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Boundary Controllability of Neutral Integrodifferential Systems in Banach Spaces

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Abstract: Sufficient conditions for boundary controllability of neutral integrodifferential systems in Banach spaces are established. The results are obtained by using the strongly continuous semigroup theory and the Schaefer fixed point theorem.

Key Words: Boundary controllability, neutral integrodifferential system, semigroup theory, fixed point theorem.

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1. Introduction

The theory of nonlinear neutral integrodifferential systems in Banach spaces has been studied by several authors [7,10,12]. Ntouyas and Tsamatos [12] discussed the existence results for neutral functional integrodifferential equations by means of the Schaefer fixed point theorem. Arino et al. [1] studied the existence results for initial value problem for neutral functional differential equations. Controllability of neutral functional integrodifferential systems in abstract spaces was first studied by Balachandran et al. [3]. Recently, Balachandran and Marshal Anthoni [2] discussed the controllability of second order neutral functional differential systems using the strongly continuous cosine family of bounded linear operators. Several authors [4,6,16] have developed many abstract settings to describe the boundary control systems in which the control must be taken in sufficiently smooth functions for the existence of regular solutions to state space system. Barbu [5] and Fattorini [8] discussed the general theory for boundary control systems. Lasiecka [11] established the regularity of optimal boundary controls for parabolic equations. Han and Park [9] derived a set of sufficient conditions for the boundary controllability of a semilinear system with nonlocal conditions. The aim of this paper is to derive a set of sufficient conditions for the boundary controllability of neutral integrodifferential systems in Banach spaces by using the semigroup theory and the Schaefer fixed point theorem.

2. Preliminaries

Let E and U be a pair of real Banach spaces with the norms $\|\cdot\|$ and $\|\cdot\|_U$ respectively. Let σ be a linear, closed and densely defined operator with $D(\sigma) \subseteq E$

and $R(\sigma) \subseteq E$ and let θ be a linear operator with $D(\theta) \subseteq E$ and $R(\theta) \subseteq X$, a Banach space.

Consider the boundary control neutral integrodifferential system

$$\begin{aligned} \frac{d}{dt}[x(t) - h(t, x_t)] &= \sigma x(t) + f(t, x_t, \int_0^t g(t, s, x_s)ds), \\ \theta x(t) &= B_1 u(t), \quad t \in J = [0, b], \\ x_0 &= \phi \text{ on } [-r, 0], \end{aligned} \tag{1}$$

where the control function $u \in L^1(J, U)$, a Banach space of admissible control functions, $B_1 : U \to X$ is a linear continuous operator, the nonlinear operators $f : J \times Y \times E \to E$, $h : J \times Y \to E$ and $g : \Delta \times Y \to E$ are continuous and $\Delta = \{(t, s); \quad 0 \le s \le t \le b\}$. Here $Y = C([-r, 0], E)$ is the Banach space of all continuous functions $\phi : [-r, 0] \to E$ endowed with the norm $|\phi| = \sup\{\|\phi(s)\|; \quad -r \le s \le 0\}$. Also for $x \in C([-r, b], E)$, we have $x_t \in Y$ for $t \in [0, b]$, $x_t(s) = x(t + s)$ for $s \in [-r, 0]$.

Let $A : E \to E$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ with domain

$$D(A) = \{x \in D(\sigma); \quad \theta x = 0\}, \quad Ax = \sigma x, \quad \text{for } x \in D(A).$$

We shall make the following hypotheses:

(H_1) $D(\sigma) \subset D(\theta)$ and the restriction of θ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

(H_2) There exists a linear continuous operator $B : U \to E$ such that $\sigma B \in L(U, E)$, $\theta(Bu) = B_1 u$, for all $u \in U$. Also $Bu(t)$ is continuously differentiable and $\|Bu\| \le C\|B_1 u\|_X$ for all $u \in U$, where C is a constant.

(H_3) For all $t \in [0, b]$ and $u \in U$, $T(t)Bu \in D(A)$. Moreover, there exists a positive function $\nu \in L^1(0, b)$ such that $\|AT(t)B\|_{L(U,E)} \le \nu(t)$, a.e. for $t \in (0, b)$.

