Nihonkai Math. J.
Vol.15(2004), 169-176

On Four-dimensional Generalized Complex Space Forms

Richard S. Lemence

Abstract

F. Tricerri and L. Vanhecke [8] proved that a $2n$ $(n \geq 3)$-dimensional generalized complex space is a real space form or a complex space form. In this note, we shall show that this result is extendable to 4-dimensional case.

1 Introduction

Let (V, g) be an n-dimensional real vector space with positive definite inner product g and denote by $\mathcal{R}(V)$ the subspace of $V^* \otimes V^* \otimes V^* \otimes V^*$ consisting of all tensors having the same symmetries as the curvature tensor of a Riemannian manifold, including the first Bianchi identity. F. Tricerri and L. Vanhecke [8] gave the complete and irreducible decomposition of $\mathcal{R}(V)$ under the action of $\mathcal{U}(n)$. They then applied these algebraic results to the curvature tensors of almost Hermitian manifolds.

A $2n$ $(n \geq 2)$ - dimensional almost Hermitian manifold $M = (M, J, g)$ is called a *generalized complex space form* if the curvature tensor R takes the following form:

$$
\begin{aligned}
(1.1) \qquad R &= \frac{\tau + 3\tau^*}{16n(n+1)}(\pi_1 + \pi_2) + \frac{\tau - \tau^*}{16n(n-1)}(3\pi_1 - \pi_2) \\
&= \frac{(2n+1)\tau - 3\tau^*}{8n(n-1)(n+1)}\pi_1 + \frac{(2n-1)\tau^* - \tau}{8n(n-1)(n+1)}\pi_2
\end{aligned}
$$

for some smooth functions τ and τ^* and here

$$\pi_1(x, y)z = g(y, z)x - g(x, z)y$$

and

$$\pi_2(x, y)z = g(Jy, z)Jx - g(Jx, z)Jy - 2g(Jx, y)Jz$$

for all x, y, $z \in T_pM$, $p \in M$.

The concept of generalized complex space form is a natural generalization of a complex space form (i.e. Kähler manifold of constant holomorphic sectional curvature) which has been introduced by F. Tricerri and L. Vanhecke [8]. They showed that an almost Hermitian manifold is a generalized complex space form if and only if Einsteinian and weakly $*$- Einsteinian and Bochner flat,i.e., $B(R) = 0$ and further proved that a $2n$ ($n \geq 3$)-dimensional generalized complex space is a real space form or a complex space form.

In this paper, we shall show that the result of F. Tricerri and L. Vanhecke [8] is partially extendable to 4-dimensional case under compactness hypothesis, namely, we shall prove the following:

Theorem A. *Let $M = (M, J, g)$ be a 4-dimensional generalized complex space form. Then M is locally a real space form or globally conformal Kähler manifold. In the latter case, (M, J, g^*) with $g^* = (3\tau^* - \tau)^{\frac{2}{3}} g$ is a Kähler manifold, where τ and τ^* are the scalar curvature and the $*$-scalar curvature of M, respectively.*

Theorem B. *Let $M = (M, J, g)$ be a compact 4-dimensional generalized complex space form. Then M is a real space form of constant non-positive sectional curvature or compact complex space form.*

Remark. There is an example of 4-dimensional compact non-Hermitian, almost Hermitian flat manifold (cf. [1]). Further, there does not exist 4-dimensional compact Hermitian manifold of negative constant sectional curvature (cf. [5]). However, the author does not know whether there exist a 4-dimensional compact non-Hermitian almost Hermitian manifolds of negative constant sectional curvature or not.

