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A NEW CHARACTERIZATION OF HOMOGENEOUS REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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ABSTRACT. The purpose of this paper is to give a new characterizations of homogeneous real hypersurfaces M in complex space forms $M_n(c)$ when the covariant derivative and the Lie derivative of the Ricci tensor of M are equal to each other along the direction of the structure vector ξ.

1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. The complete and simply connected complex space form is isometric to a complex projective space P_nC, a complex Euclidean space C^n, or a complex hyperbolic space H_nC according as $c > 0$, $c = 0$ or $c < 0$ respectively. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

Now, there exist many studies about real hypersurfaces of $M_n(c)$, $c\neq 0$. One of the first researches is the classification of homogeneous real hypersurfaces of a complex projective space P_nC by Takagi [14], who showed that these hypersurfaces of P_nC could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. This result is generalized by many authors (See [3], [5], [8], [9], [11] and [13]).

On the other hand, real hypersurfaces of H_nC have been also investigated by many authors (See [1], [6], [10] and [12]) from different points of view. In particular, Berndt [1] proved the following.

Theorem A. *Let M be a real hypersurface of H_nC, $n \geqq 3$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the followings :*

(A_0) *a horosphere in H_nC, that is, a Montiel tube,*
(A_1) *a tube over a totally geodesic hyperplane H_kC ($k = 0$ or $n-1$),*
(A_2) *a tube over a totally geodesic H_kC ($1 \leqq k \leqq n-2$),*

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(B) *a tube over a totally real hyperbolic space H_nR.*

Among the classification of homogeneous real hypersurfaces in $M_n(c)$ real hypersurfaces of type A_1 or type A_2 in P_nC or those of type A_0, A_1 or A_2 in H_nC are said to be of *type A*. By a theorem due to Okumura [13] and to Montiel and Romero [12] we have

Theorem B. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. If it satisfies*

$$A\phi - \phi A = 0, \tag{1.1}$$

where A denotes the shape operator of M, then M is locally congruent to one of type A.

Now let us denote by $\mathcal{L}_\xi$ the Lie derivative with respect to the structure vector field ξ. As is easily seen, the condition (1.1) is equivalent to

$$\mathcal{L}_\xi g = 0. \tag{1.2}$$

In this paper we also consider the Lie derivative and the covariant derivative of the Ricci tensor S of M in $M_n(c)$. So the purpose of this paper is to give a new characterization of homogeneous real hypersurfaces M in complex space forms $M_n(c)$ when the covariant derivative of the Ricci tensor coincides with the Lie derivative along the direction of the structure vector ξ. We prove the following.

Theorem 1. *Let M be a real hypersurface in a complex projective space P_nC, $n \geqq 3$. If it satisfies*

$$\mathcal{L}_\xi S = \nabla_\xi S, \tag{$*$}$$

where S denotes the Ricci tensor on M, then the structure vector field ξ is principal. Moreover, if it satisfies $\alpha^2 > (n-2)c/2$ and $(*)$, then M is locally congruent to a tube of radius r over one of the following Kähler manifolds ;

(A_1) *a hyperplane P_mC, where $m = n-1$, $0 < r < \pi/2$,*
(A_2) *a totally geodesic P_kC, where $1 \leqq k \leqq n-2$, $0 < r < \pi/2$,*
(B) *a complex quadric Q_{n-1}, where the radius r satisfies $\cot^2 2r = n-2$.*

By the definition of the Lie derivative, it is easily seen that condition $(*)$ is equivalent to

$$S\phi A - \phi A S = 0. \tag{$**$}$$

Now let us consider a real hypersurface in a complex hyperbolic space H_nC satisfying the condition $(**)$. Then in this case by virtue of Theorem A we have

Theorem 2. *Let M be a real hypersurface in a complex hyperbolic space H_nC, $n \geqq 3$. If it satisfies $(**)$ and the structure vector field ξ is principal, then M is locally congruent to one of real hypersurfaces of type A_0, A_1 and A_2.*

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2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$, $c \neq 0$, $n \geqq 3$ and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmertic transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M.

