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ON MULTIVALENT FUNCTIONS WITH NEGATIVE AND MISSING COEFFICIENTS

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1. Introduction

Let S(p) be the class of functions $f(z)=z^p-\sum_{n=k}^{\infty}a_{n+p}z^{n+p}$ which are analytic in the unit disc $D=\{z:|z|<1\}$. For $0\leqslant\alpha\leqslant 1$, $0\leqslant\beta<1$ and $0<\gamma\leqslant 1$, let $P_k(\alpha,\beta,\gamma)$ be the class of those functions $f(z)$ of S(p) which satisfy the condition

$$\left|\frac{z\frac{f'(z)}{f(z)}-p}{\alpha z\frac{f'(z)}{f(z)}+[1-(1+\alpha)\beta]p}\right|<\gamma \tag{1.1}$$

for $z\in D$. Let T_p denote the subclass of S(p) consisting of P-valent functions in D and having Taylor expansion of the form

$$f(z)=z^p-\sum_{n=k}^{\infty}a_{n+p}z^{n+p}\qquad (a_{n+p}\geqslant 0,\ k\geqslant 2)\ . \tag{1.2}$$

Let $P_k[\alpha,\beta,\gamma]=P_k(\alpha,\beta,\gamma)\cap T_p$.

Goel and Sohi [1], Sarangi and Uralagaddi [2], Shukla and Dashrath [3] and Silverman [4] have studied certain subclasses of analytic functions with negative coefficients. Kumar [5], Sarangi and Patil [6] have studied the class of univalent functions with negative and missing coefficients. In this paper, we obtain integral representation formula, coefficient estimate, distortion theorem, covering theorem and radius of convexity for $P_k[\alpha,\beta,\gamma]$.

We also obtain the class preserving integral operators of the form

$$F(z)=\frac{p+c}{z^c}\int_0^z t^{c-1}f(t)dt,\qquad c>-p \tag{1.3}$$

for $f(z) \in P_k[\alpha, \beta, \gamma]$. Consequently, when $f(z) \in P_k[\alpha, \beta, \gamma]$ we obtain the radius of P-valence of $f(z) \in P_k[\alpha, \beta, \gamma]$. Lastly, we show that the class $P_k[\alpha, \beta, \gamma]$ is closed under "Arithmetic mean" and "convex linear combination".

2. Representation Formula

THEOREM 1. The function $f(z)=z^p - \sum_{n=k}^{\infty} a_{n+p} z^{n+p}$ belongs to $P_k[\alpha, \beta, \gamma]$ if and only if it can be expressed in the form

$$f(z)=z^p \exp\{-(1+\alpha)(1-\beta)p \int_0^z \frac{t^{k-1}\Phi(t)}{1+\alpha t^k \Phi(t)} dt\} \tag{2.1}$$

where $\Phi(z)$ is analytic in D and satisfies $|\Phi(z)| \leqslant \gamma$ for $z \in D$.

Proof. Let $f(z) \in P_k[\alpha, \beta, \gamma]$. Then

$$\left| \frac{z\frac{f'(z)}{f(z)} - p}{\alpha z\frac{f'(z)}{f(z)} + [1-(1+\alpha)\beta]p} \right| < \gamma \tag{2.2}$$

for $z \in D$ and by the Schwarz Lemma, we have

$$\frac{p - z\frac{f'(z)}{f(z)}}{\alpha z\frac{f'(z)}{f(z)} + [1-(1+\alpha)\beta]p} = z^k \Phi(z) \tag{2.3}$$

where $\Phi(z)$ is analytic in D and satisfies $|\Phi(z)| \leqslant \gamma$ for $z \in D$. (2.3) implies $z\frac{f'(z)}{f(z)} = \frac{p-[1-(1+\alpha)\beta]pz^k\Phi(z)}{1+\alpha z^k \Phi(z)}$. Hence we have $\frac{f'(z)}{f(z)} - \frac{p}{z} = -\frac{(1+\alpha)(1-\beta)pz^{k-1}\Phi(z)}{1+\alpha z^k \Phi(z)}$, which at once gives (2.1) on integration from 0 to z.