2 Preliminaries

Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold with the almost complex structure J and the metric g. We denote by ∇, R, ρ and τ the Levi-Civita connection, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature tensor, respectively. We assume that the Riemannian curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{2.1}$$

for X, Y and $Z \in \mathfrak{X}(M)$ ($\mathfrak{X}(M)$ denotes Lie algebra of all smooth vector fields on M). Further, we denote by ρ^* and τ^* the Ricci $*$-tensor and the $*$-scalar curvature of M, respectively. The tensor ρ^* is defined pointwisely by

$$\begin{aligned} \rho^*(x, y) &= trace(z \mapsto R(Jz, x)Jy) \\ &= -\sum_{i=1}^{2n} R(x, e_i, Jy, Je_i) \\ &= -\frac{1}{2}\sum_{i=1}^{2n} R(x, Jy, e_i, Je_i), \end{aligned} \tag{2.2}$$

for $x,\ y,\ z \in T_pM,\ p \in M$, where $R(x,y,z,w) = g(R(x,y)z,w)$ and $\{e_i\}$ is an orthonormal basis of T_pM. The $*$-scalar curvature of M is defined by $\tau^* =$ trace of Q^*, where Q^* is the Ricci $*$ - operator defined by $\rho^*(x,y) = g(Q^*x,y)$, for x, $y \in T_pM,\ p \in M$. We note that ρ^* satisfies $\rho^*(Jx,Jy) = \rho^*(y,x)$ for $x,\ y \in T_pM$, $p \in M$, but is not symmetric in general. An almost Hermitian manifold M is called a *weakly $*$-Einstein manifold* if $\rho^* = \frac{\tau^*}{2n}g$ $(dimM = 2n)$ holds, and in addition, if τ^* is constant-valued, then M is called *$*$-Einstein manifold.* There exist many examples of weakly $*$ - Einstein but not $*$ - Einstein manifolds (cf. [9], [10] and [11]).

Now we return to 4-dimensional almost Hermitian manifold $M = (M, J, g)$ under consideration. We denote by $\wedge^2 M$ the real vector bundle of all the real 2-forms on M. The $\wedge^2 M$ inherits a natural inner product coming from the Riemannian metric g and we have the following orthogonal decomposition:

$$\wedge^2 M = \mathbb{R}\Omega \oplus LM \oplus \wedge_0^{1,1} M \tag{2.3}$$

where LM (resp. $\wedge_0^{1,1}M$) is the bundle of J-skew invariant (J-invariant) effective 2-forms on M. We can identify the bundle $\mathbb{R}\Omega \oplus LM$ (resp. $\wedge_0^{1,1}M$) with the bundle $\wedge_+^2 M$ (resp. $\wedge_-^2 M$) of the self-dual (resp. anti-self dual) 2-forms on M. The bundle LM is endowed with the complex structure (denoted also by J) given by $(J\Phi)(X,Y) = -\Phi(JX,Y)$, for any local section of Φ of LM and any $X, Y \in \mathfrak{X}(M)$. We note that the almost complex structure J acts also on 1-form σ by $(J\sigma) = -\sigma(JX)$, for any $X \in \mathfrak{X}(M)$. Corresponding to the decomposition (2.3), we may set

$$\nabla\Omega = \alpha \otimes \Phi + \beta \otimes J\Phi \tag{2.4}$$

for some local 1-forms α and β, where $\Phi, J\Phi$ is a local orthonormal basis of LM. It is well-known that the almost complex structure J of M is integrable if and only if $(\nabla_X J)Y = (\nabla_{JX} J)JY$ holds for $X, Y \in \mathfrak{X}(M)$. So, from (2.4), we see that J is integrable if and only if $\beta = J\alpha$ holds on a neighborhood of any point of M. Since the dim $M = 4$, we see that there does not exist effective 3-forms on M and hence, any 3-form η is represented as $\eta = \sigma \wedge \Omega$ for some 1-form σ. Thus, we may set especially

$$d\Omega = \omega \wedge \Omega \tag{2.5}$$

for some 1-form ω on M. The 1-form ω is called the *Lee form* of M, and is given by

$$\omega = -\delta\Omega \circ J. \tag{2.6}$$

Let $\{e_i\} = \{e_1, e_2 = Je_1, e_3, e_4 = Je_3\}$ be any (local) unitary basis of $T_pM(p \in M)$ and $\{e^i\} = \{e^1, e^2 = Je^1, e^3, e^4 = Je^3\}$ be the dual basis of $\{e_i\}$. Then, the Kähler form Ω is represented by $\Omega = e^1 \wedge e^2 + e^3 \wedge e^4$. Further, we see that