Let $x(t)$ be the solution of (1). Then we can define a function $z(t) = x(t) - Bu(t)$ and, from our assumption, we see that $z(t) \in D(A)$. Hence (1) can be written in terms of A and B as

$$\begin{aligned} \frac{d}{dt}[x(t) - h(t, x_t)] &= Az(t) + \sigma Bu(t) + f(t, x_t, \int_0^t g(t, s, x_s)ds), \quad t \in J, \\ x(t) &= z(t) + Bu(t), \\ x_0 &= \phi \text{ on } [-r, 0]. \end{aligned}$$

If u is continuously differentiable on $[0, b]$, then z can be defined as a mild solution to the Cauchy problem

$$\dot{z}(t) = Az(t) + \frac{d}{dt}h(t, x_t) + \sigma Bu(t) - B\dot{u}(t)$$

$$
\begin{aligned}
& + f(t, x_t, \int_0^t g(t, s, x_s) ds), \quad t \in J, \\
z(0) &= \phi(0) - Bu(0)
\end{aligned}
$$

and the solution of (1) is given by

$$
\begin{aligned}
x(t) &= T(t)[\phi(0) - Bu(0)] + Bu(t) + \int_0^t T(t-s) \frac{d}{ds} h(s, x_s) ds \\
&+ \int_0^t T(t-s)[\sigma Bu(s) - B\dot{u}(s) + f(s, x_s, \int_0^s g(s, \tau, x_\tau) d\tau)] ds. \quad (2)
\end{aligned}
$$

Since the differentiability of the control u represents an unrealistic and severe requirement, it is necessary to extend the concept of the solution for the general inputs $u \in L^1(J, U)$. Integrating (2) by parts, we get

$$
\begin{aligned}
x(t) &= T(t)[\phi(0) - h(0, \phi)] + h(t, x_t) + \int_0^t AT(t-s) h(s, x_s) ds \\
&+ \int_0^t [T(t-s)\sigma - AT(t-s)] Bu(s) ds \\
&+ \int_0^t T(t-s) f(s, x_s, \int_0^s g(s, \tau, x_\tau) d\tau) ds, \quad t \in J, \\
x(t) &= \phi(t) \text{ on } [-r, 0]. \quad (3)
\end{aligned}
$$

Thus (3) is well defined and it is called a mild solution of the system (1).

Definition. The system (1) is said to be *null controllable* on the interval J, if for every continuous initial function $\phi \in Y$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = 0$.

We need the following fixed point theorem due to Schaefer [15].

Schaefer's Theorem. Let Z be a normed linear space. Let $F : Z \to Z$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let

$$\zeta = \{x \in Z; \ \ x = \lambda Fx \ \text{ for some } \ 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

Let A be the infinitesimal generator of a bounded analytic semigroup $T(t)$ with bounded inverse A^{-1} on the Banach space E. The operator $(-A)^\alpha$ can be defined for $0 \leq \alpha \leq 1$ as the inverse of the bounded linear operator

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_o^\infty t^{\alpha-1} T(t) dt$$

and $(-A)^\alpha$ is a closed linear invertible operator with domain $D((-A)^\alpha)$ dense in E. The closedness of $(-A)^\alpha$ implies that $D((-A)^\alpha)$ endowed with the graph norm of $(-A)^\alpha$, that is $|||x||| = \|x\| + \|(-A)^\alpha x\|$, is a Banach space. Since $(-A)^\alpha$ is invertible its graph norm $|||.|||$ is equivalent to the norm $\|x\|_\alpha = \|(-A)^\alpha x\|$. Thus, $D((-A)^\alpha)$ equipped with the norm $\|.\|_\alpha$ is a Banach space which we denote by E_α. From this definition it is clear that $0 < \alpha < \beta$ implies $E_\alpha \supset E_\beta$ and that the imbedding of E_β in E_α is compact whenever the resolvent operator of A is compact. For more results on fractional powers of operators one can refer [13].