Conversely suppose

$$f(z)=z^p \exp\{-(1+\alpha)(1-\beta)p \int_0^z \frac{t^{k-1}\Phi(t)}{1+\alpha t^k \Phi(t)} dt\}, \tag{2.4}$$

then (2.4) implies $\log[z^{-p}f(z)] = -(1+\alpha)(1-\beta)p \int_0^z \frac{t^{k-1}\Phi(t)}{1+\alpha t^k \Phi(t)} dt$. So differentiating and simplifying we get (2.3) and

$$\left| z\frac{f'(z)}{f(z)} - p \right| = (1+\alpha)(1-\beta)p \frac{|z^k \Phi(z)|}{|1+\alpha z^k \Phi(z)|}$$

$$\leqslant \frac{(1+\alpha)(1-\beta)p\gamma}{|1+\alpha z^k \Phi(z)|}$$

since $|z^k\Phi(z)|<\gamma$ for $z\in D$.

Substituting $z^k\Phi(z)$ from (2.3) and simplyfying we get

$$\left|\frac{z\frac{f'(z)}{f(z)}-p}{\alpha z\frac{f'(z)}{f(z)}+[1-(1+\alpha)\beta]p}\right|<\gamma .$$

Hence $f(z)\in P_k[\alpha,\beta,\gamma]$.

THEOREM 2. A function $f(z)=z^p-\sum_{n=k}^{\infty}a_{n+p}z^{n+p}$ is in $P_k[\alpha,\beta,\gamma]$ if and only if

$$\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a_{n+p}\leqslant\gamma(1+\alpha)(1-\beta)p. \tag{2.5}$$

Proof. Suppose that $f(z)\in P_k[\alpha,\beta,\gamma]$, then

$$\left|\frac{z\frac{f'(z)}{f(z)}-p}{\alpha z\frac{f'(z)}{f(z)}+[1-(1+\alpha)\beta]p}\right| \tag{2.6}$$

$$=\left|\frac{-\sum_{n=k}^{\infty}na_{n+p}z^{n+p}}{(1+\alpha)(1-\beta)pz^p-\sum_{n=k}^{\infty}[(1+\alpha)(1-\beta)p+\alpha n]a_{n+p}z^{n+p}}\right|<\gamma,\quad \text{for } z\in D.$$

Since $|Re(z)|\leqslant|z|$ for all z, we have

$$\left|Re\left\{\frac{-\sum_{n=k}^{\infty}na_{n+p}z^{n+p}}{(1+\alpha)(1-\beta)pz^p-\sum_{n=k}^{\infty}[(1+\alpha)(1-\beta)p+\alpha n]a_{n+p}z^{n+p}}\right\}\right|<\gamma. \tag{2.7}$$

Choose values of z on real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator of (2.7) and letting $z\to 1^-$ through real axis, we obtain that

$$\sum_{n=k}^{\infty}na_{n+p}\leqslant\gamma\{(1+\alpha)(1-\beta)p-\sum_{n=k}^{\infty}[(1+\alpha)(1-\beta)p+\alpha n]a_{n+p}\}.$$

So

$$\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a_{n+p}\leqslant\gamma(1+\alpha)(1-\beta)p .$$

Conversely suppose $\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a_{n+p}\leqslant\gamma(1+\alpha)(1-\beta)p$ is true, then

$$\left|z\frac{f'(z)}{f(z)}-p\right|-\gamma\left|\alpha z\frac{f'(z)}{f(z)}+[1-(1+\alpha)\beta]p\right|<0$$

provided

$$\left|-\sum_{n=k}^{\infty}na_{n+p}z^{n+p}\right|-\gamma\left|(1+\alpha)(1-\beta)pz^p-\sum_{n=k}^{\infty}[(1+\alpha)(1-\beta)p+\alpha n]a_{n+p}z^{n+p}\right|<0 .$$

