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ON A COMPOSITION OPERATOR AND HARDY SPACE

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ABSTRACT. Characterizing a geometric property of the self map that induces a bounded composition operator on Blochs to a Hardy-Sobolev space, we give a way of constructing examples of Bloch functions f whose derivative is in H^p for all p : $0 < p < 1$ but $f \notin BMOA$. The hyperbolic version of such an example is also given.

1. Introduction.

Let $U = \{z : |z| < 1\}$ be the open unit disc of the complex plane and let T be the boundary of U identified with $[-\pi, \pi]$. Let $\sigma(z)$ denotes the hyperbolic distance of z and 0 in U:

$$\sigma(z) = \frac{1}{2}\, log\, \frac{1+|z|}{1-|z|}.$$

For $0 < p \leq \infty$ and for f subharmonic in U, we set

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f),$$

where

$$M_p(r,f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} \qquad \text{if } p < \infty$$

and

$$M_\infty(r,f) = \sup_\theta |f(re^{i\theta})|.$$

If $f(z)$ is subharmonic in U, then it has a harmonic majorant if and only if $\|f\|_1 < \infty$.

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The Hardy space H^p is the class of those functions f holomorphic in U for which $\|f\|_p < \infty$. The Yamashita hyperbolic Hardy class H^p_σ is the class of those holomorphic self maps ϕ of U for which $\|\sigma(\phi)\|_p < \infty$. Though H^p_σ is not a linear space, it has, as hyperbolic counterparts, many properties analogous to those of H^p. See [D] and [G] for H^p spaces, and [Y1], [Y3] and [Kw4] for H^p_σ spaces.

$BMOA$, analytic functions of bounded mean oscillation, consists of those $f \in H^1$ for which

$$|f'(z)|^2(1-|z|^2)dxdy$$

is a Carleson measure, that is to say,

$$\int\int_{S_{\delta,\theta}} |f'(z)|^2(1-|z|^2)dxdy = O(\delta),$$

where $0 < \delta \leq 1$ and

$$S_{\delta,\theta} = \{re^{it} : |\theta - t| \leq \delta, 1-\delta \leq r < 1\}, \quad \theta \in T.$$

$BMOA_\sigma$, the Yamashita $BMOA$ class, consists of those holomorphic self maps ϕ of U for which

$$(\phi^{\#}(z))^2(1-|z|^2)\,dxdy$$

is a Carleson measure on U, where $\phi^{\#}$ is the hyperbolic derivative :

$$\phi^{\#}(z) = \frac{|\phi'|}{1-|\phi|^2}(z), \quad z \in U.$$

See [G] for $BMOA$ and [Y2] for $BMOA_\sigma$.

The Bloch space $\mathcal{B}$ consists of those f holomorphic in U for which

$$\|f\|_{\mathcal{B}} = \sup_{z\in U} |f'(z)|(1-|z|^2) < \infty.$$

While, by Schwarz-Pick's lemma stating that

$$\phi^{\#}(z)(1-|z|^2) \leq 1, \quad z \in U,$$

every holomorphic self map of U is hyperbolically Bloch. To keep up the parallism between H^p and H^p_σ more closely, we introduce a weighted subspace of Hardy-Sobolev space $H^{p,1}_1$, which is defined to consist of those f for which $f' \in H^p$ and $f'(z) = O(1-|z|)^{-1}$. See [KKw] and [Kw3] for the weighted subspaces of Hardy spaces.

We consider the sharpness of the following two parallel results.

