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ON PSEUDO-UMBILICAL SURFACES WITH NONZERO PARALLEL MEAN CURVATURE VECTOR IN $\mathbb{C}P^3(\tilde{c})$

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Abstract. Any pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$ is a totally real isotropic surface in $\mathbb{C}P^3(\tilde{c})$.

1. INTRODUCTION

Let $\mathbb{C}P^m(\tilde{c})$be a complex m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $\tilde{c}$.

Chen and Ogiue [1]classified totally umbilical submanifolds in $\mathbb{C}P^m(\tilde{c})$. However, it is well known that the class of pseudo-umbilical submanifolds in $\mathbb{C}P^m(\tilde{c})$ is too wide to classify. Thus, it is reasonable to study pseudo-umbilical submanifolds in $\mathbb{C}P^m(\tilde{c})$ under some additional condition.

Recently,the author proved that any pseudo-umbilical submanifold $M^n$with nonzero parallel mean curvature vector in $\mathbb{C}P^m(\tilde{c})$ is a totally real submanifold and satisfies $2m > n$ ([3]). Thus, we see that $\mathbb{C}P^2(\tilde{c})$ admits no pseudo-umbilical surfaces with nonzero parallel mean curvature vector.

In the previous paper [4], the author showed that any complete pseudo-umbilical isotropic surface of $P(\mathbb{R})$-type (see Preliminaries) with nonzero parallel mean curvature vector in $\mathbb{C}P^4(\tilde{c})$ is an extrinsic hypersphere in a 3-dimensional real projective space $\mathbb{R}P^3(\tilde{c}/4)$ of $\mathbb{C}P^3(\tilde{c})$.

The aim of this paper is to prove the following result.

Theorem1.1. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. Then M is a totally real isotropic surface in $\mathbb{C}P^3(\tilde{c})$.*

Corollary1.1. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. If the surface is of $P(\mathbb{R})$-type, then M is an extrinsic hypersphere in a 3-dimensional real projective space $\mathbb{R}P^3(\tilde{c}/4)$ of $\mathbb{C}P^3(\tilde{c})$.*

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2.PRELIMINARIES

Let M be an n-dimensional submanifold of a complex m-dimensional Kaehler manifold $\tilde{M}$ with complex structure J and Kaehler metric g. A submanifold M of a Kaehler manifold $\tilde{M}$ is said to be *totally real* if each tangent space of M is mapped into the normal space by the complex structure of $\tilde{M}$.

Let ∇(resp.$\tilde{\nabla}$) be the covariant differentiation on M(resp.$\tilde{M}$). We denote by σ the second fundamental form of M in $\tilde{M}$. Then the Gauss formula and the Weingarten formula are given respectively by

$$\sigma(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y, \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$$

for vector fields X, Y tangent to M and a normal vector field ξ normal to M, where $-A_\xi X$ (resp.$D_X\xi$) denotes the tangential (resp.normal) component of $\tilde{\nabla}_X \xi$. A normal vector field ξ is said to be *paralle* if $D_X\xi = 0$ for any vector field X tangent to M.

The covariant derivative $\bar{\nabla}\sigma$ of the second fundamental form σ is defined by

$$(\bar{\nabla}_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

for all vector fields X, Y and Z tangent to M. The second fundamental form σ is said to be *parallel* if $\bar{\nabla}_X\sigma = 0$.

Let $\zeta = 1/n$trace σ and $H = |\zeta|$ denote the mean curvature vector and the mean curvature of M in $\tilde{M}$,respectively. If the second fundamental form σ satisfies $\sigma(X,Y) = g(X,Y)\zeta$, then M is said to be *totally umbilical* submanifold in $\tilde{M}$. If the second fundamental form σ satisfies $g(\sigma(X,Y),\zeta) = g(X,Y)g(\zeta,\zeta)$, then M is said to be *pseudo-umbilical* submanifold of $\tilde{M}$. The submanifold M in $\tilde{M}$ is said to be a λ-*isotropic* submanifold if $|\sigma(X,X)| = \lambda$ for all unit tangent vectors X at each point. In particular, if the function is constant, then M is called a *constant isotropic* submanifold of $\tilde{M}$.