For $|z|=r<1$, the left side of the above inequality is bounded by

$$\sum_{n=k}^{\infty} n a_{n+p} r^{n+p} - \gamma(1+\alpha)(1-\beta)p r^{p} + \gamma \sum_{n=k}^{\infty}[(1+\alpha)(1-\beta)p+\alpha n] a_{n+p} r^{n+p}$$

$$= \sum_{n=k}^{\infty}[(1+\alpha\gamma)+(1+\alpha)(1-\beta)p\gamma] a_{n+p} r^{n+p} - \gamma(1+\alpha)(1-\beta)p r^{p}$$

$$< \sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma] a_{n+p} - \gamma(1+\alpha)(1-\beta)p \leqslant 0 \ .$$

Hence $f(z) \in P_k[\alpha, \beta, \gamma]$.

Finally, we note that the function:

$$f_k(z)=z^p - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} z^{k+p}, \qquad k=2, 3, \cdots$$

is the extremal function of (2.5).

By using the inequality (2.5) we have

COROLLARY 1. If $f(z)=z^p - \sum_{n=k}^{\infty} a_{n+p} z^{n+p}$ is in $P_k[\alpha, \beta, \gamma]$, then

$$a_{n+k} \leqslant \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}, \qquad k=2, 3, \cdots .$$

This result is sharp.

3. Distortion Properties

THEOREM 3. If $f(z) \in P_k[\alpha, \beta, \gamma]$, then for $|z|<1$,

$$|z|^p - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} |z|^{k+p} \leqslant |f(z)| \tag{3.1}$$

$$\leqslant |z|^p + \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} |z|^{k+p}$$

and

$$p|z|^{p-1} - \frac{(k+p)(1+\alpha)(1-\beta)p\gamma}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} |z|^{k+p-1} \leqslant |f'(z)| \tag{3.2}$$

$$\leqslant p|z|^{p-1} + \frac{(k+p)(1+\alpha)(1-\beta)p\gamma}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} |z|^{k+p-1} .$$

These results are sharp.

Proof: From Theorem 2, we have

$$\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma] a_{n+p} \leqslant \gamma(1+\alpha)(1-\beta)p \ .$$

Now since $n \geqslant k$, we obtain that

$$[(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma]\sum_{n=k}^{\infty}a_{n+p}$$

$$\leqslant \sum_{n=k}^{\infty}[(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma]a_{n+p} \leqslant \gamma(1+\alpha)(1-\beta)p \,.$$

So

$$\sum_{n=k}^{\infty}a_{n+p} \leqslant \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma} \,. \tag{3.3}$$

Now we have

$$|f(z)| = |z^p - \sum_{n=k}^{\infty}a_{n+p}z^{n+p}| \leqslant |z|^p + |z|^{k+p}\sum_{n=k}^{\infty}a_{n+p} \tag{3.4}$$

$$\leqslant |z|^p + \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}|z|^{k+p}$$

and

$$|f(z)| \geqslant |z|^p - |z|^{k+p}\sum_{n=k}^{\infty}a_{n+p} \tag{3.5}$$

$$\geqslant |z|^p - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}|z|^{k+p} \,.$$

From (3.4) and (3.5) we have (3.1).

Further

$$|f'(z)| = |p z^{p-1} - \sum_{n=k}^{\infty}(n+p)a_{n+p}z^{n+p-1}| \tag{3.6}$$

$$\leqslant p|z|^{p-1} + |z|^{k+p-1}\sum_{n=k}^{\infty}(n+p)a_{n+p}$$

and

$$|f'(z)| \geqslant p|z|^{p-1} - |z|^{k+p-1}\sum_{n=k}^{\infty}(n+p)a_{n+p} \,. \tag{3.7}$$

Using (3.3) in the result $\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a_{n+p} \leqslant \gamma(1+\alpha)(1-\beta)p$ of Theorem 2 and simplifying, we obtain that

$$\sum_{n=k}^{\infty}(n+p)a_{n+p} \leqslant \frac{(k+p)\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} \,.$$