Further we consider the following assumptions:

(H_4) A is the infinitesimal generator of an analytic semigroup of compact linear operators $T(t)$ in E such that $\|T(t)\| \leq K_1$ for some $K_1 > 0$ and for any $\alpha \geq 0$, there exists a positive constant $K_2(\alpha) > 0$ such that $\|(-A)^\alpha T(t)\| \leq K_2 t^{-\alpha}$.

(H_5) For each $(t,s) \in \Delta$, the function $g(t,s,\cdot) : Y \to E$ is continuous and for each $x \in Y$, the function $g(\cdot,\cdot,x) : \Delta \to E$ is strongly measurable.

(H_6) For each $t \in J$, the function $f(t,\cdot,\cdot) : Y \times E \to E$ is continuous and for each $(x,y) \in Y \times E$, the function $f(\cdot,x,y) : J \to E$ is strongly measurable.

(H_7) For every positive integer k, there exists $\mu_k \in L^1(0,b)$ such that

$$\sup_{|x|,\|y\| \leq k} \|f(t,x,y)\| \leq \mu_k(t), \quad \text{for} \quad t \in J \ a.e.$$

(H_8) The function $h : J \times Y \to E$ is completely continuous and for any bounded set Q in $C([-r,b],E)$, the set $\{t \to h(t,x_t) : x \in Q\}$ is equicontinuous in $C(J,E)$.

(H_9) There exists $\beta \in (0,1)$ and a constant $b_1 > 0$ such that

$$\|(-A)^\beta h(t,x)\| \leq b_1, \quad t \in J, \quad x \in Y.$$

(H_{10}) There exists an integrable function $q : J \to [0,\infty)$ such that

$$\|f(t,x,y)\| \leq q(t)\Omega(|x| + \|y\|), \quad t \in J, \quad x \in Y, \quad y \in E,$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H_{11}) There exists an integrable function $m : J \to [0,\infty)$ such that

$$\|g(t,s,x)\| \leq m(s)\Omega_0(|x|), \quad 0 \leq s \leq t \leq b, \quad x \in Y,$$

where $\Omega_0 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H_{12}) There exist constants $M_1, M_2 > 0$ such that $\|\sigma B\|_{L(U,E)} \leq M_1$ and $\int_0^b \nu(t)dt \leq M_2$.

(H_{13}) The linear operator W from $L^2(J, U)$ into E defined by

$$Wu = \int_0^b [T(b-s)\sigma - AT(b-s)]Bu(s)ds$$

induces a bounded invertible operator $\tilde{W}$ defined on $L^2(J, U)/kerW$ and there exists a positive constant $M > 0$ such that $\|\tilde{W}^{-1}\| \leq M$(see [14]).

(H_{14}) $\displaystyle\int_0^b p(s)ds < \int_{a^*}^{\infty} \frac{ds}{\Omega(s) + \Omega_0(s)}$,

$$\begin{aligned}
\text{where } a^* &= K_1[|\phi| + M_0 b_1] + M_0 b_1 + \frac{K_2 b_1 b^\beta}{\beta} + (bK_1 M_1 + M_2)N, \\
p(t) &= \max\{K_1 q(t), m(t)\}, \quad M_0 = \|(-A)^{-\beta}\| \text{ and} \\
N &= M\Big[\|x_1\| + K_1[|\phi| + M_0 b_1] + M_0 b_1 + \frac{K_2 b_1 b^\beta}{\beta} \\
&\quad + K_1 \int_0^b q(s)\Omega[|x_s| + \int_0^s m(\tau)\Omega_0(|x_\tau|)d\tau]ds\Big].
\end{aligned}$$

3. Main Result

Theorem: If the hypotheses $(H_1) - (H_{14})$ are satisfied, then the boundary control neutral integrodifferential system (1) is controllable on J.

