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ON GENERALIZED HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES INVOLVING RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, some generalization integral inequalities of Hermite-Hadamard type for functions whose derivatives are convex in modulus are given by using fractional integrals.

1. Introduction

Definition 1.1. A function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [9], [16, p.137]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 9, 10, 12, 16] and the references cited therein).

In [17], Pearce and Pečarić proved the following theorem:

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Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$. If the mapping $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \tag{2}$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [11, 13, 15].

Definition 1.2. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Meanwhile, Sarikaya et al. [18] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha \geq 0$.

Lemma 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L\,[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + J_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) \right] - f\left(\frac{a+b}{2}\right) \\ = \;& \frac{b-a}{4} \left\{ \int_0^1 t^{\alpha} f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) dt - \int_0^1 t^{\alpha} f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) dt \right\} \end{aligned} \tag{3}$$

with $\alpha > 0$.

It is remarkable that Sarikaya et al. [18] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1\,[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J_{\left(\frac{a+b}{2}\right)+}^{\alpha} f(b) + J_{\left(\frac{a+b}{2}\right)-}^{\alpha} f(a) \right] \leq \frac{f(a) + f(b)}{2} \tag{4}$$

with $\alpha > 0$.

For some recent results connected with fractional integral inequalities see [3]-[8], [14], [19]-[22].

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the generalized identity is obtained for fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

2. Main Results

We give an important fractional integral identity for differentiable convex functions:

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\le a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \lambda^{\alpha}(1-\lambda)^{\alpha} f(\lambda a+(1-\lambda)b) \\ & -\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)+(1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b)\right] \qquad (5) \\ = {} & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\int_0^1 t^{\alpha} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\right. \\ & \left.-\int_0^1 (1-t)^{\alpha} f'\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]dt\right\}, \end{aligned}$$

where $\lambda\in(0,1)$ and $\alpha\ge 0$.

Proof. Integrating by parts

$$\begin{aligned} & \int_0^1 t^{\alpha} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt \\ = {} & \left.\frac{t^{\alpha} f\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]}{(1-\lambda)(b-a)}\right|_0^1 \\ & -\frac{\alpha}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt \\ = {} & \frac{f(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)}-\frac{\alpha}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt \end{aligned}$$

$$= \frac{f(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\alpha}{(1-\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \int\limits_a^{\lambda a+(1-\lambda)b} (x-a)^{\alpha-1} f(x)dx$$

$$= \frac{f(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(b-a)^{\alpha+1}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)$$

that is

$$\int\limits_0^1 t^{\alpha} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right] dt$$
$$= \frac{f(\lambda a+(1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(b-a)^{\alpha+1}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a) \tag{6}$$

and similarly we get

$$-\int\limits_0^1 (1-t)^{\alpha} f'\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right] dt$$

$$= \frac{f(\lambda a+(1-\lambda)b)}{\lambda(b-a)} - \frac{\alpha}{\lambda^{\alpha+1}(b-a)^{\alpha+1}} \int\limits_{\lambda a+(1-\lambda)b}^{b} (b-x)^{\alpha-1} f(x)dx$$

$$= \frac{f(\lambda a+(1-\lambda)b)}{\lambda(b-a)} - \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}(b-a)^{\alpha+1}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b)$$

i.e.

$$-\int\limits_0^1 (1-t)^{\alpha} f'\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right] dt$$
$$= \frac{f(\lambda a+(1-\lambda)b)}{\lambda(b-a)} - \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}(b-a)^{\alpha+1}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b). \tag{7}$$

Adding (6) and (7), we obtain (5). This completes the proof. □

Remark 2.1. *If we take* $\lambda = \frac{1}{2}$ *in Lemma 2.1, then it follows that*

$$f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right]$$
$$= \frac{(b-a)}{4}\left\{\int\limits_0^1 t^{\alpha} f'\left(t\frac{a+b}{2}+(1-t)a\right) dt - \int\limits_0^1 (1-t)^{\alpha} f'\left(tb+(1-t)\frac{a+b}{2}\right) dt\right\}. \tag{8}$$

If we choose $\alpha = 1$ in (8), it follows that

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{(b-a)}{4}\left\{\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)a\right)dt - \int_0^1 (1-t)f'\left(tb + (1-t)\frac{a+b}{2}\right)dt\right\}.$$

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $0 \leq a < b$. If $|f'|^q$, $q \geq 1$, is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\lambda^\alpha (1-\lambda)^\alpha f(\lambda a + (1-\lambda)b)\right.$$

$$\left. -\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\lambda^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\right|$$

$$\leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{\alpha+1}\left\{\left(\frac{(\alpha+1)\,|f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{(\alpha+1)\,|f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right\} \tag{9}$$

where $\lambda \in (0,1)$ and $\alpha \geq 0$.

