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REDUCING SUBSPACES OF WEIGHTED HARDY SPACES ON POLYDISKS

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ABSTRACT. We consider weighted Hardy spaces on polydisk $\mathbf{D}^n$ with $n > 1$. Let $z_1, z_2, \ldots, z_n$ be coordinate functions and $N_j \in \mathbf{N}$. In this paper, we determine common reducing subspaces of $M_{z_1}^{N_1}, M_{z_2}^{N_2}, \ldots, M_{z_n}^{N_n}$.

1. Introduction

Let α be a multi-index of non-negative integers and we put $\omega = \{\omega_\alpha\}$ a set of positive numbers. Let $H^2_\omega(\mathbf{D}^n)$ be the weighted Hardy space on $\mathbf{D}^n$ with the weight ω consisting of analytic functions

$$f(z) = \sum_\alpha a_\alpha z^\alpha$$

such that

$$\|f\|^2 = \sum_\alpha \omega_\alpha |a_\alpha|^2 < \infty.$$

Suppose the case of $n = 1$. Stessin and Zhu [5] showed that every reducing subspace of M_{z^N} in $H^2_\omega(\mathbf{D})$ is a direct sum of no more than N special reducing subspaces, and these subspaces in $H^2_\omega(\mathbf{D})$ are singly generated by a polynomial of degree less than N. In this paper we generalize the results in the case of $n = 1$.

Throughout the paper we consider the case of $n = 2$ because we can prove our statement for any n as well as $n = 2$. We fix $N_1, N_2 \in \mathbf{N}$ and a weight sequence ω so that the multiplications by the coordinate functions are bounded on $H^2_\omega(\mathbf{D}^2)$. And we put the lexicographic order on a set of multi-indices. For $(z, w) \in \mathbf{C}^2$ and a multi-index $\alpha = (\alpha_1, \alpha_2)$, we define $(z, w)^\alpha = z^{\alpha_1} w^{\alpha_2}$. Let S_1, S_2 be the operators of multiplication by z^{N_1}, w^{N_2} respectively. We say a closed subspace X in $H^2_\omega(\mathbf{D}^2)$ is an invariant subspace of operators S_i if $S_i X \subset X$ for $i = 1, 2$. X is a reducing subspace of S_i if X is invariant under both S_i and its adjoint S_i^* for $i = 1, 2$.

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2. Transparent Polynomials

Now we define a class of polynomials related on N_1, N_2 and ω. Let I be a subset of multi-indices such that $I := \{(m_1, m_2); 0 \leq m_1 \leq N_1 - 1 \text{ and } 0 \leq m_2 \leq N_2 - 1\}$. We say that $(m_1, m_2) \in I$ and $(n_1, n_2) \in I$ are equivalent if

$$\frac{\omega_{m_1+k_1N_1\ m_2+k_2N_2}}{\omega_{m_1\ m_2}} = \frac{\omega_{n_1+k_1N_1\ n_2+k_2N_2}}{\omega_{n_1\ n_2}}$$

for all non-negative integers k_1, k_2. In this case we write $(m_1, m_2) \sim (n_1, n_2)$.

We assume that p is a polynomial in the form of

$$p(z, w) = \sum \{a_\alpha (z, w)^\alpha; \alpha \in I\}.$$

We say that p is *transparent* if we have $\alpha \sim \beta$ for any two nonzero coefficients a_α, a_β of p. We partition the set I into equivalent classes $\Omega_1, \ldots, \Omega_K$. We see that the polynomial

$$q_k(z, w) = \sum \{a_\alpha (z, w)^\alpha; \alpha \in \Omega_k\}$$

is transparent for each $1 \leq k \leq K$. We put the sequence $\{p_1, \ldots, p_K\}$ which we sort $\{q_1, \ldots, q_K\}$ in the lexicographic order of the minimal multi-index of the polynomials. Then the decomposition

$$p = p_1 + \cdots + p_K$$

is called the *canonical decomposition* of p.