Proof: Using the hypothesis (H_{13}), for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned}
u(t) &= -\tilde{W}^{-1}[T(b)[\phi(0) - h(0,\phi)] + h(b, x_b) \\
&\quad + \int_0^b AT(b-s)h(s, x_s)ds + \int_0^b T(b-s)f(s, x_s, \int_0^s g(s, \tau, x_\tau)d\tau)ds](t).
\end{aligned}$$

Let $Y_b = C([-r, b]; E)$ be the Banach space endowed with the norm

$$|||x||| = \sup\{\|x(t)\|; \quad -r \leq t \leq b\}.$$

First we obtain a priori bounds for the following equation,

$$\begin{aligned}
x(t) &= \lambda T(t)[\phi(0) - h(0,\phi)] + \lambda h(t, x_t) + \lambda \int_0^t AT(t-s)h(s, x_s)ds \\
&\quad - \lambda \int_0^t [T(t-\eta)\sigma - AT(t-\eta)]B\tilde{W}^{-1}[T(b)[\phi(0) - h(0,\phi)] \\
&\quad + h(b, x_b) + \int_0^b AT(b-s)h(s, x_s)ds \\
&\quad + \int_0^b T(b-s)f(s, x_s, \int_0^s g(s, \tau, x_\tau)d\tau)ds]d\eta \\
&\quad + \lambda \int_0^t T(t-s)f(s, x_s, \int_0^s g(s, \tau, x_\tau)d\tau)ds.
\end{aligned}$$

We have

$$\begin{aligned}\|x(t)\| &\le K_1[|\phi|+M_0b_1]+M_0b_1+\frac{K_2b_1b^\beta}{\beta}+\int_0^t[K_1M_1+\nu(\eta)]M\Big[K_1[|\phi|+M_0b_1]\\ &\quad +M_0b_1+\frac{K_2b_1b^\beta}{\beta}+K_1\int_0^b q(s)\Omega[|x_s|+\int_0^s m(\tau)\Omega_0(|x_\tau|)d\tau]ds\Big]d\eta\\ &\quad +K_1\int_0^t q(s)\Omega[|x_s|+\int_0^s m(\tau)\Omega_0(|x_\tau|)d\tau]ds. \qquad (4)\end{aligned}$$

Consider the function β given by

$$\beta(t)=\sup\{\|x(s)\|;\ -r\le s\le t\},\quad 0\le t\le b.$$

Let $t^*\in[-r,t]$ be such that $\beta(t)=\|x(t^*)\|$. If $t^*\in J$, then by (4),

$$\begin{aligned}\beta(t) &\le K_1[|\phi|+M_0b_1]+M_0b_1+\frac{K_2b_1b^\beta}{\beta}+[bK_1M_1+M_2]N\\ &\quad +K_1\int_0^{t^*} q(s)\Omega[\beta(s)+\int_0^s m(\tau)\Omega_0(\beta(\tau))d\tau]ds\\ &\le K_1[|\phi|+M_0b_1]+M_0b_1+\frac{K_2b_1b^\beta}{\beta}+[bK_1M_1+M_2]N\\ &\quad +K_1\int_0^{t} q(s)\Omega[\beta(s)+\int_0^s m(\tau)\Omega_0(\beta(\tau))d\tau]ds. \qquad (5)\end{aligned}$$

If $t^*\in[-r,0]$, then $\beta(t)=|\phi|$ and the inequality (5) holds since $K_1\ge 1$. Denoting the right hand side of the above inequality by $v(t)$, we have

$$\beta(t)\le v(t),\quad 0\le t\le b,$$

$$a^*=v(0)=K_1[|\phi|+M_0b_1]+M_0b_1+\frac{K_2b_1b^\beta}{\beta}+(bK_1M_1+M_2)N$$

and

$$\begin{aligned}v'(t) &= K_1q(t)\Omega[\beta(t)+\int_0^t m(s)\Omega_0(\beta(s))ds]\\ &\le K_1q(t)\Omega[v(t)+\int_0^t m(s)\Omega_0(v(s))ds].\end{aligned}$$

Let $w(t)=v(t)+\int_0^t m(s)\Omega_0(v(s))ds$. Then $w(0)=v(0)$, $\quad v(t)\le w(t)$ and

$$\begin{aligned}w'(t) &= v'(t)+m(t)\Omega_0(v(t))\\ &\le K_1q(t)\Omega(w(t))+m(t)\Omega_0(w(t))\\ &\le p(t)[\Omega(w(t))+\Omega_0(w(t))].\end{aligned}$$