By properties of the almost complex structure J, the set (ϕ, ξ, η, g) of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation and X denotes any vector field tangent to M. Accordingly, this set (ϕ, ξ, η, g) defines the *almost contact metric structure* on M. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows :

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X. Let T_0 be a distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$ of the tangent space T_xM of the hypersurface M at x. It is called a *holomorphic distribution* on M.

Next we suppose that the structure vector field ξ is principal with corresponding principal curvature α. Then it is seen in [5] and [11] that α is locally constant on M and it satisfies

$$2A\phi A = \frac{1}{2}c\phi + \alpha(A\phi + \phi A). \tag{2.4}$$

Therefore if a vector field X orthogonal to ξ is principal with principal curvature λ and if $2\lambda - \alpha \neq 0$, then ϕX is also principal with principal curvature $\mu = (2\alpha\lambda + c)/2(2\lambda - \alpha)$, namely we have

$$A\phi X = \mu\phi X, \quad \mu = \frac{2\lambda\alpha + c}{4\lambda - 2\alpha}. \tag{2.5}$$

3. The Ricci tensor

Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$. This section is to investigate a sufficient condition for the structure vector ξ to be principal in terms of the Ricci tensor. It is closely related with another characterization of real hypersurfaces of type A concerning the Lie derivative with respect to the structure vector ξ. Its Ricci tensor S of M is given by

$$S = \frac{1}{4}c\{(2n+1)I - 3\xi \otimes \eta\} + hA - A^2,$$

where I denotes the identity transformation on M and h is the trace of the shape operator A. Then the Lie derivative of the Ricci tensor S with respect to the structure vector ξ on M is given by

$$\mathcal{L}_\xi S = \nabla_\xi S + S\phi A - \phi AS,$$

with the help of (2.1).

Assume that M is a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geqq 3$, whose Ricci tensor S satisfies $(**)$. Namely we assume that

$$S\phi A - \phi AS = 0. \tag{$**$}$$

From the definition of S it follows that the condition $(**)$ is equivalent to

$$h(A\phi A - \phi A^2) + \phi A^3 - A^2\phi A + \frac{3}{4}c\phi A\xi \otimes \eta = 0, \tag{3.1}$$

that is,

$$h(A\phi - \phi A)A + (\phi A - A\phi)A^2 + A(\phi A - A\phi)A + \frac{3}{4}c\phi A\xi \otimes \eta = 0. \tag{3.2}$$

Let X be a principal vector with principal curvature λ. Taking the inner product of X with (3.1) and taking account of the skew-symmetry of the structure tensor ϕ, we have

$$A^3\phi X - hA^2\phi X + \lambda(h-\lambda)A\phi X - \frac{3}{4}cg(X, \phi A\xi)\xi = 0. \tag{3.3}$$

At any point x in the real hypersurface M the tangent space of M at x is denoted by T_xM. Now let us denote by $L(X_1, \ldots, X_m)$ a linear subspace of T_xM spanned by the vectors $X_1, \ldots, X_m$ in T_xM. When the subspace $L(X_1, \ldots, X_m)$ is invariant by the shape operator A of M, we say that the subspace $L(X_1, \ldots, X_m)$ is *A-invariant.*

If we assume that ξ is not principal, there is a vector Y orthogonal to ξ such that

$$A\xi = \alpha\xi + Y,$$

where $\alpha = g(A\xi, \xi)$. Since Y is non zero, the vector AY can be expressed as

$$AY = \beta\xi + \gamma Y + Y_1,$$

where Y_1 is orthogonal to ξ and Y, and $\beta = g(Y, Y)$ and $\gamma = g(AY, Y)/\beta$.

Now in order to get our result let us prove the following.