Let $\mathbb{S}_2$ be an algebra over $\mathbf{C}$ generated by the operators S_1, S_1^*, S_2, and S_2^*. For any nonzero function $f \in H^2_\omega(\mathbf{D}^2)$, we put $\mathbb{S}_2 f = \{Tf; T \in \mathbb{S}_2\}$. We set X_f the closure of $\mathbb{S}_2 f$ in $H^2_\omega(\mathbf{D}^2)$. We call X_f the reducing subspace generated by f. We see that X_f is the smallest reducing subspace containing f. Now we denote that $\mathrm{Span} X$ is the closed linear span of a set X in $H^2_\omega(\mathbf{D}^2)$.

Lemma 1. *If* $f(z, w) = \sum_{\alpha \in I} b_\alpha (z, w)^\alpha$ *is a transparent polynomial, then*

$$X_f = \mathrm{Span}\{f z^{k_1 N_1} w^{k_2 N_2} : k_1, k_2 = 0, 1, 2, \ldots\}.$$

Proof. Let $X = \mathrm{Span}\{f z^{k_1 N_1} w^{k_2 N_2} : k_1, k_2 = 0, 1, 2, \ldots\}$. Then $f \in X \subset X_f$. From the definition of X_f, it is sufficient to show that X is a reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$. The definition of X follows that X is invariant under S_1 and S_2. We will calculate that X is invariant under S_1^*. We fix some positive integer k_1 and write $k_1 = k_1' + 1$. Then

$$S_1^*(f z^{k_1 N_1} w^{k_2 N_2}) = S_1^* S_1 (f z^{k_1' N_1} w^{k_2 N_2}).$$

Let (m_1, m_2) be the minimal multi-index of nonzero coefficients of f. Then for any multi-index (α_1, α_2) of nonzero coefficients of f, we prove

$$\begin{aligned}
S_1^*(fz^{k_1N_1}w^{k_2N_2}) &= S_1^*S_1(\sum_{\alpha\in I} b_\alpha z^{\alpha_1+k_1'N_1}w^{\alpha_2+k_2N_2}) \\
&= \sum_{\alpha\in I} b_\alpha \frac{\omega_{\alpha_1+k_1N_1\ \alpha_2+k_2N_2}}{\omega_{\alpha_1+k_1'N_1\ \alpha_2+k_2N_2}} z^{\alpha_1+k_1'N_1}w^{\alpha_2+k_2N_2} \\
&= \sum_{\alpha\in I} b_\alpha \frac{\omega_{m_1+k_1N_1\ m_2+k_2N_2}}{\omega_{m_1+k_1'N_1\ m_2+k_2N_2}} z^{\alpha_1+k_1'N_1}w^{\alpha_2+k_2N_2} \\
&= \frac{\omega_{m_1+k_1N_1\ m_2+k_2N_2}}{\omega_{m_1+k_1'N_1\ m_2+k_2N_2}} \left(\sum_{\alpha\in I} b_\alpha z^{\alpha_1}w^{\alpha_2}\right) z^{k_1'N_1}w^{k_2N_2} \\
&= \frac{\omega_{m_1+k_1N_1\ m_2+k_2N_2}}{\omega_{m_1+k_1'N_1\ m_2+k_2N_2}} fz^{k_1'N_1}w^{k_2N_2} \in X,
\end{aligned}$$

because p is transparent. This shows that X is invariant under S_1^*. The same argument shows that X is invariant under S_2^*. $\square$

For any subspace X of $H^2_\omega(\mathbf{D}^2)$ with $X \neq \{0\}$, let (m_1, m_2) be the minimal multi-index such that there exists some $f \in X$ with $f^{(m_1,m_2)}(0,0) \neq 0$ but $g^{(k_1,k_2)}(0,0) = 0$ for all $g \in X$ and $(k_1, k_2) < (m_1, m_2)$. We will call (m_1, m_2) the order of X at the origin.

Proposition 2. *Let X be a nonzero reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$ and let (m_1, m_2) be the order of X at the origin. Then the extremal problem*

$$\sup\{\mathtt{Re}f^{(m_1,m_2)}(0,0) : f \in X, \|f\| \leq 1\}$$

has a unique solution G with $\|G\| = 1$ and $G^{(m_1,m_2)}(0,0) > 0$. Furthermore, G is a polynomial in the form of $G(z,w) = \sum_{\alpha\in I} b_\alpha (z,w)^\alpha$.