Remark 2.2. *We want to note that $\left(\frac{(\alpha+1)|f'(\lambda a+(1-\lambda)b)|^q + |f'(a)|^q}{(\alpha+2)}\right)^{\frac{1}{q}}$ denotes the q-power mean of $|f'(\lambda a + (1-\lambda)b)|$ and $|f'(a)|$ with respect to some probability measure. It is similar for $\left(\frac{(\alpha+1)|f'(\lambda a+(1-\lambda)b)|^q + |f'(b)|^q}{(\alpha+2)}\right)^{\frac{1}{q}}$.*

Proof of Theorem 2.1. Firstly, we suppose that $q = 1$. Using Lemma 2.1 and convexity of $|f'|$, we find that

$$\left|\lambda^\alpha (1-\lambda)^\alpha f(\lambda a + (1-\lambda)b)\right.$$

$$\left. -\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\lambda^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^\alpha_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\right|$$

$$\leq \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\int_0^1 t^\alpha \left|f'\left[t(\lambda a + (1-\lambda)b) + (1-t)a\right]\right| dt\right.$$

$$+\int_0^1 (1-t)^\alpha \left|f'\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right| dt \Bigg\}$$

$$\leq \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\Bigg\{\int_0^1 t^\alpha \left[t\left|f'(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f'(a)\right|\right] dt$$

$$+\int_0^1 (1-t)^\alpha \left[t\left|f'(b)\right|+(1-t)\left|f'(\lambda a+(1-\lambda)b)\right|\right] dt\Bigg\}$$

$$=\frac{2\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+2)}\left(\left|f'(\lambda a+(1-\lambda)b)\right|+\frac{|f'(a)|+|f'(b)|}{2(\alpha+1)}\right).$$

This implies that (9) holds for the case of $q=1$.

Secondly, we suppose that $q>1$. Using Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^1 t^\alpha \left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a)\right]\right| dt \\
&+\int_0^1 (1-t)^\alpha \left|f'\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right| dt \\
\leq &\left(\int_0^1 t^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^\alpha \left|f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a)\right]\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1 (1-t)^\alpha dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^\alpha \left|f'\left[tb+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{10}$$

Hence, using convexity of $|f'|^q$, (10) and Lemma 2.1 we obtain

$$\Bigg|\lambda^\alpha(1-\lambda)^\alpha f(\lambda a+(1-\lambda)b)$$

$$-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\lambda^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\Bigg|$$

$$\leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+1)^{1-\frac{1}{q}}}\left\{\left(\int_0^1 t^\alpha\left[t\left|f'(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 (1-t)^\alpha \left[t\left|f'(b)\right|^q+(1-t)\left|f'(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}$$

$$\leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+1)^{1-\frac{1}{q}}}\left\{\left(\frac{(\alpha+1)\left|f'(\lambda a+(1-\lambda)b)\right|^q+\left|f'(a)\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\alpha+1)\left|f'(\lambda a+(1-\lambda)b)\right|^q+\left|f'(b)\right|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 2.1. *Under the assumption of Theorem 2.1 with $\lambda=\frac{1}{2}$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right|$$

$$\leq \frac{(b-a)}{4(\alpha+1)}\left\{\left(\frac{(\alpha+1)\left|f'(\frac{a+b}{2})\right|^q+\left|f'(a)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\alpha+1)\left|f'(\frac{a+b}{2})\right|^q+\left|f'(b)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right\}, \tag{11}$$

where $q\geq 1$.

Remark 2.3. *If we take $\alpha=1$ in Corollary 2.1, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left\{\left(\frac{2\left|f'(\frac{a+b}{2})\right|^q+\left|f'(a)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'(\frac{a+b}{2})\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right\},$$

where $q\geq 1$. Choosing $q=1$ in the last inequality, it follows that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{12}\left(2\left|f'(\frac{a+b}{2})\right|+\frac{|f'(a)|+|f'(b)|}{2}\right).$$

In the last inequality, if we take a function f with $2\left|f'(\frac{a+b}{2})\right|=\frac{|f'(a)|+|f'(b)|}{2}$, then we obtain the following better inequality:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{12}\left(|f'(a)|+|f'(b)|\right).$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^q$ is convex on $[a,b]$ for some $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\Big|\lambda^{\alpha}(1-\lambda)^{\alpha}f(\lambda a+(1-\lambda)b)\\
&-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\Big|\\
\leq\ &\frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha p+1)^{\frac{1}{p}}}\left\{\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$, $\alpha\geq 0$ *and* $\lambda\in(0,1)$.