Lemma 3.1. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geqq 3$. If it satisfies the condition $S\phi A - \phi AS = 0$, then the subspace $L(\xi, A\xi)$ is A-invariant.*

Proof. Now let us consider the transpose of (3.1). Then it follows

$$h(A\phi A - A^2\phi) + A^3\phi - A\phi A^2 + \frac{3}{4}c\xi\otimes\eta\circ A\phi = 0. \tag{3.4}$$

Let us transform the shape operator A to (3.1) to the left and to (3.4) to the right respectively. Then the combination of these two equations yields

$$A\phi A\xi\otimes\eta + \xi\otimes\eta\circ A\phi A = 0.$$

From this, by applying ξ we have

$$A\phi A\xi = 0. \tag{3.5}$$

Now applying ξ to (3.1) and using (3.5), we have

$$\phi(A^3\xi - hA^2\xi + \frac{3}{4}cA\xi) = 0.$$

So it follows

$$A^3\xi - hA^2\xi + \frac{3}{4}cA\xi \equiv 0 \quad (mod\ \xi). \tag{3.6}$$

Next, let us apply ξ to (3.4) and use (3.5). Then

$$A\phi A^2\xi = 0. \tag{3.7}$$

By operating $A\xi$ to (3.1) to the right and using (3.7) we obtain

$$\phi(A^4\xi - hA^3\xi + \frac{3}{4}cg(A\xi,\xi)A\xi) = 0. \tag{3.8}$$

From this it follows

$$A^4\xi - hA^3\xi + \frac{3}{4}cg(A\xi,\xi)A\xi \equiv 0 \quad (mod\ \xi). \tag{3.9}$$

Finally, let us transform the shape operator A to (3.6). Then by substracting (3.9) from this, we have

$$cA^2\xi \equiv 0 \quad (mod\ \xi,\ A\xi).$$

This means that the linear subspace $L(\xi, A\xi)$ is A-*invariant.* It completes the proof of Lemma 3.1. □

Lemma 3.2. *Let M be a real hypersurface in a complex projective space P_nC, $n \geqq 3$. If it satisfies $S\phi A - \phi AS = 0$, then the structure vector ξ is principal.*

Proof. Suppose that the structure vector ξ is not principal. Namely suppose that $A\xi = \alpha\xi + Y$, where Y is a vector in T_0 and $\beta = g(Y,Y)$ is a smooth non-negative function on M. Let M_0 be a subset in M consisting of points x at which $\beta(x) \neq 0$. Suppose that the subset M_0 is not empty.

By Lemma 3.1 we have proved that the subspace $L(\xi, Y)$ is A-invariant. Taking the inner product (3.1) with the vector ϕY and making use of the property $A\phi Y = 0$ derived from the formula (3.5) in the proof of Lemma 3.1, we have

$$A^3Y - hA^2Y + \frac{3}{4}c\beta\xi = 0, \tag{3.10}$$

where we have used the properties that A is symmetric, ϕ is skew-symmetric and $\phi^2 Y = -Y$. Then Lemma 3.1 gives that

$$\begin{aligned} &AY = \beta\xi + \gamma Y, \quad A^2Y = \beta(\alpha+\gamma)\xi + (\beta+\gamma^2)Y, \\ &A^3Y = \beta\{\alpha(\alpha+\gamma) + (\beta+\gamma^2)\}\xi + \{\beta(\alpha+\gamma) + \gamma(\beta+\gamma^2)\}Y, \end{aligned}$$

from the above equation, (3.10) is equivalent to

$$\begin{aligned} \beta\{(\alpha^2 + \alpha\gamma + \gamma^2 + \beta) - h(\alpha+\gamma) + \frac{3}{4}c\}\xi \\ + \{(\alpha\beta + 2\beta\gamma + \gamma^3) - h(\beta+\gamma^2)\}Y = 0. \end{aligned}$$

This implies that we have

$$(3.11)\qquad \begin{aligned}&(\alpha^2+\alpha\gamma+\gamma^2+\beta)-h(\alpha+\gamma)+\frac{3}{4}c=0,\\&(\alpha\beta+2\beta\gamma+\gamma^3)-h(\beta+\gamma^2)=0.\end{aligned}$$