The first normal space at x, $N_x^1(M)$ is defined to be the vector space spanned by all vectors $\sigma(X,Y)$. The first osculating space $O_x^1(M)$ at x is defined by

$$O_x^1(M) = T_x(M) + N_x^1(M)$$

The submanifold M of $\tilde{M}$ is called a submanifold of $P(\mathbb{R})$-*type* (resp. $P(\mathbb{C})$-*type*) if $JT_x(M) \subset (N_x^1(M))^{\perp}$ (resp.$JT_x(M) \subset N_x^1(M)$) for every point $x \in M$.

Let R (resp.$\tilde{R}$) be the Riemannian curvature for ∇(resp.$\tilde{\nabla}$). Then the Gauss equation is given by

$$\begin{aligned} g(\tilde{R}(X,Y)Z,W) = g(R(X,Y)Z,W) &+ g(\sigma(X,Z),\sigma(Y,W)) \\ &- g(\sigma(Y,Z),\sigma(X,W)) \end{aligned}$$

for all vector fields X, Y, Z and W tangent to M.

3.LEMMAS

Let M^2 be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^m(\tilde{c})$.

We recall the following results.

Theorem3.1[3]. *Let M be an n-dimensional pseudo-umbilical submanifold with nonzero parallel mean curvature vector in $\mathbb{C}P^m(\tilde{c})$. Then $2m > n$ and M^n is immersed in $\mathbb{C}P^m(\tilde{c})$ as a totally real submanifold.*

Since M is a totally real surface in $\mathbb{C}P^m(\tilde{c})$, the normal space $T_x^{\perp}(M)$ is decomposed in the following way; $T_x^{\perp}(M) = JT_x(M) \oplus \nu_x$ at each point x of M, where ν_x denotes the orthogonal complement of $JT_x(M)$ in $T_x^{\perp}(M)$.

Lemma3.1[4]. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^m(\tilde{c})$. Then we have*

(1) $\zeta \in \nu_x$
(2) $g(\sigma(X,Y), J\zeta) = 0$
(3) $g((\tilde{\nabla}_X \sigma)(Y,Z), \zeta) = 0$

for all vector fields X, Y and Z tangent to M.

We prepare the following fundamental fact without proof.

Lemma3.2. *Let M^n be a totally real submanifold in $\mathbb{C}P^m(\tilde{c})$. Then we have*

$$g(\sigma(X,Y),JZ) = g(\sigma(X,Z),JY)$$

for all vector fields X, Y and Z tangent to M.

Lemma3.3. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^m(\tilde{c})$. If the surface is of $P(\mathbb{R})$-type, then we have*

(1) $g((\bar{\nabla}_X\sigma)(Y,Z),J\zeta) = 0$
(2) $g((\bar{\nabla}_X\sigma)(Y,Z),JW) = g(J\sigma(Y,Z),\sigma(X,W))$

for all vector fields X, Y, Z and W tangent to M.

Proof. By Lemma3.1(2), we get

$$\begin{aligned} g((\bar{\nabla}_X\sigma)(Y,Z),J\zeta) &= g(D_X(\sigma(Y,Z)),J\zeta) \\ &= g(\tilde{\nabla}_X(\sigma(Y,Z)),J\zeta) \\ &= -g(\sigma(Y,Z),\tilde{\nabla}_X(J\zeta)) \\ &= g(J\sigma(Y,Z),\tilde{\nabla}_X\zeta) \\ &= g(J\sigma(Y,Z),D_X\zeta) \\ &= 0 \end{aligned}$$

And this Lemma3.3(2) has been proved in [4]. □

We recall the following results.

Theorem3.2[2]. *If M^n is an $n(\geq 2)$-dimensional complete nonzero isotropic totally real submanifold of $P(\mathbb{R})$-type with parallel second fundamental form in $\mathbb{C}P^m(\tilde{c})$, there exists a unique totally geodesic submanifold $\mathbb{R}P^r(c)$ such that M^n is a submanifold in $\mathbb{R}P^r(c)$ and that $O_x^1(M) = T_x(\mathbb{R}P^r(c))$ for every point $x \in M$.*