This implies $\displaystyle\int_{w(0)}^{w(t)}\frac{ds}{\Omega(s)+\Omega_0(s)}\le\int_0^b p(s)ds<\int_{a^*}^\infty\frac{ds}{\Omega(s)+\Omega_0(s)},\quad t\in J.$

The above inequality implies that there is a constant K^* such that $v(t) \leq K^*$, $t \in J$. Then $\beta(t) \leq K^*$ for all $t \in J$ and hence

$$|||x||| = \sup\{\|x(t)\|;\ -r \leq t \leq b\} \leq K^*,$$

where K^* depends only on b and on the functions m, Ω and Ω_0.
Let $Y_0 = \{x \in Y_b;\ x_0 = 0\}$. For $\phi \in Y$, define $\hat{z} \in Y_b$ by

$$\hat{z}(t) = \begin{cases} \phi(t), & -r \leq t \leq 0, \\ T(t)\phi(0), & 0 \leq t \leq b. \end{cases}$$

If $x(t) = \hat{z}(t) + y(t)$, $t \in [-r, b]$, it is easy to see that y satisfies

$$\begin{aligned} y_0 &= 0, \\ y(t) &= -T(t)h(0,\phi) + h(t, y_t + \hat{z}_t) + \int_0^t AT(t-s)h(s, y_s + \hat{z}_s)ds \\ &\quad - \int_0^t [T(t-\eta)\sigma - AT(t-\eta)]B\tilde{W}^{-1}\big[T(b)[\phi(0) - h(0,\phi)] \\ &\quad + h(b, x_b) + \int_0^b AT(b-s)h(s, y_s + \hat{z}_s)ds \\ &\quad + \int_0^b T(b-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds\big]d\eta \\ &\quad + \int_0^t T(t-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds, \quad 0 \leq t \leq b. \end{aligned}$$

We shall now prove that the operator $\Psi : Y_0 \to Y_0$ defined by

$$\begin{aligned} (\Psi y)(t) &= 0, \quad -r \leq t \leq 0, \\ &= -T(t)h(0,\phi) + h(t, y_t + \hat{z}_t) + \int_0^t AT(t-s)h(s, y_s + \hat{z}_s)ds \\ &\quad - \int_0^t [T(t-\eta)\sigma - AT(t-\eta)]B\tilde{W}^{-1}\big[T(b)[\phi(0) - h(0,\phi)] \\ &\quad + h(b, x_b) + \int_0^b AT(b-s)h(s, y_s + \hat{z}_s)ds \\ &\quad + \int_0^b T(b-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds\big]d\eta \\ &\quad + \int_0^t T(t-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds, \quad t \in J \end{aligned}$$

is a completely continuous operator. Let $B_R = \{y \in Y_0;\ |||y||| \leq R\}$ for some $R \geq 1$. First we show that Ψ maps B_R into an equicontinuous family. Let $y \in B_R$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq b$,

$$\begin{aligned} &\|(\Psi y)(t_1) - (\Psi y)(t_2)\| \\ &\leq \|T(t_1) - T(t_2)\|\|h(0,\phi)\| + \|h(t_1, y_{t_1} + \hat{z}_{t_1}) - h(t_2, y_{t_2} + \hat{z}_{t_2})\| \end{aligned}$$

