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ON THETA PAIRS OF MAXIMAL SUBGROUPS OF SOLVABLE GROUPS

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ABSTRACT For a maximal subgroup M of a finite group G, a θ-pair is defined as a pair of subgroups (C, D) of G such that $D \trianglelefteq G$, $D \subsetneq C$, $\langle M, C\rangle = G$, $\langle M, D\rangle = M$ and C/D has no proper normal subgroup of G/D. We obtain several results on the maximal θ-pairs, which characterize solvable groups.

1. INTRODUCTION

Many authors have found interest to investigate how various conditions given on maximal subgroups of a finite group determine the structure of the group. In [5], we have introduced a characteristic subgroup $S_{\mathcal{D}}(G)$, which is a generalization of the Frattini subgroup $\phi(G)$ of G, and studied its influence on solvable group. We also introduced another characteristic subgroup $B_p(G)$ in [6]. In [9], Mukherjee and Bhattacharya have introduced the concept of maximal θ-pairs for a maximal subgroup of a finite group and studied their effects on solvable, supersolvable and nilpotent groups. In this paper, our aim is to find out some more conditions on maximal θ-pairs for a maximal subgroup M of a finite group G, which characterize the solvability of the group. All groups considered here are finite and we use standard notation as in [7]. In addition, the notation $M \lessdot G$ is sometimes used to denote that M is a maximal subgroup of G.

2. PRELIMINARIES

DEFINITION For a maximal subgroup M of a group G, let $\theta(M) = \{(C,D) : C \leqslant G,\ D \triangleleft G,\ D \subsetneq C,\ \langle M,C\rangle = G,\ \langle M,D\rangle = M$ and C/D contains properly no non-trivial normal subgroup of $G/D\}$. This family of subgroups introduced by Mukherjee and Bhattacharya [9] is motivated by the interesting concept of the Index complex defined in Deskins [3-4].

Any pair (C,D) in $\theta(M)$ is called a θ-pair. A partial order relation $\leq$ is defined on $\theta(M)$ as follows :

$(C,D) \leq (C',D')$ if $C \subseteq C'$; no condition is placed on the second component of the pairs (From the definition of $\theta(M)$, it follows that $D \subset D'$. Also $C = C'$ implies $D = D'$).

Any maximal element in $\theta(M)$ with respect to this ordering is called a maximal θ-pair. If a maximal θ-pair (C,D) is such that $C/D \trianglelefteq G/D$ then we call it a normal maximal θ-pair. The index of a θ-pair (C,D) is defined to be $[C : D]$.

(2.1) [9, Lemma 2.1] If (C,D) is a maximal θ-pair in $\theta(M)$ and $N \triangleleft G$, $N \subset D$ then $(C/N,\ D/N)$ is a maximal θ-pair in $\theta(M/N)$. Conversely if $(C/N,\ D/N)$ is a maximal θ-pair in $\theta(M/N)$ then (C,D) is a maximal θ-pair in $\theta(M)$.

(2.2) [1, Lemma 3] If a group G possesses a maximal subgroup with trivial core then the following properties of G are equivalent.

(i) The indices in G of all the maximal subgroups with trivial core are powers of a prime p.

(ii) There exists a unique minimal normal subgroup of G, and there exists a common prime divisor of all the indices in G of all the maximal subgroups with trivial core.

(iii) There exists a non-trivial solvable normal subgroup of G.

(2.3) If a group G has an abelian maximal subgroup then G is solvable.

This result follows directly from a result of Huppert [8, Satz 2]

<u>DEFINITION</u> Let H and K be two normal subgroups of a group G with $K \subset H$. Then the factor group H/K is called a chief factor of G if there is no normal subgroup N of G such that $K \subset N \subset H$ with proper inclusion. Let M be a maximal subgroup of G. H is said to be a normal supplement of M in G if $MH = G$. The normal index of M in G is defined as the order of a chief factor H/K, where H is minimal in the set of all normal supplements of M in G, and is denoted by $\eta(G : M)$.

It was proved that $\eta(G : M)$ is uniquely determined by M [3, 2.1] (or [2, Lemma 1]).

If (C, D) is a normal θ-pair in $\theta(M)$ then $\eta(G : M) = [C : D]$.