Theorem A ([K] or [KKw]). *If $f \in H_1^{p,1}$ for some p: $0 < p < 1$, then $f \in H^q$ for all q: $0 < q < \infty$.*

Theorem B ([Kw2]). *If ϕ is a holomorphic self map of U and $(\phi^{\#})^p$ admits a harmonic majorant in U for some $p : 0 < p < 1$, then $\phi \in H_\sigma^q$ for all $q : 0 < q < \infty$.*

The example $f(z) = log(1-z)$ shows that we can improve the conclusion of Theorem A neither up to $f \in H^\infty$ nor up to the Dirichlet finite functions. In a parallel direction, the bound on q in Theorem B is sharp in the sense that there is a holomorphic self map ϕ such that $(\phi^{\#})^p$ admits harmonic majorants in U for arbitrary p less than 1, but neither $\phi \in H_\sigma^\infty$ nor ϕ a function of hyperbolically Dirichlet finite, that is,

$$\int\int_U (\phi^{\#})^2(z)dxdy = \infty.$$

Next step to the sharpness problem may be concerned with $BMOA$ and $BMOA_\sigma$ respectively. We prove

Theorem 1. *There is an f such that $f \in H_1^{p,1}$ for all $p : 0 < p < 1$ but $f \notin BMOA$.*

Theorem 2. *There exists a holomorphic self map ϕ such that $(\phi^{\#})^p$ has harmonic majorants for all $p : 0 < p < 1$ but $\phi \notin BMOA_\sigma$.*

Theorem 1 is sharp because if $f' \in H^1$ then, by a well known result of Privalov(see [D, pp 42-52]), f is continuous on $\bar{U}$ and absolutely continuous on T so that $f \in H^\infty \subset BMOA$. Theorem 2 is sharp by a parallel reason (see [Y1, Theorem 1]).

It seems that there are other ways of showing the existence of functions requested in Theorem 1. Our point is that we can transfer, by use of a composition operator, the problem to a parallel problem of holomorphic self maps on U with hyperbolic geometry. We show that Theorem 1 is a consequence of Theorem 2. This will be done in Section 2 and Section 3. We prove Theorem 2 in Section 4 by giving an example which is connected with the order of contact. See [S] for the concept on contact.

2. A composition operator on the Bloch space.

The following represents one of the relationships between H^p and H_σ^p via composition operators.

Theorem 3. *Let ϕ be a holomorphic selfmap of U and $0 < p < \infty$. Then the following (1) and (2) are equivalent.*

(1) $(g \circ \phi)' \in H^p$ *for all* $g \in \mathcal{B}$.

(2) $(\phi^{\#})^p$ *has a harmonic majorant.*

Proof. Suppose that $(\phi^{\#})^p$ has a harmonic majorant. Then, for $g \in \mathcal{B}$, the lemma of Schwarz-Pick gives

$$\|(g \circ \phi)'\|_p^p \leq \|g\|_{\mathcal{B}} \, \|\phi^{\#}\|_p^p < \infty,$$

so that $(g \circ \phi)' \in H^p$.

Conversely, suppose that $(g \circ \phi)' \in H^p$ for all $g \in \mathcal{B}$. By [RU, Proposition 5.4], there is $\{g_j\}_{j=1,2} \subset \mathcal{B}$ such that

$$\sum_{j=1}^{2} |g_j'(z)| \geq \frac{1}{(1 - |z|^2)}, \quad z \in U.$$

Hence

$$M_p(r, \phi^{\#}) \leq C(p) \sum_{j=1}^{2} M_p(r, (g_j \circ \phi)') < \infty$$

for some constant $C(p)$ depending on p. Therefore $(\phi^{\#})^p$ has a harmonic majorant. $\square$

On the same vein we need is the following

Theorem C ([RU]). *Let* ϕ *be a holomorphic self map of* U. *Then the following (1) and (2) are equivalent.*

(1) $g \circ \phi \in BMOA$ *for all* $g \in \mathcal{B}$.

(2) $\phi \in BMOA_\sigma$.

See [Kw1] for a similar results on H^p. Though the proof is simple, Theorem 3 has many applications. We see one of them in the next section.

3. Theorem 2 implies Theorem 1.

Assuming Theorem 2, we can prove Theorem 1 by help of Theorem 3.