Substituting this values of $\sum_{n=k}^{\infty}(n+p)a_{n+p}$ in (3.6) and (3.7) we have

$$p|z|^{p-1} - \frac{(k+p)\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}|z|^{k+p-1} \leqslant |f'(z)|$$

$$\leqslant p|z|^{p-1} + \frac{(k+p)\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}|z|^{k+p-1} .$$

Equalities in (3.1) and (3.2) are obtained if we take

$$f(z) = z^{p} - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} z^{k+p}, \quad k=2,\ 3,\ \cdots.$$

COROLLARY 2. If $f(z) \in P_k[\alpha, \beta, \gamma]$, then the disc D is mapped by $f(z)$ onto a domain that contains the disc

$$|w| \leqslant \frac{(1+\alpha\gamma)k}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}. \tag{3.8}$$

The result is sharp with the extremal function

$$f(z) = z^{p} - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} z^{k+p}, \qquad k=2,\ 3,\ \cdots.$$

Proof. By letting $r=|z| \to 1^{-}$ in inequality (3.1), we have

$$1 - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma} \leqslant |f(z)|$$

$$1 + \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}.$$

So $f(z)$ maps the disc D onto a domain that contains the disc

$$|w| \leqslant 1 - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}$$

$$= \frac{(1+\alpha\gamma)k}{(1+\alpha\gamma)k+(1+\alpha)(1-\beta)p\gamma}.$$

4. Integral Operator

THEOREM 4. Let c be a real number such that $c > -p$. If $f(z) \in P_k[\alpha, \beta, \gamma]$, then the function $F(z)$ defined by

$$F(z) = \frac{p+c}{z^{c}} \int_0^z t^{c-1} f(t)\,dt$$

also belongs to $P_k[\alpha, \beta, \gamma]$.

Proof. Let $f(z) = z^{p} - \sum_{n=k}^{\infty} a_{n+p} z^{n+p} \in P_k[\alpha, \beta, \gamma]$, then

$$F(z) = \frac{p+c}{z^{c}} \int_0^z t^{c-1} f(t)\,dt$$

$$= \frac{p+c}{z^{c}} \int_0^z \left(t^{p+c-1} - \sum_{n=k}^{\infty} a_{n+p} t^{n+p+c-1}\right) dt$$

$$=z^p-\sum_{n=k}^{\infty}a_{n+p}\frac{p+c}{n+p+c}z^{n+p}$$

$$=z^p-\sum_{n=k}^{\infty}b_{n+p}z^{n+p}$$

where $b_{n+p}=\frac{p+c}{n+p+c}a_{n+p}$. Therefore using Theorem 2 for the coefficients of $F(z)$ we have

$$\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]b_{n+p}$$

$$=\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]\frac{p+c}{n+p+c}a_{n+p}$$

$$\leqslant\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a_{n+p}\leqslant\gamma(1+\alpha)(1-\beta)p$$

since $0<\frac{p+c}{n+p+c}<1$ and $f(z)\in P_k[\alpha,\beta,\gamma]$. Hence $F(z)\in P_k[\alpha,\beta,\gamma]$.

5. Radius of Convexity

THEOREM 5. If $f(z)\in P_k[\alpha,\beta,\gamma]$, then $f(z)$ is P-valently convex in the $|z|<R$, where

$$R=\inf_{n>k}\left\{\frac{(1+\alpha\gamma)n+(1+\alpha)(1-\gamma)p\gamma}{(1+\alpha)(1-\beta)p\gamma}\left(\frac{p}{n+p}\right)^2\right\}^{1/n}. \tag{5.1}$$

Proof. It is sufficient to show that

$$\left|1+z\frac{f''(z)}{f'(z)}-p\right|\leqslant p,\qquad \text{for } |z|<R. \tag{5.2}$$