$$\begin{aligned}
\{\Phi, J\Phi\} &= \{\frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^4 + e^2 \wedge e^3)\}, \\
\{\psi_1, \psi_2, \psi_3\} &= \{\frac{1}{\sqrt{2}}(e^1 \wedge e^2 - e^3 \wedge e^4), \frac{1}{\sqrt{2}}(e^1 \wedge e^3 - e^2 \wedge e^4), \\
&\quad \frac{1}{\sqrt{2}}(e^1 \wedge e^4 - e^2 \wedge e^3)\}
\end{aligned} \tag{2.7}$$

are (locally) orthonormal bases of LM and $\wedge_0^{1,1}M = \wedge_-^2 M$, respectively. In this paper, for any (local) unitary basis $\{e_i\}$ of T_pM at any point $p \in M$, we shall adopt the following notational convention:

$$\begin{aligned}
J_{ij} &= g(Je_i, e_j) \\
\nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k), &\dots, \nabla_{\bar{i}} J_{\bar{j}\bar{k}} = g((\nabla_{Je_i} J)Je_j, Je_k) \\
R_{ijkl} = g(R(e_i, e_j)e_k, e_l), &\dots, R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g(R(Je_i, Je_j)Je_k, Je_l),
\end{aligned} \tag{2.8}$$

and so on where the latin indices ranges over $1, 2, 3, 4$. Following this notational convention, from (2.6), we have

$$\omega_k = \sum_{i,j} (\nabla_i J_{ij}) J_{kj}. \tag{2.9}$$

3 Proofs of Theorem A and B

First we prove Theorem A.

Let $M = (M, J, g)$ be a 4 - dimensional generalized complex space form. Then (1.1) reduces to

$$\begin{aligned}
R(x, y, z, w) = &\frac{5\tau - 3\tau^*}{48}\{g(y, z)g(x, w) - g(x, z)g(y, w)\} \\
&+ \frac{3\tau^* - \tau}{48}\{g(Jx, w)g(Jy, z) - g((Jy, w)g(Jx, z) \\
&-2g(Jx, y)g(Jz, w)\}
\end{aligned} \tag{3.1}$$

for x, y, z, $w \in T_pM (p \in M)$. First of all, from (3.1), we may note that M is Einstein($\rho = \frac{\tau}{4}g$) and weakly $*$-Einstein ($\rho^* = \frac{\tau^*}{4}g$); and, further a space of pointwise constant holomorphic sectional curvature $\frac{1}{24}(3\tau^* + \tau)$. Now, from (3.1), we also get

$$
\begin{aligned}
(3.2) \qquad & (\nabla_u R)(x,y,z,w) \\
= & -\frac{1}{16}u(\tau^*)\{g(x,w)g(y,z) - g(y,w)g(x,z) \\
& -g(Jx,w)g(Jy,z) + g(Jy,w)g(Jx,z) \\
& +2g(Jx,y)g(Jz,w)\} + \frac{3\tau^* - \tau}{48}\{g(Jy,z)g((\nabla_u J)x,w) \\
& +g(Jx,w)g((\nabla_u J)y,z) - g(Jx,z)g((\nabla_u J)y,w) \\
& -g(Jy,w)g((\nabla_u J)x,z) - 2g(Jx,y)g((\nabla_u J)z,w) \\
& -2g(Jz,w)g((\nabla_u J)x,y)\}
\end{aligned}
$$

for $u, x, y, z, w \in T_pM (p \in M)$.

Let $\{e_i\}$ be any unitary basis of T_pM at any poiny $p \in M$. Then, since M is Einstein, from (3.2), we get

$$
\begin{aligned}
(3.3) \qquad 0 = & (\nabla_w \rho)(y,z) - (\nabla_z \rho)(y,w) \\
= & \sum_{i=1}^{4}(\nabla_{e_i} R)(e_i,y,z,w) \\
= & -\frac{1}{16}\{w(\tau^*)g(y,z) - z(\tau^*)g(y,w) + (Jw)(\tau^*)g(Jy,z) \\
& -(Jz)(\tau^*)g(Jy,w) - 2(Jy)(\tau^*)g(Jz,w)\} \\
& +\frac{3\tau^* - \tau}{48}\{-g((\nabla_{Jw}J)y,z) + g((\nabla_{Jz}J)y,w) \\
& +(J\omega)(w)g(Jy,z) - (J\omega)(z)g(Jy,w) \\
& -2(J\omega)(y)g(Jz,w) + 2g((\nabla_{Jy}J)z,w)\}
\end{aligned}
$$

for $y, z, w \in T_pM (p \in M)$.