Mutiplying $(\beta+\gamma^2)$ to the first of (3.11) and $(\alpha+\gamma)$ to the second of (3.11) and substracting the obtained second equation from the first, we have

$$(3.12)\qquad (\alpha\gamma-\beta)^2+\frac{3}{4}c(\beta+\gamma^2)=0.$$

From this we have

$$\alpha\gamma=\beta \text{ and } \beta+\gamma^2=0,$$

because $\beta=g(Y,Y)\geq 0$ and $c>0$. So it follows $\beta=\gamma=0$. This implies that the open subset M_0 should be empty. That is, ξ is principal. $\square$

Theorem 3.3. *Let M be a real hypersurface in a complex hyperbolic space H_nC, $n\geqq 3$. If it satisfies the condition $S\phi A-\phi AS=0$ and the structure vector field ξ is principal, then M has at most five distinct constant principal curvatures.*

Proof. By the assumption the structure vector ξ is principal, namely we have $A\xi=\alpha\xi$.

Suppose that a principal unit vector X in T_0 with principal curvature $\lambda\neq\alpha/2$. Then the vector ϕX is also principal with principal curvature $\mu=(2\alpha\lambda+c)/(4\lambda-2\alpha)$ by (2.5). By (3.3) and $A\xi=\alpha\xi$, we get

$$(3.13)\qquad \mu(\mu-\lambda)(\mu+\lambda-h)=0.$$

Suppose that $\mu=0$. By (3.1) we see $\lambda^2(\lambda-h)=0$. Then in this case, because of $2\alpha\lambda=-c\neq 0$, we have $\lambda=h$ and therefore we have $\lambda+\mu=h$. Combining the above situation with the case $\mu\neq 0$, we have

$$(3.14)\qquad (\lambda-\mu)(\lambda+\mu-h)=0,$$

namely we have

$$(3.15)\qquad \lambda=\mu \quad\text{or}\quad \lambda+\mu=h.$$

For the principal curvature $\lambda(\neq\alpha/2)$ the corresponding principal curvature μ is given by $(2\alpha\lambda+c)/(4\lambda-2\alpha)$ and hence if $\lambda=\mu$, then the principal curvature λ satisfies the equation

$$(3.16)\qquad 4x^2-4\alpha x-c=0.$$

Here we note that the constant α can not vanish. Now the roots of (3.16) give

$$(3.17)\qquad \lambda=\frac{1}{2}(\alpha+\sqrt{\alpha^2+c}) \quad\text{or}\quad \lambda=\frac{1}{2}(\alpha-\sqrt{\alpha^2+c}).$$

We denote by λ_+ and λ_- the above two principal curvatures, respectively. Let α, λ_a, λ_r, λ_x, μ_x be all principal curvatures on M, where the indices run over the following ranges : $1 \leqq a \leqq p$, $p+1 \leqq r \leqq 2q$ and $2q+1 \leqq x \leqq n+q-1$ and $\lambda_x + \mu_x = h$. Because α is constant, the principal curvatures λ_+ and λ_- are constant. The trace h of the shape operator A is given by

$$h = \alpha + p\lambda_+ + (2q-p)\lambda_- + (n-1-q)h, \tag{3.18}$$

and therefore we have

$$(q+1)\alpha + (p-q)\sqrt{\alpha^2+c} + (n-q-2)h = 0. \tag{3.19}$$

From this we assert that the trace h is constant. In fact, we suppose that $q = n-2$. Then we have

$$(q+1)^2\alpha^2 = (p-q)^2(\alpha^2+c),$$

namely

$$(p+1)(2q-p+1)\alpha^2 = (p-q)^2 c \neq 0.$$

Because of $2q > p$ the constant c must be positive, a contradiction. Thus we have $q < n-2$, which yields that the trace h is constant.