It follows from the definition of normal index that $[G : M]$ divides $\eta(G : M)$. But under some conditions, $[G : M] = \eta(G : M)$. For example, if $[G : M]$ is a square-free integer then $\eta(G : M) = [G : M]$ [10, Lemma 3.1].

<u>DEFINITION</u> Let G be any group and p be any prime. Define two characteristic subgroups of G as follows :

$$B_p(G) = \bigcap \{M : M \in \beta_p(G)\}$$

$$\phi_p(G) = \bigcap \{M : M \in \Upsilon_p(G)\}$$

where $\beta_p(G) = \{M \lessdot G : [G : M]_p = 1 \text{ and } \eta(G : M) \text{ is composite}\}$

and $\Upsilon_p(G) = \{M \lessdot G : [G : M]_p = 1\}$.

In case $\beta_p(G)$ is empty then we define $B_p(G) = G$ and the same thing is done for the subgroup $\phi_p(G)$. Note that $\phi_p(G) \subseteq B_p(G)$.

(2.4) [6, Theorem 3.6] $B_p(G)$ is solvable and so $\phi_p(G)$ is solvable.

3. SOLVABILITY CONDITIONS.

THEOREM 3.1 For a group G, each of the following conditions implies the solvability of G :

(a) $[C : D] = [G : M]$ for each maximal θ-pair (C, D) in $\theta(M)$ and any maximal subgroup M in $\beta_p(G)$.

(b) $[C : D]_2 = [G : M]_2$ for each maximal θ-pair (C, D) in $\theta(M)$ and any maximal subgroup M of G.

(c) G is p-solvable and $[C : D]_2 = [G : M]_2$ for each maximal θ-pair (C, D) in $\theta(M)$ and any maximal subgroup M in $\beta_p(G)$.

Proof (a) Let G be a group satisfying the condition stated in (a). We shall show that G is solvable. We may assume that $\beta_p(G)$ is non-empty. For otherwise, $G = B_p(G)$ and so G is solvable by (2.4). If G is simple, $(G : 1)$ is a maximal pair in $\theta(M)$ for any maximal subgroup M. Then we have $|G| = [G : M]$ by the hypothesis, and therefore any maximal subgroup of G is trivial. This implies that G is a cyclic group of prime order, sothat it is solvable. Thus we may assume that G is not simple. Let N be a minimal normal subgroup of G. By induction, G/N is solvable. If N_1 and N_2 are two distinct minimal normal subgroups of G, we have $N_1 \cap N_2 = 1$ and so $G = G/N_1 \cap N_2$ is isomorphic to a subgroup of $(G/N_1) \times (G/N_2)$. Since G/N_i $(i = 1, 2)$ are solvable by the argument above, G is also solvable. Thus we may assume that there is a unique minimal normal subgroup N of G. Since $B_p(G)$ is solvable by (2.4), we may assume that $B_p(G) \neq G$. If $N \subseteq B_p(G)$ then N is solvable and hence G is solvable. If $N \not\subseteq B_p(G)$ then there exists M_o in $\beta_p(G)$ such that $N \not\subseteq M_o$. So $G = M_oN$ and $\mathrm{core}_G(M_o) = \langle 1 \rangle$, since N is the unique minimal normal subgroup of G. Thus G possesses a core-free maximal subgroup. Let M be any maximal subgroup of G with trivial core. Then $N \not\subseteq M$ and so $G = MN$. It can be verified that $(N, \langle 1 \rangle)$ is a θ-pair in $\theta(M)$. If $(N, \langle 1 \rangle)$ is not a maximal θ-pair, then $(N, \langle 1 \rangle) < (C, D)$ for some pair (C, D) in $\theta(M)$. Since M is core-free and $\langle M, D \rangle = M$ it follows that $D = \langle 1 \rangle$. But then $C/\langle 1 \rangle$ has no proper normal subgroup of $G/\langle 1 \rangle$, which is impossible, since $N \subset C$. Thus $(N, \langle 1 \rangle)$ is a maximal θ-pair in $\theta(M)$. Similarly it can be verified as above that $(N, \langle 1 \rangle)$ is a maximal θ-pair in $\theta(M_o)$. By hypothesis $[G : M_o] = |N|$ and so $|N|_p = 1$. Since $\eta(G : M_o) = |N|$, $|N|$ is composite. Also the relation $\eta(G : M) = |N|$ implies that $\eta(G : M)_p = 1$ and $\eta(G : M)$ is composite. Since $[G : M]$ divides $\eta(G : M)$, $[G : M]_p = 1$. Hence M belongs to $\beta_p(G)$. By hypothesis $[G : M] = |N|$. This implies that there exists a common prime divisor of all the indices in G of all the maximal subgroups with trivial core. So by (2.2), N is solvable and hence G is solvable.