Proof. If f is a function in X with Taylor expansion $f(z,w) = \sum_\alpha a_\alpha (z,w)^\alpha$, then $f^{(m_1,m_2)}(0,0) = a_{(m_1,m_2)} m_1! m_2!$. Then $|a_{m_1,m_2} m_1! m_2!|^2 \leq \frac{(m_1! m_2!)^2}{\omega_{m_1\ m_2}} \sum_\alpha \omega_\alpha |a_\alpha|^2$ so the mapping $f \mapsto f^{(m_1,m_2)}(0,0)$ is a bounded linear functional on $H^2_\omega(\mathbf{D}^2)$. It follows that the extremal problem has a unique solution G with $\|G\| = 1$ and $G^{(m_1,m_2)}(0,0) > 0$. To show that G is the above polynomial, we prove $S_1^* G = 0$ and $S_2^* G = 0$. We put $g_f = \dfrac{G + S_1 f}{\|G + S_1 f\|}$ for $f \in X$. Since $\mathtt{Re}g_f^{(m_1,m_2)}(0,0) \leq G^{(m_1,m_2)}(0,0)$, it is easy to see that $\|G + S_1 f\| \geq 1$ for all $f \in X$. From this inequality we obtain $G \perp S_1 X$. Since $S_1^* G \in X$, we have $\langle S_1 S_1^* G, G\rangle = 0$, or $S_1^* G = 0$. Similarly we see that $S_2^* G = 0$. Therefore the degree of G is less than N_1 in z-valuable and N_2 in w-valuable. $\square$

The function G in Proposition 2 will be called the *extremal function* of X.

Lemma 3. *Suppose X is a reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$ and*

$$p(z,w) = \sum_{\alpha \in I} b_\alpha (z,w)^\alpha = p_1 + p_2 + \cdots + p_n$$

is the canonical decomposition of the polynomial p. If $p \in X$, then $p_i \in X$ for each $i = 1, 2, \cdots, n$.

Proof. Let $(m_1^{(i)}, m_2^{(i)})$ be the minimal multi-index of p_i. We note that if $i < j$, then $(m_1^{(i)}, m_2^{(i)})$ and $(m_1^{(j)}, m_2^{(j)})$ are not equivalent, and $(m_1^{(i)}, m_2^{(i)}) < (m_1^{(j)}, m_2^{(j)})$. We will show that $p_1 \in X$. Choose positive integers k_1 and k_2 such that

$$\frac{\omega_{m_1^{(1)}+k_1N_1\ m_2^{(1)}+k_2N_2}}{\omega_{m_1^{(1)}\ m_2^{(1)}}} \neq \frac{\omega_{m_1^{(n)}+k_1N_1\ m_2^{(n)}+k_2N_2}}{\omega_{m_1^{(n)}\ m_2^{(n)}}}.$$

Then

$$\begin{aligned} &\frac{\omega_{m_1^{(n)}+k_1N_1\ m_2^{(n)}+k_2N_2}}{\omega_{m_1^{(n)}\ m_2^{(n)}}} p - (S_1^*)^{k_1}(S_2^*)^{k_2}(S_1)^{k_1}(S_2)^{k_2} p \\ = &\sum_{k=1}^{n-1} \Big(\frac{\omega_{m_1^{(n)}+k_1N_1\ m_2^{(n)}+k_2N_2}}{\omega_{m_1^{(n)}\ m_2^{(n)}}} - \frac{\omega_{m_1^{(k)}+k_1N_1\ m_2^{(k)}+k_2N_2}}{\omega_{m_1^{(k)}\ m_2^{(k)}}} \Big) p_k, \end{aligned}$$

because

$$(S_1^*)^{k_1}(S_2^*)^{k_2}(S_1)^{k_1}(S_2)^{k_2} p = \sum_{k=1}^{n} \frac{\omega_{m_1^{(k)}+k_1N_1\ m_2^{(k)}+k_2N_2}}{\omega_{m_1^{(k)}\ m_2^{(k)}}} p_k.$$