Proof (Theorem 2 $\Longrightarrow$ Theorem 1). Suppose that ϕ is a self map of U such that $(\phi^{\#})^p$ has a harmonic majorant for all $p : 0 < p < 1$ and $\phi \notin BMOA_\sigma$. Then by

Theorem 3, $(g \circ \phi)' \in H^p$ for all $p : 0 < p < 1$ and for all $g \in \mathcal{B}$. Also by Theorem C,

$$g \circ \phi \notin BMOA \tag{3.1}$$

for some $g \in \mathcal{B}$. Now $f = g \circ \phi$ with g satisfying (3.1) is a required function of Theorem 1. $\square$

We are left, therefore, to prove Theorem 2 in the next section.

4. Proof of Theorem 2 by an example.

Recall, for a holomorphic self map ϕ, that $(\phi^{\#})^p$ has a harmonic majorant if and only if

$$\sup_r \int_T \left(\frac{|\phi'(re^{i\theta})|}{1 - |\phi(re^{i\theta})|^2} \right)^p d\theta < \infty, \tag{4.1}$$

and that $\phi \in BMOA_\sigma$ if and only if

$$\int\int_{S_{\delta,\theta}} \left(\frac{|\phi'(z)|}{1 - |\phi(z)|^2} \right)^2 (1 - |z|^2)\, dxdy = O(\delta) \tag{4.2}$$

for all $\theta \in T$.

Properties (4.1) and (4.2) are connected with the order of contact and the angular derivative. See [S], for example, for these two concepts. For a theoretical background, we invoke a theorem of Tsuji-Warschawski(see [T, p 366] or [S, p 72]) stating a necessary and sufficient contact condition for a univalent map to have an angular derivative. Without regarding the growth of the derivative, a holomorphic self map ϕ need to have worse order of contacts to satisfy (4.1) but need to have a smooth contact to fail (4.2). We prove Theorem 2 by giving the following example.

Example.

There is a holomorphic self map ϕ of U such that

(1) ϕ does not satisfy (4.2) for $\theta = 0$.

(2) ϕ is univalent and has angular derivative nowhere on T.

(3) ϕ satisfies (4.1) for all $p : 0 < p < 1$.

We employ a function ϕ whose form was introduced in [ST]. Let

$$h(z) = \frac{i}{k} \frac{\sqrt{\frac{1+iz}{1-iz}i} - i}{\sqrt{\frac{1+iz}{1-iz}i} + i}, \quad z \in U,$$

where the branch of square root is taken with negative imaginary axis omitted, and k is a large positive constant, say $k > 9$. Then h maps U conformally onto

$$\Delta = \{w : |w| < \frac{1}{k}, Re(w) > 0\}.$$

Since h is conformal and extensible analytically across T in a neighborhood of $z = 1$ with $h(1) = 0$, $h'(1) = -\frac{1}{4k}$, h maps the (Polar) Carleson square $S_\delta = S_{\delta,0}$ onto a roughly rectangular (Carleson) square $R_\delta = h(S_\delta)$ of the right half plane around 0 if δ is sufficiently small. Hence we can take δ_0 small enough such that $|h'(z)| > \frac{1}{8k}$ for all $z \in S_{\delta_0}$ and

$$Re(w) < \frac{1}{k}(1 - |h^{-1}(w)|), \quad w \in R_\delta$$

along with

$$\left\{w : |w| < \frac{\delta}{8k}, Re(w) > 0\right\} \subset R_\delta$$

for all $\delta : \delta < \delta_0$.