This completes the proof of (a).

The proofs of (b) and (c) are analogous to that of (a) and so we omit them.

<u>THEOREM 3.2</u> A group G is solvable if it has a solvable maximal subgroup M such that the index of each maximal pair (C, D) in $\theta(M)$ is equal to $[G : M]$.

<u>Proof</u> If possible, let G be a counter example of minimal order. By the arguments in the proof of Theorem 3.1, we may assume that G is not simple. Assume that $H = \mathrm{core}_G(M) \neq \langle 1 \rangle$. Then by (2.1), we see that G/H satisfies the hypothesis of the theorem. By minimality of G, G/H is solvable. Also, since $H \subsetneq M$, it follows that H is solvable and hence G is solvable, a contradiction. Thus M is core-free. Let N be a minimal normal subgroup of G. Then $N \not\subset M$ and so $G = MN$. Since $(N, \langle 1 \rangle)$ is a maximal pair in $\theta(M)$ (see the argument in the proof of Theorem 3.1), we have $[G : M] = |N|$ by hypothesis and so $M \cap N = \langle 1 \rangle$. Now M is not simple. For, otherwise M is commutative and so by (2.3), G is solvable, a contradiction. Let L be a minimal normal subgroup of the solvable group M. Then L is an elementary abelian p-group for some prime p. Let $A = C_N(L) = \{x \in N : y^{-1}xy = x, \forall y \in L\}$. Then A is an M-invariant subgroup of N and so $M \subseteq N_G(A) \subseteq G$. This implies that either $M = N_G(A)$ or $N_G(A) = G$. If $M = N_G(A)$ then $A \subseteq M \cap N$ and so $A = \langle 1 \rangle$. This implies that $M = G$, a contradiction.

If $N_G(A) = G$ then $A \triangleleft G$ and so either $A = \langle 1 \rangle$ or $A = N$. But $A = N$ implies that $L \triangleleft G$ and consequently $\text{core}_G(M) \neq \langle 1 \rangle$, a contradiction. Hence $A = C_N(L) = \langle 1 \rangle$. We claim that $(|L|, |N|)=1$. If not, there is a prime p dividing $|N|$. Let P be a Sylow p-subgroup of LN containing L. Then $P \cap N$ is a non-trivial normal subgroup of the nilpotent group P and consequently $Z(P) \cap N \neq \langle 1 \rangle$. Now $Z(P) \cap N \subseteq C_N(L)$ and so $Z(P) \cap N = \langle 1 \rangle$, a contradiction. Hence $(|L|, |N|) = 1$. Since $C_N(L) = \langle 1 \rangle$, it follows from Theorem 2.2 [7] that for each prime q dividing $|N|$, there exists a unique L-invariant Sylow q-subgroup Q of N. Then for any $g \in M$ $g^{-1}Qg = Q$ and thus Q is an M-invariant q-subgroup of N. Since $M \lessdot G$, it can be verified as above that the only M-invariant subgroups of N are N and $\langle 1 \rangle$. and consequently $Q = N$. This implies that N is solvable and hence G is solvable, a contradiction. This completes the proof.

<u>**THEOREM 3.3**</u> For a group G, the following conditions are equivalent to the solvability of G :

(a) G has a solvable maximal subgroup M such that for each maximal pair (C, D) in $\theta(M)$, C/D is solvable.

(b) C/D is solvable for any maximal θ-pair (C, D) in $\theta(M)$ and any M in $\beta_p(G)$.

<u>**Proof (a)**</u> If G is simple, (G, 1) is a maximal pair in $\theta(M)$ and then $G = G/\langle 1 \rangle$ is solvable by the hypothesis. Thus we may assume that G is not simple. Assume that $H = \text{core}_G(M) \neq \langle 1 \rangle$. By induction, G/H is solvable. As $H \subseteq M$, H is solvable and hence G is solvable. Thus M is core-free. Let N be a minimal normal subgroup of G. Then $N \not\subseteq M$ and so $G = MN$. By the arguments in

the proof of Theorem 3.1, we obtain that $(N, \langle 1 \rangle)$ is a maximal θ-pair in $\theta(M)$. So by hypothesis, N is solvable. Also since $G/N \cong M/M \cap N$, it follows that G/N is solvable. Hence G is solvable.