We see that the above polynomial is in X and the coefficient of p_1 is nonzero. If some of the coefficients of $p_2, \ldots, p_{n-1}$ are nonzero, then we can vanish the coefficients of these polynomials in the same way. After at most $n-1$ steps, we will have a nonzero constant multiple of p_1, which belongs to X. Thus $p_1 \in X$. For $i = 2, \ldots, n$, we see $p_i \in X$ in the same way. □

Proposition 4. *The extremal function of any reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$ is transparent.*

Proof. Let X be a reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$. We get G, the extremal function of X by Proposition 2. If $G = g_1 + \cdots + g_n$ is the canonical decomposition of G into transparent polynomials, then g_1 contains the term $(G^{(m_1,m_2)}(0,0)/m_1!m_2!)z^{m_1}w^{m_2}$, where (m_1, m_2) is the order of zero of X at the origin. The polynomial g_1 satisfies the condition of extremal problem in Proposition 2; $\|g_1\| \le \|G\| = 1$, $g_1^{(m_1,m_2)}(0,0) = G^{(m_1,m_2)}(0,0)$, and $g_1 \in X$ by Lemma 3. The fact that G is extremal implies that G is equal to g_1 and is transparent. □

Proposition 5. *If p is a transparent polynomial and $Y \subset X_p$ is a reducing subspace, $Y = \{0\}$ or X_p.*

Proof. We assume $Y \neq \{0\}$. Let G_Y be its extremal function of Y. Then G_Y is a polynomial of degree less than (N_1, N_2) from Proposition 2. On the other hand, from the definition of X_p, there is some function $f(z^{N_1}, w^{N_2})$ in $\mathrm{Hol}(\mathbf{D}^2)$ such that $pf = G_Y$. We consider the degree of these polynomials, we see that f is constant therefore $p \in Y$. This implies $X_p \subset Y$ or $X_p = Y$. □

3. Main Result

We remark that we can extend results proved by Stessin and Zhu. Here we show a part of our result.

Theorem 6. *Every reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$ is generated by no more than N_1N_2 transparent polynomials.*

Proof. Let X be a nonzero reducing subspace of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$. Let G be the extremal function of X. From Proposition 4, G is transparent. And also X contains the reducing subspace X_G which is minimal from Proposition 5. Let $Y = X \ominus X_G$. We note that the term of $z^{m_1}w^{m_2}$ is contained in X where (m_1, m_2) is the minimal multi-index of G, but the term of $z^{m_1}w^{m_2}$ is not contained in Y from the definition of Y. Therefore through this process, we can make the order of zero of Y at the origin strictly greater than the order of zero of X at the origin. If $Y \neq \{0\}$, then we find the extremal function G' which is transparent and we consider $Y \ominus X_{G'}$. We continue these processes no more than N_1N_2 times because the number of the terms in the extremal functions is no more than N_1N_2 by Proposition 2. □

Corollary 7. *The reducing subspaces of the operators of multiplication by z, w in $H^2_\omega(\mathbf{D}^2)$ is $\{0\}$ and $H^2_\omega(\mathbf{D}^2)$.*

Proof. Let X be a nonzero reducing subspace of these operators in $H^2_\omega(\mathbf{D}^2)$. Then the extremal function of X is constant. It is easy to see that $X = H^2_\omega(\mathbf{D}^2)$. □

A weight sequence ω is of type I if for each (m_1, m_2),$(n_1, n_2) \in I$ with $(m_1, m_2) \neq (n_1, n_2)$ there exist some integers $k_1, k_2 > 0$ such that

$$\frac{\omega_{m_1+k_1N_1\ m_2+k_2N_2}}{\omega_{m_1\ m_2}} \neq \frac{\omega_{n_1+k_1N_1\ n_2+k_2N_2}}{\omega_{n_1\ n_2}}.$$

A weight sequence ω is of type II if it is not of type I.