Theorem3.3[2]. *If M^n is an $n(\geq 2)$-dimensional complete nonzero isotropic totally real submanifold of $P(\mathbb{C})$-type with parallel second fundamental form in $\mathbb{C}P^m(\tilde{c})$, there exists a unique totally geodesic Kaehler submanifold $\mathbb{C}P^r(\tilde{c})$ such that M^n is a submanifold in $\mathbb{C}P^r(\tilde{c})$ and that $O_x^1(M) = T_x(\mathbb{C}P^r(\tilde{c}))$ for every point $x \in M$.*

4.Proof of Theorem1.1

Let M^2 be a pseudo-umbilical surface with nonzero parallel mean curvature vector ζ in $\mathbb{C}P^3(\tilde{c})$. We choose a local orthonormal frame field

$$e_1, e_2, e_3, e_4 = Je_1, e_5 = Je_2, e_6 = Je_3$$

of $\mathbb{C}P^3(\tilde{c})$ such that e_1, e_2 are tangent to M. By Lemma3.1(1), we choose e_3 in such a way that its direction coincides with that of the mean curvature vector ζ. Since M is a pseudo-umbilical surface, it is umbilic with respect to the direction of the mean curvature vector ζ. Thus, by Lemma3.1(2), the surface satisfies

$$\begin{cases} \sigma(e_1, e_1) = He_3 + ae_4 + be_5 \\ \sigma(e_1, e_2) = \qquad fe_4 + ge_5 \\ \sigma(e_2, e_2) = He_3 - ae_4 - be_5 \end{cases} \tag{4.1}$$

for some functions a, b, f, g with respect to the orthonormal local frame field $\{e_i\}$. By Lemma3.2,we get

$$g(\sigma(e_1, e_2), Je_1) = g(\sigma(e_1, e_1), Je_2) \tag{4.2}$$

$$g(\sigma(e_2, e_1), Je_2) = g(\sigma(e_2, e_2), Je_1) \tag{4.3}$$

Thus by (4.1),(4.2) and (4.3) we obtain $f = b$ and $g = -a$. Therefore we have the following.

Proposition4.1. *Let M be a pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. Then the surface satisfies*

$$\begin{cases} \sigma(e_1, e_1) = He_3 + ae_4 + be_5 \\ \sigma(e_1, e_2) = \qquad be_4 - ae_5 \\ \sigma(e_2, e_2) = He_3 - ae_4 - be_5 \end{cases} \tag{4.4}$$

for some functions a, b with respect to the orthonormal local frame field $\{e_i\}$.

By Proposition4.1,for any unit tangent vector $(ke_1+le_2)/\sqrt{k^2+l^2}$, where k, l are some real numbers, we get

$$|\sigma((ke_1+le_2)/\sqrt{k^2+l^2}, (ke_1+le_2)/\sqrt{k^2+l^2})|^2 = H^2+a^2+b^2 \tag{4,5}$$

Thus we see that the surface is isotropic. This completes the proof of Theorem1.1.

Remark4.1. By Proposition4.1 and (2.1), we get the Gauss curvature $K = \tilde{c}/4 + H^2 - 2(a^2 + b^2)$. If the Gauss curvature is constant, then $a^2 + b^2$ is constant. By (4.5), we see that the surface in Theorem1.1 is constant isotropic.

Now we prove Corollary1.1. If the surface M is of $P(\mathbb{R})$-type, then by (4.4) we see that the surface is immersed in $\mathbb{C}P^3(\tilde{c})$ as a totally umbilical submanifold. Immediately, by Lemma3.1(3) and Lemma3.3 , we have $\bar{\nabla}\sigma \equiv 0$. The assertion of Corollary1.1 follows immediately from Theorem3.2.

Finally, we remark the following Proposition4.2. If a pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$ is not totally umbilical, then we see that $ab \neq 0$ in (4.4). Thus, by Proposition4.1 we get $\dim N_x^1(M) = 3$ and $\dim O_x^1(M) = 5$. By Theorem3.3, if $\bar{\nabla}\sigma \equiv 0$, then there exists a real 5-dimensional complex projective space. This is a contradiction. Therefore we get

Proposition4.2. *Let M be a complete pseudo-umbilical surface with nonzero parallel mean curvature vector in $\mathbb{C}P^3(\tilde{c})$. If M is not totally umbilical, then the surface is a totally real isotropic surface in $\mathbb{C}P^3(\tilde{c})$ whose second fundamental form is not parallel.*

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