$$
\begin{aligned}
&+\|\int_0^{t_1} A[T(t_1-s)-T(t_2-s)]h(s,y_s+\hat{z}_s)ds\| \\
&+\|\int_{t_1}^{t_2} AT(t_2-s)h(s,y_s+\hat{z}_s)ds\| \\
&+\|\int_0^{t_1}[T(t_1-\eta)-T(t_2-\eta)]\sigma B\tilde{W}^{-1}\big[T(b)[\phi(0)-h(0,\phi)] \\
&\quad +h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad +\int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\big]d\eta\| \\
&+\|\int_{t_1}^{t_2} T(t_2-\eta)\sigma B\tilde{W}^{-1}\big[T(b)[\phi(0)-h(0,\phi)] \\
&\quad +h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad +\int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\big]d\eta\| \\
&+\|\int_0^{t_1} A[T(t_1-\eta)-T(t_2-\eta)]B\tilde{W}^{-1}\big[T(b)[\phi(0)-h(0,\phi)] \\
&\quad +h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad +\int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\big]d\eta\| \\
&+\|\int_{t_1}^{t_2} AT(t_2-\eta)B\tilde{W}^{-1}\big[T(b)[\phi(0)-h(0,\phi)] \\
&\quad +h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad +\int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\big]d\eta\| \\
&+\|\int_0^{t_1}[T(t_1-s)-T(t_2-s)]f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\| \\
&+\|\int_{t_1}^{t_2} T(t_2-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\| \\
\le\ & \|T(t_1)-T(t_2)\|\|h(0,\phi)\|+\|h(t_1,y_{t_1}+\hat{z_{t_1}})-h(t_2,y_{t_2}+\hat{z_{t_2}})\| \\
&+\frac{K'b_1b^{\beta-\gamma}}{\beta-\gamma}(t_2-t_1)^\gamma+\frac{K_2b_1(t_2-t_1)^\beta}{\beta} \\
&+\int_0^{t_1}\|T(t_1-\eta)-T(t_2-\eta)\|M_1M\big[K_1\|\phi(0)-h(0,\phi)\| \\
&\quad +M_0b_1+\frac{K_2b_1b^\beta}{\beta}+K_1\int_0^b \mu_{k^*}(s)ds\big]d\eta \\
&+\int_{t_1}^{t_2}\|T(t_2-\eta)\|M_1M\big[K_1\|\phi(0)-h(0,\phi)\|
\end{aligned}
$$

$$
\begin{aligned}
&\quad +M_0 b_1 + \frac{K_2 b_1 b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s)ds\Big]d\eta \\
&+ \int_0^{t_1} \|A[T(t_1-\eta) - T(t_2-\eta)]B\| M\Big[K_1\|\phi(0) - h(0,\phi)\| \\
&\quad +M_0 b_1 + \frac{K_2 b_1 b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s)ds\Big]d\eta \\
&+ \int_{t_1}^{t_2} \|AT(t_2-\eta)B\| M\Big[K_1\|\phi(0) - h(0,\phi)\| \\
&\quad +M_0 b_1 + \frac{K_2 b_1 b^\beta}{\beta} + K_1 \int_0^b \mu_{k^*}(s)ds\Big]d\eta \\
&+ \int_0^{t_1} \|T(t_1-s) - T(t_2-s)\| \mu_{k^*}(s)ds + \int_{t_1}^{t_2} \|T(t_2-s)\| \mu_{k^*}(s)ds
\end{aligned}
$$

where $K' > 0$ and $k^* \geq R + \|\hat{z}\| + \|m\|_{L^1(0,b)}\Omega_0(R + \|\hat{z}\|)$. The right hand side is independent of $y \in B_R$ and tends to zero as $t_2 - t_1 \to 0$, since h is completely continuous and the compactness of $T(t)$, for $t > 0$ implies continuity in the uniform operator topology. Thus Ψ maps B_R into an equicontinuous family of functions.

Next we show that $\overline{\Psi B_R}$ is compact. Since we have shown ΨB_R is equicontinuous, by Ascoli Arzela theorem it suffices to show that Ψ maps B_R into precompact set in E. Let $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon \leq b$. For $y \in B_R$, we define