Now $f(z)=z^p-\sum_{n=k}^{\infty}a_{n+p}z^{n+p}$, so

$$1+z\frac{f''(z)}{f'(z)}-p=\frac{(1-p)f'(z)+zf''(z)}{f'(z)}$$

$$=\frac{(1-p)[pz^{p-1}-\sum_{n=k}^{\infty}(n+p)a_{n+p}z^{n+p-1}]+z[p(p-1)z^{p-2}-\sum_{n=k}^{\infty}(n+p)(n+p-1)a_{n+p}z^{n+p-2}]}{pz^{p-1}-\sum_{n=k}^{\infty}(n+p)a_{n+p}z^{n+p-1}}$$

$$=\frac{-\sum_{n=k}^{\infty}n(n+p)a_{n+p}z^n}{p-\sum_{n=k}^{\infty}(n+p)a_{n+p}z^n}.$$

Therefore, if

$$\frac{\sum_{n=k}^{\infty}n(n+p)a_{n+p}|z|^n}{p-\sum_{n=k}^{\infty}(n+p)a_{n+p}|z|^n}\leqslant p \tag{5.3}$$

or

$$\sum_{n=k}^{\infty}\left(\frac{n+p}{p}\right)^2 a_{n+p}|z|^n \leqslant 1 \quad , \tag{5.4}$$

then we have (5.2). From Theorem 2 we have

$$\sum_{n=k}^{\infty} \frac{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}{(1+\alpha)(1-\beta)p\gamma}\, a_{n+p} \leqslant 1 \ .$$

Hence (5.4) will be satisfied if

$$\left(\frac{n+p}{p}\right)^2|z|^n \leqslant \frac{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}{(1+\alpha)(1-\beta)p\gamma}$$

or if

$$|z| \leqslant \left\{ \frac{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}{(1+\alpha)(1-\beta)p\gamma} \left(\frac{p}{n+p}\right)^2 \right\}^{1/n} .$$

So f(z) is P-valently convex in the disc

$$|z| < R = \inf_{n \geqslant k} \left\{ \frac{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}{(1+\alpha)(1-\beta)p\gamma} \left(\frac{p}{n+p}\right)^2 \right\}^{1/n} .$$

6. Closure Properties

In this section, we show that the class $P_k[\alpha, \beta, \gamma]$ is closed under "Arithemetic mean" and "Convex linear combinations".

THEOREM 6. If $f_j(z) = z^p - \sum_{n=k}^{\infty} a^j_{n+p} z^{n+p} \in P_k[\alpha, \beta, \gamma]$, $(j=1,2,\cdots,m)$, then $g(z) = z^p - \sum_{n=k}^{\infty} b_{n+p} z^{n+p}$ also belongs to $P_k[\alpha, \beta, \gamma]$, where $b_{n+p} = \frac{1}{m}\sum_{j=1}^{m} a^j_{n+p}$.

Proof. Since $f_j(z) \in P_k[\alpha, \beta, \gamma]$ it follows from Theorem 2 that

$$\sum_{n=k}^{\infty} [(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a^j_{n+p} \leqslant \gamma(1+\alpha)(1-\beta)p \tag{6.1}$$

$j=1,2,\cdots,m$. Therefore we obtain that

$$\begin{aligned} &\sum_{n=k}^{\infty} [(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]b_{n+p} \\ &= \sum_{n=k}^{\infty} [(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]\frac{1}{m}\sum_{j=1}^{m} a^j_{n+p} \\ &= \frac{1}{m}\sum_{j=1}^{m}\left\{\sum_{n=k}^{\infty} [(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]a^j_{n+p}\right\} \\ &\leqslant \frac{1}{m}\sum_{j=1}^{m} \gamma(1+\alpha)(1-\beta)p = \gamma(1+\alpha)(1-\beta)p \ . \end{aligned} \tag{6.2}$$

Hence from Theorem 2, g(z) belongs to $P_k[\alpha, \beta, \gamma]$.