Proof. Using Lemma 2.1, the convexity of $|f'|^q$ and Hölder's inequality, we obtain

$$\begin{aligned}
&\Big|\lambda^{\alpha}(1-\lambda)^{\alpha}f(\lambda a+(1-\lambda)b)\\
&-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\Big|\\
\leq\ &\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(t(\lambda a+(1-\lambda)b)+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(tb+(1-t)(\lambda a+(1-\lambda)b)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\leq\ &\frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha p+1)^{\frac{1}{p}}}\left\{\left(\int_0^1\left[t\left|f'(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1\left[t\left|f'(b)\right|^q+(1-t)\left|f'(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

$$= \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha p+1)^{\frac{1}{p}}}\left\{\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{|f'(\lambda a+(1-\lambda)b)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

Corollary 2.2. *Under the assumption of Theorem 2.2 with $\lambda=\frac{1}{2}$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right|$$

$$\leq \frac{(b-a)}{4(\alpha p+1)^{\frac{1}{p}}}\left\{\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Remark 2.4. *If we take $\alpha=1$ in Corollary 2.2, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left\{\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $|f'|^q$ is convex on $[a,b]$ for some $q\geq 1$, then the following inequality for fractional integrals holds:*

$$|\lambda^\alpha(1-\lambda)^\alpha f(\lambda a+(1-\lambda)b)$$

$$-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\lambda^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\Big|$$

$$\leq \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\left(\frac{(\alpha q+1)|f'(\lambda a+(1-\lambda)b)|^q+|f'(a)|^q}{(\alpha q+1)(\alpha q+2)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\alpha q+1)|f'(\lambda a+(1-\lambda)b)|^q+|f'(b)|^q}{(\alpha q+1)(\alpha q+2)}\right)^{\frac{1}{q}}\right\},$$

where $\lambda\in(0,1)$ and $\alpha\geq 0$.

Proof. In case of $q > 1$, take $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Using Lemma 2.1, the convexity of $|f'|^q$, and Hölder's inequality, we have

$$\begin{aligned}
&\Big|\lambda^\alpha(1-\lambda)^\alpha f(\lambda a + (1-\lambda)b) \\
&-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\lambda^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^-}f(a) + (1-\lambda)^{\alpha+1}J^\alpha_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\Big| \\
\leq\ & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\alpha q}\left|f'\left(t(\lambda a+(1-\lambda)b)+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{\alpha q}\left|f'\left(tb+(1-t)(\lambda a+(1-\lambda)b)\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\left(\int_0^1 t^{\alpha q}\left[t\left|f'(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 (1-t)^{\alpha q}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)\left\{\left(\frac{(\alpha q+1)\left|f'(\lambda a+(1-\lambda)b)\right|^q+\left|f'(a)\right|^q}{(\alpha q+1)(\alpha q+2)}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{(\alpha q+1)\left|f'(\lambda a+(1-\lambda)b)\right|^q+\left|f'(b)\right|^q}{(\alpha q+1)(\alpha q+2)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Therefore we obtain the desired inequality. In case of $q = 1$, we also obtain the desired inequality using the same method above. □

Corollary 2.3. *Under the assumption of Theorem 2.3 with $\lambda = \frac{1}{2}$, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left[J^\alpha_{(\frac{a+b}{2})^-}f(a)+J^\alpha_{(\frac{a+b}{2})^+}f(b)\right]\right| \\
\leq\ & \frac{(b-a)}{4}\left\{\left(\frac{(q\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q}{(q\alpha+1)(q\alpha+2)}\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\frac{(q\alpha+1)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(b)|^{q}}{(q\alpha+1)(q\alpha+2)}\right)^{\frac{1}{q}}\Bigg\}.$$

Remark 2.5. *If we take $\alpha=1$ in Corollary 2.3, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{4}\left\{\left(\frac{(q+1)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(a)|^{q}}{(q+1)(q+2)}\right)^{\frac{1}{q}}+\left(\frac{(q+1)\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(b)|^{q}}{(q+1)(q+2)}\right)^{\frac{1}{q}}\right\}.$$

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