The converse is obvious.

The proof of (b) is similar to that of (a).

<u>THEOREM 3.4</u> For a group G, the following conditions are equivalent to the solvability of G :

(a) For each M in $\beta_p(G)$, there exists a normal maximal pair (C, D) in $\theta(M)$ such that C/D is solvable.

(b) G has a solvable maximal subgroup M such that there exists a normal maximal pair (C, D) in $\theta(M)$ with C/D solvable.

(c) For each M in $\beta_p(G)$, there exists a maximal pair (C, D) in $\theta(M)$ such that C/D is abelian.

(d) G has a solvable maximal subgroup M such that there exists a maximal pair (C, D) in $\theta(M)$ with C/D abelian.

<u>Proof (a)</u> Assume that G satisfies the condition stated in (a). We have to show that G is solvable. As in the proof of Theorem 3.1 and Theorem 3.3, we may assume that $\beta_p(G)$ is non-empty and G is not simple. Let N be a minimal normal subgroup of G.

We now show that G/N is solvable. We may suppose that $\beta_p(G/N)$ is non-empty by (2.4). Let M/N be a maximal subgroup in $\beta_p(G/N)$. Then M belongs to $\beta_p(G)$. By hypothesis, there exists a normal maximal pair (C, D) in $\theta(M)$ such that C/D is solvable. If $N \subseteq D$ then (C/N, D/N) is a normal maximal pair in $\theta(M/N)$ and $C/N \,/\, D/N$ is solvable. Thus G/N is solvable by the hypothesis of induction.

If $N \not\subset D$ then we claim that $N \not\subset C$. For if $N \subseteq C$ then $ND \subseteq C$ and so either $ND = C$ or $ND \subsetneq C$. If $ND = C$ then $C \subseteq M$ and consequently $G = \langle M, C \rangle = M$, a contradiction. If $ND \subsetneq C$ then ND/D is a proper non-trivial normal subgroup of G/D in C/D, which contradicts the definition of the θ-pair (C, D). Now since C/D is solvable, CN/DN is also solvable. Let K be a maximal proper normal subgroup of G contained in $CN \cap M$ and containing DN. We now claim that CN/K is not a minimal normal subgroup of G/K. For if CN/K is a minimal normal subgroup of G/K then (CN, K) belongs to $\theta(M)$ and $(C, D) \leqslant (CN, K)$ and hence $C = CN$ by the maximality of (C, D), a contradiction.

Let H/K be a minimal normal subgroup of G/K such that $H/K \subset CN/K$. Then from the choice of K, we obtain that $H \not\subset M$ and so $G = MH$. Therefore (H, K) is a pair in $\theta(M)$. Also H/K is solvable. If (H, K) is a maximal pair in $\theta(M)$ then $(H/N, K/N)$ is a maximal pair in $\theta(M/N)$ and $H/N\big/K/N$ is solvable. Thus G/N is solvable by the hypothesis of induction.

If on the other hand, (H,K) is not a maximal pair in $\theta(M)$ then let $(H,K) < (H_1,K_1)$, where (H_1,K_1) is a maximal pair in $\theta(M)$ and consequently $H \subset H_1$. Since H_1/K_1 contains properly no non-trivial normal subgroup of G/K_1, K_1 is a maximal proper normal subgroup of G in H_1, that is contained in M and $H \not\subset K_1$. If $HK_1 \neq H_1$ then HK_1/K_1 is a proper normal subgroup in H_1/K_1, a contradiction. Hence $HK_1 = H_1$. If $K = K_1$ then $H_1 = HK = H$ and so (H, K) is a maximal pair in $\theta(M)$, a contradiction. So $K \subsetneq K_1$. Also H_1/K_1 is solvable. Thus $(H_1/N, K_1/N)$ is a maximal pair in $\theta(M/N)$ such that $H_1/N\big/K_1/N$ is solvable. By induction, G/N is solvable. As in the proof of Theorem 3.1, we may assume that there is a unique minimal normal subgroup N of G. If $N \subseteq B_p(G)$ then N is solvable by (2.4) and hence G is solvable. If $N \not\subseteq B_p(G)$ then there exists M in $\beta_p(G)$ such that $G = MN$ and $\mathrm{core}_G(M) = \langle 1 \rangle$, by the uniqueness of N (see the proof of Theorem 3.1). By hypothesis, there exists a normal maximal pair (C, D) in $\theta(M)$ such that C/D is solvable. Since $\mathrm{core}_G(M) = \langle 1 \rangle$, it follows that $D = \langle 1 \rangle$ and consequently C is solvable. Thus N is solvable, since $N \subseteq C$ by the uniqueness of the minimal normal subgroup N. So G is solvable. The converse holds trivially.