Now, let

$$f(z) = z\, log\, \frac{1}{z}, \quad z \in \Delta$$

with the principal branch of $log\, z$ and let

$$\phi(z) = 1 - f \circ h(z), \quad z \in U.$$

Then it follows from the conformality of h with an easy calculation that, for all

$\delta < \delta_0$,

$$
\begin{aligned}
&\int\int_{S_\delta} \left(\frac{|\phi'(z)|}{1-|\phi(z)|^2}\right)^2 (1-|z|^2)\, dxdy \\
\gtrsim &\int\int_{R_\delta} \left(\frac{|f'(w)|}{1-|1-f(w)|^2}\right)^2 u\, dudv \\
= &\int\int_{R_\delta} \frac{u\,|1+\log w|^2}{(1-|1+w\log w|^2)^2}\, dudv \\
\gtrsim &\int_0^{\delta/8k}\int_0^{\pi/2} \frac{(\log r)^2\cos\theta}{(\theta\sin\theta)^2+(\log r\cos\theta)^2}\, d\theta dr \\
\gtrsim &\int_0^{\delta/8k}\int_0^{\pi/2} \frac{\sec\theta\, d\theta dr}{1+\left(\frac{\theta\tan\theta}{\log r}\right)^2},
\end{aligned}
$$

whence

$$
\begin{aligned}
&\limsup_{\delta\to 0}\frac{1}{\delta}\int\int_{S_\delta} \left(\frac{|\phi'(z)|}{1-|\phi(z)|^2}\right)^2 (1-|z|^2)\, dxdy \\
\gtrsim &\lim_{r\to 0}\int_0^{\pi/2} \frac{\sec\theta\, d\theta}{1+\left(\frac{\theta\tan\theta}{\log r}\right)^2} \\
= &\infty.
\end{aligned}
$$

This proves (1). Here $A \gtrsim B$ means that there is a positive constant C such that $CA \geq B$.

It is not difficult to see that f' has positive real part on $\bar{\Delta}\setminus 0$, so that ϕ is univalent. Concerning the order of contact, from

$$h(e^{i\theta}) = i\left\{-\frac{\theta}{4k} + O\left(|\theta|^2\right)\right\}, \tag{4.3}$$

we see that

$$
\begin{aligned}
&1-|\phi(e^{i\theta})|^2 \\
&=1-|1-f\circ h(e^{i\theta})|^2 \\
&=\frac{\pi}{4k}|\theta| + O\left(\theta^2\log|\theta|\right)
\end{aligned}
\tag{4.4}
$$

in a neighborhood of $\theta = 0$. By (4.4),

$$\int_0^\epsilon \frac{1-|\phi(e^{i\theta})|}{\theta^2}\, d\theta = \infty$$

for $\epsilon > 0$, so that ϕ can not have an angular derivative at $z = 1$ by Tsuji-Warchauski's Theorem [T]. Hence ϕ has no angular derivatives on T. This with univalency proves (2).

By (4.3),

$$|\phi'(e^{i\theta})| = |1 + log\, h(e^{i\theta})|\,|h'(e^{i\theta})| = O(|1 + log\, h(e^{i\theta})|) = O\,(log\,|\theta|)$$

for θ near 0. Thus, by (4.4), the function

$$\frac{|\phi'(e^{i\theta})|}{1 - |\phi(e^{i\theta})|^2}$$

is p-integrable with respect to θ on a small neighborhood of $\theta = 0$ for all p less than 1. Now since $\overline{\phi(U)} \subset U \cup \{1\}$, to prove (3) it is enough to check that the radial limit function $\lim_{r \to 1} |\phi'(re^{i\theta})|$ is p-integrable with respect to θ for all p less than 1 on any compact subset of T that does not contain $\theta = 0$. Since

$$|\phi'(e^{i\theta})| = |f \circ h(e^{i\theta})\, h'(e^{i\theta})| = O(|h'(e^{i\theta})|)$$

on such a set, we are enough to check the behavior of h'. Note that $h'(re^{i\theta})$ fails to have finite radial limits only at $\theta = \pm\frac{\pi}{2}$, and in a small neighborhood of those points

$$|h'(z)| = O\left(|1 \pm iz|^{-1/2}\right).$$

Hence $|h'(e^{i\theta})|^p$ is integrable for all p less than 1 with respect to θ on the compact subset. Therefore we have (3). □

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