If ω is of type I, then the only transparent polynomials are the monomials in the form of $a_\alpha(z, w)^\alpha$ where $\alpha \in I$, hence there are $2^{N_1N_2} - 2$ proper reducing subspaces of S_1 and S_2 in $H^2_\omega(\mathbf{D}^2)$, and they are the direct partial sums of X_{m_1,m_2}'s, where

$$X_{m_1,m_2} = \mathrm{Span}\{z^{m_1+k_1N_1}w^{m_2+k_2N_2}; k_1, k_2 = 0, 1, 2, \cdots\}.$$

If ω is of type II, then every reducing subspace is generated by no more than N_1N_2 transparent polynomials.

Example 8. Let $N_1 = N_2 = 2$. For a real number β with $-1 < \beta < \infty$, we put $\gamma_n = \dfrac{n!\Gamma(2+\beta)}{\Gamma(2+\beta+n)}$. We see that the weighted Bergman space $A^2_\beta(\mathbf{D}^2)$ has the weight of type I, where $\omega_{\alpha_1\ \alpha_2} = \gamma_{\alpha_1}\gamma_{\alpha_2}$. A direct calculation shows that $z - w$ is not transparent. Concretely

$$\frac{\omega_{3\ 0}}{\omega_{1\ 0}} \neq \frac{\omega_{2\ 1}}{\omega_{0\ 1}}.$$

This expression shows that the multi-indices $(0,1)$ and $(1,0)$ are not equivalent. Moreover

$$\frac{\omega_{2\ 1}}{\omega_{0\ 1}}(z-w) - S_1^* S_1(z-w) = \left(\frac{\omega_{2\ 1}}{\omega_{0\ 1}} - \frac{\omega_{3\ 0}}{\omega_{1\ 0}}\right) z \in \mathbb{S}_2(z-w).$$

We also see that the monomial w is in $\mathbb{S}_2(z-w)$. Therefore the reducing subspace X_{z-w} contains the transparent polynomials z and w, and we get $X_{z-w} = X_z \oplus X_w$.

4. Reducing subspaces of M_z^N

In this section, we consider $N_1 = 0$ or $N_2 = 0$. Without loss of generality, we can put $N_2 = 0$. The problem is determining the reducing subspaces of S_1 in $H^2_\omega(\mathbf{D}^2)$.

Proposition 9. *Suppose the weight ω is of type I. Every reducing subspace of S_1 in $H^2_\omega(\mathbf{D}^2)$ is the direct partial sums of X_m's, where*

$$X_m = \mathrm{Span}\{z^{m+kN_1} f(w); k = 0,1,2,\ldots, f \in H^2_\omega(\mathbf{D})\}.$$

Proof. We can show this result in the same way as above. □

We can extend this result to the weighted Hardy space $H^2_\omega(\mathbf{D}^n)$.

Theorem 10. *Suppose the weight ω is of type I. We fix $N_1, \ldots, N_l \in \mathbf{N}$. Every reducing subspace of $M_{z_1}^{N_1}, \ldots, M_{z_l}^{N_l}$ in $H^2_\omega(\mathbf{D}^n)$ is the direct partial sums of X_m's, where*

$$X_{m_1,\ldots,m_l} = \mathrm{Span}\{z_1^{m_1+k_1N_1} \cdots z_l^{m_l+k_lN_l} f(w); k_1, \ldots, k_l = 0,1,2,\ldots, f \in H^2_\omega(\mathbf{D}^{n-l})\}.$$

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References

[1] A. Brown and P. R. Halmos, *Algebraic Properties of Toeplitz Operators*, J. Reine Angew. Math. **213** (1963), 89–102.

[2] R. G. Douglas, *Banach Algebra Techniques in Operator Theory. Second Edition*, Graduate Texts in Mathematics **179**, Springer-Verlag, New York, 1998.
[3] P. R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.
[4] Yanyue Shi and Yufeng Lu, *Reducing Subspaces for Toeplitz Operators on the polydisk*, Bull. Korean Math. Soc. **50** (2013), 687–696.
[5] M. Stessin and K. Zhu, *Reducing subspace of weighted shift operators*, Proc. Amer. Math. Soc. **130** (2002), 2631–2639.

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