THEOREM 7. The extreme points of $P_k[\alpha, \beta, \gamma]$ are given by

$$f_p(z)=z^p \tag{6.3}$$

and

$$f_{n+p}(z)=z^p-\frac{(1+\alpha)(1-\beta)p\gamma}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}z^{n+p} \tag{6.4}$$

$$n=k,\ k+1,\cdots\ ;\quad k=2,\ 3,\cdots .$$

Proof. We must show that $f(z)\in P_k[\alpha,\beta,\gamma]$ if and only if it can be expressed in the form $f(z)=\lambda_p f_p(z)+\sum_{n=k}^{\infty}\lambda_{n+p}f_{n+p}(z)$, where $\lambda_p\geq 0$, $\lambda_{n+p}\geq 0$ and $\lambda_p+\sum_{n=k}^{\infty}\lambda_{n+p}=1$.

Let suppose that

$$\begin{aligned} f(z)&=\lambda_p f_p(z)+\sum_{n=k}^{\infty}\lambda_{n+p}f_{n+p}(z)\\ &=(1-\sum_{n=k}^{\infty}\lambda_{n+p})z^p+\sum_{n=k}^{\infty}\lambda_{n+p}\left(z^p-\frac{(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}z^{n+p}\right)\\ &=z^p-\frac{\gamma(1+\alpha)(1-\beta)p\lambda_{n+p}}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}z^{n+p} . \end{aligned}$$

Then from Theorem 2, we have

$$\sum_{n=k}^{\infty}[(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma]\frac{\gamma(1+\alpha)(1-\beta)p\lambda_{n+p}}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}$$

$$=\gamma(1+\alpha)(1-\beta)p\sum_{n=k}^{\infty}\lambda_{n+p}=\gamma(1+\alpha)(1-\beta)p(1-\lambda_p)\leq\gamma(1+\alpha)(1-\beta)p .$$

Hence $f(z)\in P_k[\alpha,\beta,\gamma]$.

Conversely, suppose $f(z)=z^p-\sum_{n=k}^{\infty}a_{n+p}z^{n+p}\in P_k[\alpha,\beta,\gamma]$, we set

$$\lambda_{n+p}=\frac{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}{\gamma(1+\alpha)(1-\beta)p}a_{n+p} \tag{6.5}$$

where $n=k,\ k+1,\cdots$ and $\lambda_p=1-\sum_{n=k}^{\infty}\lambda_{n+p}$. We obtain that

$$\begin{aligned} f(z)&=z^p-\sum_{n=k}^{\infty}a_{n+p}z^{n+p}\\ &=(\lambda_p+\sum_{n=k}^{\infty}\lambda_{n+p})z^p-\sum_{n=k}^{\infty}\frac{\gamma(1+\alpha)(1-\beta)p\lambda_{n+p}}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma}z^{n+p} \end{aligned}$$

$$= \lambda_p z^p + \sum_{n=k}^{\infty} \lambda_{n+p} [z^p - \frac{\gamma(1+\alpha)(1-\beta)p}{(1+\alpha\gamma)n+(1+\alpha)(1-\beta)p\gamma} z^{n+p}]$$

$$= \lambda_p f_p(z) + \sum_{n=k}^{\infty} \lambda_{n+p} f_{n+p}(z) .$$

This proof is comleted.

REFERENCES

[1] R.M.Goel and N.S.Sohi; Indian J.Pure Appl.Math., 12(1981), 844-853.

[2] S.M.Sarangi and B.A.Uralagaddi; Rendiconti Accademia Nazionaledei Lincei., LXV, 38-42.

[3] S.L.Shukla and Dashrath; Soochow.J.Math., 8(1982), 179-188.

[4] H.Silverman; Proc.Amer.Math.Soc., 51(1975), 109-116.

[5] V.Kumar; Journal of Mathematical Research and Exposition.,Volume 4, No.1(1984), 27-33.

[6] S.M.Sarangi and V.J.Patil; Journal of Mathematical Research and Exposition., Volume 10, No.3(1990), 341-348.

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