$$
\begin{aligned}
(\Psi_\epsilon y)(t) &= -T(t)h(0,\phi) + h(t, y_t + \hat{z}_t) + \int_0^{t-\epsilon} AT(t-s)h(s, y_s + \hat{z}_s)ds \\
&\quad - \int_0^{t-\epsilon} T(t-\eta)\sigma B\tilde{W}^{-1}\Big[T(b)[\phi(0) - h(0,\phi)] \\
&\qquad + h(b, y_b + \hat{z}_b) + \int_0^b AT(b-s)h(s, y_s + \hat{z}_s)ds \\
&\qquad + \int_0^b T(b-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds\Big]d\eta \\
&\quad + \int_0^{t-\epsilon} AT(t-\eta)B\tilde{W}^{-1}\Big[T(b)[\phi(0) - h(0,\phi)] \\
&\qquad + h(b, y_b + \hat{z}_b) + \int_0^b AT(b-s)h(s, y_s + \hat{z}_s)ds \\
&\qquad + \int_0^b T(b-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds\Big]d\eta \\
&\quad + \int_0^{t-\epsilon} T(t-s)f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau, y_\tau + \hat{z}_\tau)d\tau)ds \\
&= -T(t)h(0,\phi) + h(t, y_t + \hat{z}_t) + T(\epsilon)\int_0^{t-\epsilon} AT(t-s-\epsilon)h(s, y_s + \hat{z}_s)ds \\
&\quad - T(\epsilon)\int_0^{t-\epsilon} T(t-\eta-\epsilon)\sigma B\tilde{W}^{-1}\Big[T(b)[\phi(0) - h(0,\phi)] \\
&\qquad + h(b, y_b + \hat{z}_b) + \int_0^b AT(b-s)h(s, y_s + \hat{z}_s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\Big]d\eta \\
&+T(\epsilon)\int_0^{t-\epsilon} AT(t-\eta-\epsilon)B\tilde{W}^{-1}\Big[T(b)[\phi(0)-h(0,\phi)] \\
&\quad + h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad + \int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\Big]d\eta \\
&+T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds.
\end{aligned}
$$

Since $T(t)$ is compact, the set $\{\Psi_\epsilon y(t);\ y\in B_R\}$ is precompact in E for every ϵ, $0<\epsilon<t$. Moreover for every $y\in B_R$, we have

$$
\begin{aligned}
&\|(\Psi y)(t)-(\Psi_\epsilon y)(t)\| \\
&\le \int_{t-\epsilon}^t \|AT(t-s)h(s,y_s+\hat{z}_s)\|ds \\
&\quad + \int_{t-\epsilon}^t \|T(t-\eta)\sigma B\tilde{W}^{-1}\Big[T(b)[\phi(0)-h(0,\phi)] \\
&\qquad + h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad + \int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\Big]\|d\eta \\
&\quad + \int_{t-\epsilon}^t \|AT(t-\eta)B\tilde{W}^{-1}\Big[T(b)[\phi(0)-h(0,\phi)] \\
&\qquad + h(b,y_b+\hat{z}_b)+\int_0^b AT(b-s)h(s,y_s+\hat{z}_s)ds \\
&\quad + \int_0^b T(b-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds\Big]\|d\eta \\
&\quad + \int_{t-\epsilon}^t T(t-s)f(s,y_s+\hat{z}_s,\int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)ds \\
&\le \int_{t-\epsilon}^t \|AT(t-s)h(s,y_s+\hat{z}_s)\|ds \\
&\quad + \int_{t-\epsilon}^t \|T(t-\eta)\|M_1M\Big[K_1\|\phi(0)-h(0,\phi)\| \\
&\qquad + \|h(b,y_b+\hat{z}_b)\|+\int_0^b \|AT(b-s)h(s,y_s+\hat{z}_s)\|ds \\
&\qquad + \int_0^b \|T(b-s)\|\mu_{k^*}(s)ds\Big]d\eta \\
&\quad + \int_{t-\epsilon}^t \|AT(t-\eta)B\|M\Big[K_1\|\phi(0)-h(0,\phi)\| \\
&\qquad + \|h(b,y_b+\hat{z}_b)\|+\int_0^b \|AT(b-s)h(s,y_s+\hat{z}_s)\|ds
\end{aligned}
$$

$$+\int_0^b \|T(b-s)\|\mu_{k^*}(s)ds\Big]d\eta + \int_{t-\epsilon}^t \|T(t-s)\|\mu_{k^*}(s)ds.$$

