $k \geq 0$. Combining these properties immediately yields the following lemma.

Lemma 3.9. *Let ψ be such that $\|\cdot\|_\psi$ is symmetric and strictly convex. Then ψ takes the minimum only at $(1/3,1/3,1/3)$.*

Proof. For each $(r,s,t) \in \varDelta_3$, we have

$$\begin{aligned}\psi(1/3,1/3,1/3) &= \|(1/3,1/3,1/3)\|_\psi \\ &= \frac{1}{3}\|(r,s,t)+(t,r,s)+(s,t,r)\|_\psi \\ &\le \frac{1}{3}(\|(r,s,t)\|_\psi + \|(t,r,s)\|_\psi + \|(s,t,r)\|_\psi) \\ &= \|(r,s,t)\|_\psi = \psi(r,s,t)\end{aligned}$$

since $\|\cdot\|_\psi$ is symmetric. Hence ψ takes the minimum at $(1/3,1/3,1/3)$. Moreover, if the equality occurs in the above inequality, the strict convexity of $\|\cdot\|_\psi$ assures that $(r,s,t) = k(t,r,s) = l(s,t,r)$ for some $k,l \ge 0$. By taking the norms of each vectors, we have $k = l = 1$, that is, $r = s = t = 1/3$. This proves the uniqueness. $\square$

It is well-known, for a reflexive Banach space X, that X is smooth if and only if the dual space X^* is strictly convex. From this, $\|\cdot\|_\psi$ is smooth if and only if $\|\cdot\|_{\psi^*}$ is strictly convex. Moreover, it is easy to check that $\|\cdot\|_\psi$ is symmetric if and only if $\|\cdot\|_{\psi^*}$ is. Thus the following result is now an immediate consequence of Theorem 3.8 and the preceding lemma.

Corollary 3.10. *Let $\psi \in \Psi_3$ be such that $\|\cdot\|_\psi$ is symmetric, strictly convex and smooth. Then a normed space X with $\dim X \ge 3$ is an inner product space if and only if it satisfies* (GK_ψ^s).

Needless to say, the ℓ_p-norm satisfies the assumptions in the preceding corollary for $1 < p < \infty$. Thus we obtain the result of Mendoza and Pakhrou [9, Theorem 5] as a corollary.

Corollary 3.11 (Mendoza and Pakhrou [9])**.** *Let $1 < p < \infty$. Then a normed space X with $\dim X \ge 3$ is an inner product space if and only if it satisfies* (GK_p^s).

Finally, we recall that an inner product space satisfies the condition (GK_ψ^s) for any $\psi \in \Psi_3$. Hence the following problem naturally arises. We remark that, in the two-dimensional case, (GK_ψ^s) is always satisfied even in non-inner product spaces ([5, Proposition 3.2]).

Problem 3.12. Let X be a normed space with $\dim X \ge 3$, and let $\psi \in \Psi_3$. Does (GK_ψ^s) imply that X is an inner product space?

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