By $\lambda + \mu = h$ and $\mu = (2\alpha\lambda + c)/(4\lambda - 2\alpha)$, the principal curvatures λ_x and μ_x satisfy the equation

$$4x^2 - 4hx + 2\alpha h + c = 0.$$

Consequently, these principal curvatures are also constant. This shows that all of principal curvatures on M are at most five. If there does not exist a principal vector X in T_0 with principal curvature $\lambda \neq \alpha/2$, then distinct principal curvatures are only α and $\alpha/2$. It means that M has two distinct constant principal curvatures. It completes the proof of Theorem 3.3. □

Remark. Under the condition $\mathcal{L}_\xi S = 0$ Kimura and Maeda [8] have asserted that the structure vector ξ is principal. But the method in Lemmas 3.1 and 3.2 which are used to obtain the fact that the structure vector field ξ is principal is quite different from the Kimura and Maeda's one. So it seems to be sure to the present authors that the method in above will be also useful to derive the result that ξ is principal for the case $c < 0$.

4. Proof of Theorem 1

In this section we consider the case where the ambient space is a complex projective space. Let M be a real hypersurface of $M_n(c)$, $c > 0$, $n \geqq 3$. We assume that the Ricci tensor S satisfies

$$S\phi A - \phi A S = 0. \tag{**}$$

Then, by Lemma 3.2 the structure vector ξ is principal and hence the above assumption gives us

$$h(A\phi A - \phi A^2) + \phi A^3 - A^2\phi A = 0, \tag{4.1}$$

which is reformed as

$$h(A\phi - \phi A)A + (\phi A - \phi A)A^2 + A(\phi A - A\phi)A = 0. \tag{4.2}$$

Let X be a unit principal vector in T_0 with principal curvature λ. Since c is positive, we have $2\lambda - \alpha \neq 0$, unless we have $2\alpha\lambda + c = 0$ by (2.5), that is, $\alpha^2 + c = 0$, a contradiction. So ϕX is also a unit principal vector in T_0 with corresponding principal curvature μ by (2.5). Again by (2.5) we see that $\mu = (2\alpha\lambda + c)/(4\lambda - 2\alpha)$. Thus we have

$$\lambda(\lambda - \mu)(\lambda + \mu - h) = 0$$

and then we have

$$\mu(\lambda - \mu)(\lambda + \mu - h) = 0.$$

Accordingly we see

$$(\lambda - \mu)(\lambda + \mu - h) = 0 \tag{4.3}$$

and therefore we have

$$(4\lambda^2 - 4\alpha\lambda - c)(4\lambda^2 - 4h\lambda + 2h\alpha + c) = 0. \tag{4.4}$$

Theorem 4.1. *Let M be a real hypersurface in a complex projective space P_nC, $n \geqq 3$. If it satisfies $\alpha^2 > (n-2)c/2$ and if it satisfies (**), then M has at most five distinct constant principal curvatures.*

Proof. By Lemma 3.2 the structure vector ξ is principal, namely we have $A\xi = \alpha\xi$. For a principal unit vector X in T_0 with principal curvature, the vector ϕX is also a principal unit vector X in T_0 with corresponding principal curvature $\mu = (2\alpha\lambda + c)/(4\lambda - 2\alpha)$ by (2.5). If $\lambda = \mu$, then the principal curvature λ satisfies the quadratic equation

$$4x^2 - 4\alpha x - c = 0.$$

We denote by λ_+ and λ_- these two principal curvatures. Let their multiplicities be p and $2q - p$, respectively. Then by (3.19) we have

$$(q+1)\alpha + (p-q)\sqrt{\alpha^2 + c} + (n - q - 2)h = 0. \tag{4.5}$$

It is seen that α is locally constant and moreover the fact and the above equation means that the trace h of the shape operator A is constant in the case where $q < n - 2$. So it tells us that M has at most five distinct constant principal curvatures.