Since there are precompact sets arbitrarily close to the set $\{(\Psi y)(t);\ y \in B_R\}$, the set is precompact in E. It remains to show that $\Psi : Y_0 \to Y_0$ is continuous. Let $\{y_n\}_0^\infty \subseteq Y_0$ with $y_n \to y$ in Y_0. Then there is an integer R such that $\|y_n(t)\| \le R$, for all n and $t \in J$, so $y_n \in B_R$. By (H_5) and (H_6),

$$f(t, y_{nt} + \hat{z}_t, \int_0^t g(t,s,y_{ns} + \hat{z}_s)ds) \to f(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s + \hat{z}_s)ds),$$

for each $t \in J$, and since

$$\|f(t, y_{nt} + \hat{z}_t, \int_0^t g(t,s,y_{ns} + \hat{z}_s)ds) - f(t, y_t + \hat{z}_t, \int_0^t g(t,s,y_s + \hat{z}_s)ds)\| \le 2\mu_{k^*}(t)$$

and also h is completely continuous, we have by the dominated convergence theorem,

$$\begin{aligned}
&|||\Psi y_n - \Psi y|||\\
=\ & \sup_{t\in J} \|[h(t, y_{nt} + \hat{z}_t) - h(t, y_t + \hat{z}_t)]\\
&+ \int_0^t AT(t-s)[h(s, y_{ns} + \hat{z}_s) - h(s, y_s + \hat{z}_s)]ds\\
&- \int_0^t [T(t-\eta)\sigma - AT(t-\eta)]B\tilde{W}^{-1}\big[h(b, y_{nb} + \hat{z}_b) - h(b, y_b + \hat{z}_b)\\
&\quad + \int_0^b AT(b-s)[h(s, y_{ns} + \hat{z}_s) - h(s, y_s + \hat{z}_s)]ds\\
&+ \int_0^b T(b-s)[f(s, y_{ns} + \hat{z}_s, \int_0^s g(s,\tau, y_{n\tau} + \hat{z}_\tau)d\tau)\\
&\quad - f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau,y_\tau + \hat{z}_\tau)d\tau)]ds\big]d\eta\\
&+ \int_0^t T(t-s)[f(s, y_{ns} + \hat{z}_s, \int_0^s g(s,\tau, y_{n\tau} + \hat{z}_\tau)d\tau)\\
&\quad - f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau,y_\tau + \hat{z}_\tau)d\tau)]ds\|\\
\le\ & \sup_{t\in J} \|h(t, y_{nt} + \hat{z}_t) - h(t, y_t + \hat{z}_t)\|\\
&+ \int_0^t \|AT(t-s)\|\|h(s, y_{ns} + \hat{z}_s) - h(s, y_s + \hat{z}_s)\|ds\\
&+ \int_0^t [\|T(t-\eta)\|\|\sigma B\| + \|AT(t-\eta)]B\|]\|\tilde{W}^{-1}\|\\
&\quad [\int_0^b \|AT(b-s)\|\|h(s, y_{ns} + \hat{z}_s) - h(s, y_s + \hat{z}_s)\|ds\\
&+ \int_0^b \|T(b-s)\|\|f(s, y_{ns} + \hat{z}_s, \int_0^s g(s,\tau, y_{n\tau} + \hat{z}_\tau)d\tau)\\
&\quad - f(s, y_s + \hat{z}_s, \int_0^s g(s,\tau,y_\tau + \hat{z}_\tau)d\tau)\|ds]d\eta
\end{aligned}$$

$$+\int_0^t \|T(t-s)\|\|f(s, y_{ns}+\hat{z}_s, \int_0^s g(s,\tau,y_{n\tau}+\hat{z}_\tau)d\tau)$$
$$-f(s, y_s+\hat{z}_s, \int_0^s g(s,\tau,y_\tau+\hat{z}_\tau)d\tau)\|ds \to 0, \text{ as } n \to \infty.$$

Thus Ψ is continuous and hence it is completely continuous.

Finally the set $\zeta(\Psi) = \{y \in Y_0;\ \ y = \lambda\Psi y,\ \ \lambda \in (0,1)\}$ is bounded, since for every solution y in $\zeta(\Psi)$, the function $x = y + \hat{z}$ is a mild solution of (3), for which we have proved that $|||x||| \leq K^*$ and hence $|||y||| \leq K^* + |\hat{z}|$. Hence by Schaefer's fixed point theorem the operator Ψ has a fixed point in Y_0 which satisfies $\Psi x(b) = 0$. Thus the system (1) is null controllable on J.

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