The proofs of (b), (c) and (d) are similar to the proof of (a) and so we omit them.

<u>THEOREM 3.5</u> For a group G, the following conditions are equivalent to the solvability of G :

(a) For any two distinct maximal subgroups M_1 and M_2 of G, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common maximal pair (C, D) it follows that C/D is solvable.

(b) G is p-solvable and for any two distinct maximal subgroups M_1, M_2 in $\beta_p(G)$, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common maximal pair (C, D), it follows that C/D is solvable.

<u>Proof (a)</u> We may assume that G is not simple (see the proof of Theorem 3.3). Let N be a minimal normal subgroup of G. By induction, G/N is solvable. As in the proof of Theorem 3.1, we may assume that there is a unique minimal normal subgroup N of G. If N is contained in the Frattini subgroup $\phi(G)$, then N is solvable by (2.4) and hence G is solvable. If $N \not\subseteq \phi(G)$ then there exists a maximal subgroup M_1 of G such that $G = M_1N$. Let q be a prime divisor of $[G : M_1]$. If $N \subseteq \phi_q(G)$ then N is solvable and hence G is solvable. If $N \not\subseteq \phi_q(G)$ then there exists a maximal

subgroup M_2 in $\Upsilon_q(G)$ such that $N \not\subset M_2$ and so $G = M_2N$. As in the proof of Theorem 3.1, we can show that $(N, \langle 1 \rangle)$ is a common maximal pair in $\theta(M_1)$ and $\theta(M_2)$. Since q divides $[G : M_1]$ but not $[G : M_2]$, M_1 and M_2 are distinct maximal subgroups of G. By hypothesis N is solvable and hence G is solvable.

The converse follows trivially.

The proof of (b) is similar to that of (a) and so we omit it.

<u>**THEOREM 3.6**</u> For a group G, the following conditions are equivalent to the solvability of G :

(a) $C_{G/D}(C/D) \neq \langle 1 \rangle$ for any normal maximal pair (C, D) in $\theta(M)$ and any M in $\beta_p(G)$.

(b) G has a solvable maximal subgroup M such that for each normal maximal pair (C, D) in $\theta(M)$, it follows that $C_{G/D}(C/D) \neq \langle 1 \rangle$.

(c) For any two distinct maximal subgroups M_1, M_2 of G, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common normal maximal pair (C, D), it follows that $C_{G/D}(C/D) \neq \langle 1 \rangle$.

(d) G is p-solvable and for any two distinct maximal subgroups M_1, M_2 in $\beta_p(G)$, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common normal maximal pair (C, D), it follows that $C_{G/D}(C/D) \neq \langle 1 \rangle$.

<u>**Proof (a)**</u> Since $B_p(G)$ is solvable by (2.4), we may assume that $\beta_p(G)$ is non-empty. If G is simple then $G = Z(G)$ and hence G is solvable. So we assume that G is not simple. Let N be a minimal normal subgroup of G. By induction G/N is solvable. We may assume that N is the unique minimal normal subgroup of G (see the proof of Theorem 3.1).

If $N \subseteq B_p(G)$ then N is solvable and hence G is solvable. If $N \not\subseteq B_p(G)$ then there exists M_o in $\beta_p(G)$ such that $N \not\subseteq M_o$ and so $G = M_oN$ and $\mathrm{core}_G(M_o) = \langle 1 \rangle$. Also $(N, \langle 1 \rangle)$ is a maximal pair in $\theta(M_o)$ (see the proof of Theorem 3.1). By hypothesis, $C_G(N) \neq \langle 1 \rangle$ and hence it follows that $N \subseteq C_G(N)$. Consequently N is abelian and so G is solvable.