Suppose that $q = n - 2$. Then by (4.5) α is expressed as

$$\alpha^2 = \frac{(p-q)^2 c}{(p+1)(2q - p + 1)}, \quad 0 \leqq p \leqq 2q = 2(n-2) \tag{4.6}$$

We denote by $g(p)$ the right hand side of the above equation. Then it is easily seen that we have

$$g(q) \leqq g(p) \leqq g(0) = g(2q) = (n-2)^2 c/(2n-3) < (n-2)c/2,$$

from which it follows that if $\alpha^2 > (n-2)c/2$, then the equation (4.6) does not hold. Hence the case where $q = n - 2$ cannot occur. It completes the proof. $\square$

Theorem 1. *Let M be a real hypersurface in a complex projective space P_nC, $n \geqq 3$. If it satisfies*

$$(**) \qquad S\phi A - \phi AS = 0,$$

where S denotes the Ricci tensor on M, then the structure vector field ξ is principal. Moreover, if it satisfies $\alpha^2 > (n-2)c/2$ and $(**)$, then M is locally congruent to a tube of radius r over one of the following Kähler manifolds ;

(A_1) a hyperplane P_mC, where $m = n-1$, $0 < r < \pi/2$,
(A_2) a totally geodesic P_kC, where $1 \leqq k \leqq n-2$, $0 < r < \pi/2$,
(B) a complex quadric Q_{n-1}, where the radius r satisfies $cot^2 2r = n-2$.

Proof. According to a theorem due to Kimura [7] and Theorem 4.1, M is homogeneous. By virtue of the classification theorem of Takagi, M is one of type A_1, A_2, B, C, D and E. Hence, in order to prove above theorem, we may check the condition $(**)$ one by one for the above six model spaces.

First, let M be of type C, D and E. Without loss of generality, we may put $c = 4$. Then, for the table of Takagi [15], it follows that there is a principal curvature λ different from the corresponding principal curvature μ and they satisfy

$$\lambda + \mu = -\frac{4}{\alpha}, \quad \lambda\mu = -1,$$

where we have $\lambda = cot\ (r - \pi/4)$, $\mu = -tan\ (r - \pi/4)$ and $\alpha = 2\ cot\ 2r$.

On the other hand, the other two principal curvatures λ and the corresponding principal curvature μ are given by $\lambda = cot\ r$, $\mu = -tan\ r$ and $\alpha = 2\ cot\ 2r$. So we have also

$$\lambda + \mu = \alpha, \quad \lambda\mu = -1.$$

Since the principal curvatures $\lambda = cot\ r$ and $\mu = -tan\ r$ are derived from the equation $\lambda = \mu$, namely they are the roots of the quadratic equation $x^2 - \alpha x - 1 = 0$ and hence the others $\lambda = cot(r - \pi/4)$ and $\mu = -tan(r - \pi/4)$ are derived from the equation $\lambda + \mu = h$. We denote all principal curvatures α, $\lambda_a = \lambda_+ = cot\ r$, $\lambda_r = \lambda_- = -tan\ r$, λ_x and μ_x, whose indices runs over the ranges $1 \leqq a \leqq p$, $p+1 \leqq r \leqq 2q$, $2q+1 \leqq x \leqq n+q-1$. Furthermore we see $\lambda_x + \mu_x = -4/\alpha = h$. The trace h of the shape operator A is given by

$$h = \alpha + p\lambda_a + (2q-p)\lambda_r + (n-1-q)(\lambda_x + \mu_x).$$

Namely, we have

$$\alpha + p\lambda_+ + (2q-p)\lambda_- + (n-2-q)h = 0.$$

Substituting the values of principal curvatures and the trace h into this equation we get

$$4(n-1)x^4 - \{4(n-1) + 2(p-q)\}x^2 + (p+1) = 0,$$

where $x = sin\ r$.

On the other hand, the assumption for α is equivalent to

$$2nx^4 - 2nx^2 + 1 > 0.$$

By these two relations we have

$$2n(p-q)sin^2\ r > np - n + 2. \tag{4.7}$$

On each real hypersurface M of type $C \sim E$, the multiplicity p or q of the principal curvature λ_+ or λ_- is greater than or equal to 1. Hence the right hand side of (4.7) is positive and therefore we have $p > q$. On the other hand, since the radius r is less than $\pi/4$, we see $2sin^2\ r < 1$, from which together with (4.7) it follows that

$$0 > n(q-1) + 2,$$

a contradiction. Thus the real hypersurface M of type $C \sim E$ cannot occur.