By setting $w = e_1$, $y = z = e_3$ in (3.3) we get

$$(3.4) \qquad e_1(3\tau^* - \tau) + 3(3\tau^* - \tau)g((\nabla_{e_4}J)e_1, e_3) = 0.$$

Similarly, by setting $w = e_1$, $y = z = e_4$ in (3.3) we get

$$(3.5) \qquad e_1(3\tau^* - \tau) - 3(3\tau^* - \tau)g((\nabla_{e_3}J)e_1, e_4) = 0.$$

Further, we get the following:

$$(3.6) \qquad e_2(3\tau^* - \tau) + 3(3\tau^* - \tau)g((\nabla_{e_4}J)e_2, e_3) = 0,$$

$$(3.7) \qquad e_2(3\tau^* - \tau) - (3\tau^* - \tau)g((\nabla_{e_3}J)e_2, e_4) = 0,$$

$$(3.8) \qquad e_3(3\tau^* - \tau) - 3(3\tau^* - \tau)g((\nabla_{e_2}J)e_1, e_3) = 0,$$

$$e_3(3\tau^* - \tau) + 3(3\tau^* - \tau)g((\nabla_{e_1}J)e_2, e_3) = 0, \tag{3.9}$$

$$e_4(3\tau^* - \tau) - 3(3\tau^* - \tau)g((\nabla_{e_2}J)e_1, e_4) = 0, \tag{3.10}$$

$$e_4(3\tau^* - \tau) + 3(3\tau^* - \tau)g((\nabla_{e_1}J)e_2, e_4) = 0. \tag{3.11}$$

From (3.4)~(3.11), taking the account of (2.4) and (2.7), we have

$$(3\tau^* - \tau)(\beta - J\alpha) = 0. \tag{3.12}$$

Let $M_o = \{p \in M | 3\tau^* - \tau = 0 \text{ at } p\}$. Since M is Einstein, M is real analytic as Riemannian manifold. Thus, if the interior of M_o is not empty, then M is locally a real space form of dimension 4 by (3.1). In the sequel, we assume that the interior of M_o is empty. Then we see that the complement $M_o^{'}$ of M_o in M is an open dense subset of M, and $\beta - J\alpha = 0$ holds on a neighborhood of any point $M_o^{'}$ by virtue (3.12). Thus, we see that J is integrable. Therefore, we get $(\nabla_X J)Y = (\nabla_{JX} J)JY$ holds for any $X, Y \in \mathfrak{X}(M)$.

By direct calculation, we get

$$\begin{aligned} g((\nabla_{e_4}J)e_1, e_3) &= -g((\nabla_{e_4}J)e_2, e_4) = g((\nabla_{e_4}J)e_4, e_2), \\ g((\nabla_{e_3}J)e_1, e_4) &= g((\nabla_{e_3}J)e_2, e_3) = -g((\nabla_{e_3}J)e_3, e_2), \end{aligned}$$

and hence

$$-g((\nabla_{e_4}J)e_1, e_3) = g((\nabla_{e_3}J)e_1, e_4) = -\frac{1}{2}\omega_1 \tag{3.13}$$

by virtue of (2.9). Similarly, we get

$$\begin{aligned} -g((\nabla_{e_4}J)e_2, e_3) &= g((\nabla_{e_3}J)e_2, e_4) = -\frac{1}{2}\omega_2, \\ g((\nabla_{e_2}J)e_1, e_3) &= -g((\nabla_{e_1}J)e_2, e_3) = -\frac{1}{2}\omega_3, \\ g((\nabla_{e_2}J)e_1, e_4) &= -g((\nabla_{e_1}J)e_2, e_4) = -\frac{1}{2}\omega_4. \end{aligned} \tag{3.14}$$