The converse follows directly from Theorem 3.2(i) [9]

The proofs of (b), (c) and (d) are same as that of (a).

<u>THEOREM 3.7</u> For a group G, each of the following conditions implies the solvability of G :

(a) $B_p(G/D) \neq \langle 1 \rangle$ for each maximal pair (C, D) in $\theta(M)$ and every M in $\beta_p(G)$.

(b) G has a solvable maximal subgroup M such that for each maximal pair (C, D) in $\theta(M)$, $B_p(G/D) \neq \langle 1 \rangle$.

(c) For any two distinct maximal subgroups M_1 and M_2 of G, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common maximal pair (C, D) it follows that $B_p(G/D) \neq \langle 1 \rangle$.

(d) G is p-solvable and for any two distinct maximal subgroups M_1, M_2 in $\beta_p(G)$, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common maximal pair (C, D), it follows that $B_p(G/D) \neq \langle 1 \rangle$.

<u>Proof (a)</u> We may assume that $\beta_p(G)$ is non-empty (see the proof of Theorem 3.6). If G is simple then for any maximal subgroup M in $\beta_p(G)$, $(G, \langle 1 \rangle)$ is a maximal pair in $\theta(M)$ and so by hypothesis, $B_p(G) \neq \langle 1 \rangle$. Hence $G = B_p(G)$ and consequently G is solvable, by (2.4). So we assume that G is not simple. Let N be a minimal normal subgroup of G. By induction, G/N is solvable. We may assume that N is the unique minimal normal subgroup of G. If $N \subseteq B_p(G)$ then N is solvable and hence G is solvable.

If $N \not\subseteq B_p(G)$ then there exists a maximal subgroup M in $\beta_p(G)$ such that $N \not\subseteq M$ and so $G = MN$ and $core_G(M) = \langle 1 \rangle$. Also $(N, \langle 1 \rangle)$ is a maximal pair in $\theta(M)$ (see the proof of Theorem 3.1). By hypothesis $B_p(G) \neq \langle 1 \rangle$ and so $N \subseteq B_p(G)$. Hence N is solvable and so G is solvable.

We omit the proofs of (b), (c) and (d), because they are similar to the proof of (a).

<u>THEOREM 3.8</u> For a group G, each of the following conditions implies the solvability of G :

(a) All non-normal maximal subgroups having a common maximal θ-pair are conjugate in G.

(b) G is p-solvable and all non-normal maximal subgroups belonging to $\beta_p(G)$ having a common maximal θ-pair, are conjugate in G.

<u>Proof (a)</u> Suppose that the theorem is false and let G be a counter example of minimal order. If G is simple then since all maximal subgroups of G have a maximal θ-pair $(G, 1)$ in common, they are conjugate by the hypothesis. Threfore all maximal subgroups in G have the same indices. So by Theorem 4 [11], G is solvable, a contradiction. Therefore, we assume that G is not simple. Let N be a minimal normal subgroup of G. Then since G/N inherits the conjugacy property, so by using (2.1), we can show that G/N satisfy the hypothesis of the theorem. Hence by minimality of G, G/N is solvable. We assume that there is a unique minimal normal subgroup N of G (see the proof of Theorem 3.1). If N is contained in the Frattini subgroup $\phi(G)$ then N is solvable and hence G is solvable, a contradiction. If $N \not\subseteq \phi(G)$ then there exists a maximal subgroup M_1 of G such that $G = M_1N$ and $core_G(M_1) = \langle 1 \rangle$. Let p be a prime divisor of $[G : M_1]$.

If $N \subseteq \phi_p(G)$ then N is solvable and hence G is solvable, a contradiction. If $N \not\subseteq \phi_p(G)$ then there exists M_2 in $\mathcal{M}_p(G)$ such that $N \not\subseteq M_2$ and so $G = M_2N$ and $\mathrm{core}_G(M_2) = \langle 1 \rangle$. Also $(N, \langle 1 \rangle)$ is a common maximal θ-pair in $\theta(M_1)$ and $\theta(M_2)$ (see the proof of Theorem 3.1). So by hypothesis M_1 and M_2 are conjugate in G and consequently $[G : M_1] = [G : M_2]$. This implies that p divides $[G : M_2]$, which contradicts the fact that $[G : M_2]_p = 1$. The proof of other part is similar and so we omit it.

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