Next, let M be of type B. By virtue of the table of Takagi [15], we see that any principal curvature λ is different from the corresponding principal curvature μ and they satisfy

$$\lambda + \mu = -\frac{4}{\alpha}, \quad \lambda\mu = -1,$$

where we have $\lambda = cot\ (r - \pi/4)$, $\mu = -tan\ (r - \pi/4)$ and $\alpha = 2cot\ 2r$, $0 < r < \pi/4$. Since M is supposed to be of type B, we have

$$h = \alpha + (n-1)(\lambda + \mu),$$

from which, together with the fact that $h = \lambda + \mu$, $\lambda \neq \mu$, it follows

$$\alpha + (n-2)h = 0, \quad \alpha^2 = 4cot^2 2r = 4(n-2).$$

Hence M is a tube of the radius r over a complex quadric Q_{n-1}, where the radius r satisfies $cot^2 2r = n - 2$.

It is trivial that the real hypersurface M of type A satisfies (4.2) and hence it satisfies the assumption (**). It completes the proof. $\square$

5. Proof of Theorem 2

In this section we prove Theorem 2 which is another characterization of real hypersurfaces of type A concerning the Lie derivative and the covariant derivative with respect to the structure vector ξ.

Let M be a real hypersurface in a complex hyperbolic space H_nC, $n \geqq 3$. Assume that its Ricci tensor S satisfies

$$S\phi A - \phi AS = 0. \tag{**}$$

From the definition of S it follows that the condition (**) is equivalent to

$$h(A\phi - \phi A)A + (\phi A - A\phi)A^2 + A(\phi A - A\phi)A + \frac{3}{4}c\phi A\xi \otimes \eta = 0. \tag{5.1}$$

Then let us suppose that the structure vector ξ is principal with constant principal curvature α. Let X be a principal vector in T_0 with principal curvature λ. Then ϕX is also the principal vector in T_0 with princopal curvature μ. If $\lambda \neq \alpha/2$, then $\mu = (\alpha\lambda + c)/(4\lambda - 2\alpha)$. We can consider the following two cases ;

$$\text{I.} \quad \alpha^2 + c \neq 0, \qquad \text{II.} \quad \alpha^2 + c = 0.$$

Now let us consider the first case.

The Case I. Let X be a unit principal vector in T_0 with principal curvature λ. Then we see that ϕA is also the unit principal vector in T_0 with principal curvature μ such that $\mu = (\alpha\lambda + c)/(4\lambda - 2\alpha)$. By (3.14) in Theorem 3.3, the principal curvatures satisfy

$$(4\lambda^2 - 4\alpha\lambda - c)(4\lambda^2 - 4h\lambda + 2\alpha h + c) = 0, \tag{5.2}$$

where α and h are constant. This means that M has at most five distinct constant principal curvatures. Thus, according to the theorem due to Berndt [1] M is homogeneous. Then, taking account of the classification theorem, we obtain the fact that M is locally congruent to one of the homogeneous real hypersurfaces of type A_0, A_1, A_2 and B. Thus we may check whether or not these four model spaces satisfy the condition (5.1) one by one. In this case we assume $\alpha^2 + c \neq 0$. So it is enough to check (5.1) for the type A_1, A_2 and B.

First, let M be of type B. In the sequel we suppose that $c = -4$. Then, for the table of Berndt [1], we get

$$\alpha = 2 \, tanh \, 2r, \quad \lambda = tanh \, r, \quad \mu = coth \, r,$$

which satisfy

$$\lambda \neq \mu, \quad \lambda + \mu = \frac{4}{\alpha}, \quad \lambda\mu = 1.$$

Accordingly we have $h = \alpha + (n-1)(\lambda + \mu)$. Combining this with (5.2) we see $\alpha = -(n-2)(\lambda + \mu)$, namely we have $\alpha^2 = -4(n-2)$, a contradiction. Thus the real hypersurface of type B cannot occour.

Next, let M be of type A. Then it is easily seen that it satisfies the condition (5.1) by Theorem B due to Montiel and Romero [12].