Thus, by (3.4)~(3.11), (3.13) and (3.14), we have finally the following differential equation

$$d(3\tau^* - \tau) + \frac{3}{2}(3\tau^* - \tau)\omega = 0. \tag{3.15}$$

By (3.15) and our hypothesis, we can immediately see that the function $3\tau^* - \tau$ vanishes nowhere on M. Thus, taking the exterior derivative of equality (3.15), we have further

$$d\omega = 0 \ \ on \ \ M \tag{3.16}$$

Therefore, it follows from (3.16) that M is locally conformal Kähler manifold.

Now, we consider a new Riemannian metric g^* defined by $g^* = (3\tau^* - \tau)^{\frac{2}{3}} g$. Then, we see that (M, J, g^*) is a (real) 4-dimensional Hermitian manifold with the corresponding Kähler form $\Omega^* = (3\tau^* - \tau)^{\frac{2}{3}} \Omega$. By (2.5) and (3.15), we may easily check that $d\Omega^* = 0$ holds on M, and hence, (M, J, g) is a Kähler manifold of real dimension 4. This completes the proof of Theorem A.

Next, we shall prove Theorem B. From (3.1), taking account of the result by Koda ([1], Prop. 4.1), we see that a 4-dimensional generalized complex space form is a self-dual Einstein manifold. On one hand, it is well-known that a 4-dimensional sphere does not admit almost complex structure. Therefore, we see that Theorem B follows immediately from the arguments in this section and the following result ([2] and [5])

Theorem C. *Let $M = (M, J, g)$ be a compact self-dual Einstein Hermitian surface. Then M is a compact complex space form.*

Acknowledgment. The author wishes to thank Prof. Dr. Kouei Sekigawa for his comments and suggestions.

References

[1] T. Koda, *Self-dual and anti-Self-dual Hermitian Surfaces*, Kodai Math. J. 10 (1987), 335-342

[2] T. Koda and K. Sekigawa, *Self-Dual Einstein Hermitian Surfaces*, Advanced Studies in Pure Mathematics, Vol. 22 (1993) 123-131

[3] K. Matsuo, *Pseudo-Bochner Curvature Tensor on Hermitian Manifolds*, Colloquium Mathematicum, Vol. 80, No. 2 (1999), 201-209

[4] K. Sekigawa, *On some 4-dimensional compact almost Hermitian manifolds*, J. Ramanujan Math. Soc. 2 (1987), 101-116

[5] K. Sekigawa and T. Koda, *Compact Hermitian Surfaces of Pointwise Constant Holomorphic Sectional Curvature*, Glasgow Math. J. 37 (1994), 343-349

[6] S. Tachibana, *On the Bochner Curvature Tensor*, Natural Science Report, Ochanomizu University, Vol. 18. No. 1, 1967, 15- 19

[7] S. Tachibana and R.C. Liu, *Notes on Kahlerian Metrics with Vanishing Bochner Curvature Tensor*, Kodai Math. Sem. Rep. 22 (1970), 313-321

[8] F. Tricerri and L. Vanhecke, *Curvature Tensors on Almost Hermitian Manifolds*, Transactions of the American Mathematical Society, Vol. 267, No. 2, October 1981, 365-398

[9] L. Vanhecke, *The Bochner curvature tensor on almost Hermitian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino 34 (1975-76), 21-38

[10] L. Vanhecke, *On the Decomposition of curvature tensor fields on almost Hermitian manifolds*, Proc. Conf. Differential Geometry, Michigan State Univ., East Lansing, Mich., 1976, pp. 16-33

[11] L. Vanhecke and F. Bouten, *Constant type for almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. R.S. Roumanie 20 (1976), 415-422

Department of Mathematical Science,
Graduate School of Science and Technology,
Niigata University, Niigata, 950-2181 Japan

email: f02n406h@mail.cc.niigata-u.ac.jp

Recieved 5 August, 2004 Revised 24 September, 2004