The Case II : $\alpha^2 = 4$. First we consider the subcase where $\alpha = 2$. Then by (2.4) we obtain the fact that if X in T_0 is a principal vector with principal curvatures λ, then the following equation

$$(4\lambda - 2\alpha)A\phi X = (2\alpha\lambda + c)\phi X,$$

and hence we have

$$(\lambda - 1)A\phi X = (\lambda - 1)\phi X.$$

Let M_1 be a subset in M consisting of points x at which $\lambda(x) \neq 1$. Suppose that M_1 is not empty. On M_1, ϕX is a vector with principal curvature 1. Since the structure vector is principal, it implies that ϕX is a principal vector with principal curvature μ such that $\lambda \neq \mu$. Accordingly by (5.1) we have $\lambda + \mu = h$.

On the other hand, the trace h of the shape operator A is given by $h = \alpha + p\lambda + q\mu$, where $p + q = 2n - 2$, from which together with the last equation it follows that

$$(p-1)\lambda + (q+1) = 0$$

on M_1. Thus the principal curvature λ is a constant different from 1 on M_1. By the definition the principal curvature λ is equal to 1 on the subset $M - M_1$. By the continuity of the principal curvatures the subset $M - M_1$ must be empty. Namely, the subset M_1 coincides with the whole M and every principal curvatures are constant on M. Then by a Theorem of Berndt [1] M is locally congruent to a horosphere. Thus its principal curvatures are given by $\alpha = 2, \lambda = 1$ with its multiplicities $1, 2(n-1)$ respectively. This makes a contradiction. So the subset M_1 should be empty. Then we see that the principal curvature λ satisfies $\lambda = 1$ on M. This shows that M is of type A_0.

Conversely, let M be a real hypersurface of type A_0 in $M_n(c)$, $c < 0$. Then M has two distinct principal curvatures $\alpha = 2$ and $\lambda = 1$. So it satisfies the condition (5.1).

Finally let us consider the case $\alpha = -2$. Then in this case the global unit normal vector field C on M can be oriented in such a way that α is positive, because α is the principal curvature corresponding to the principal direction $\xi = -JC$ (See Berndt [1]). Thus by the same method to the above argument we find that M is of type A_0. It completes the proof.

References

1. J. Berndt, *Real hypersurfaces with constant principal curvature in a complex hyperbolic space*, J. Reine Angew Math. **395** (1989), 132-141.
2. J. Berndt, *Real hypersurfaces with constant principal curvatures in complex space forms*, Geometry and Topology of Submanifolds II, Avignon, 1988, World Scientific, (1990), 10-19.
3. T. E. Cecil and P. J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481-498.
4. U-H. Ki, S. Maeda and Y. J. Suh, *Lie derivatives on homogeneous real hypersurfaces of type B in a complex projective space*, Yokohama Math. J. **42** (1994), 121-131.
5. U-H. Ki and Y. J. Suh, *On real hypersurfaces of a complex space form*, Math. J. Okayama Univ. **32** (1990), 207-221.
6. U-H. Ki and Y. J. Suh, *On a characterization of type A in a complex space form*, Canadian Math. Bull. **37** (1994), 238-244.
7. M. Kimura, *Real hypersurfaces and complex submanifolds in a complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137-149.
8. M. Kimura and S. Maeda, *Lie derivatives on real hypersurfaces in a complex projective space*, Checho. Math. J. **120** (1995), 135–148.
9. J.-H. Kwon and H. Nakagawa, *A note on real hypersurfaces of a complex projective space*, J. Austral. Math. Soc.(A). **47** (1989), 108-113.

10. J.-H. Kwon and Y. J. Suh, *Lie derivatives on homogeneous real hypersurfaces of type A in complex space forms*, Bull. Korean Math. Soc. **34** (1997), 459–468.
11. Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28** (1976), 529-540.
12. S. Montiel and A. Romero, *On some real hypersurfaces of a complex hyperbolic space*, Geom. Dedicata **20** (1986), 245-261.
13. M. Okumura, *Real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355-364.
14. R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495-506.
15. R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures*, J. Math. Soc. Japan **27** (1975), 43–53.

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