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On a Class of Reaction-Diffusion Systems Describing Bone Remodelling Phenomena

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Bone remodelling refers to the naturally occuring reformation of bone after some form of damage or trauma. Principally, two types of cell are involved in the processes which make up the metabolism of bone: *osteoclasts* and *osteoblasts*. Osteoclasts dissociate clacium through the secretion of acid, and degrade organic matter by releasing lysosomal enzymes, resulting in *resorption* of the bone. On the other hand, osteoblasts produce both *inorganic calcium phosphate*, which is converted to *hydroxyapatite*, and an organic matrix consisting mainly of *type I collagen*, depositing new bone at the site resorbed by osteclasts. A third kind of cell present in the bone, *osteocytes*, also play a part in the metabolism by sensing physical loads, and conveying signals to activate osteoblasts. In this way, the three types of cell interact and between them carry out the bone remodelling process.

The purpose of this paper is to investigate a new class of convective reaction-diffusion systems, which describe the phenomena outlined above. Study of this phenomena clearly has applications from a medical point of view, and is of particular importance in relation to research into bone diseases, such as osteoporosis, physiological studies of the internal architecture of bone, and design of dental implants.

Activation by osteocytes of formation and resorption processes is induced mainly by the density of calcium and the concentrations of osteclasts and osteoblasts in the marrow. Therefore we use these three parameters as the elements of the reaction-diffusion systems treated in this paper.

Let Ω be a bounded domain in $\mathbb{R}^n$, with sufficently smooth boundary $\partial\Omega$. The system of equations describing the bone remodelling phenomena is given below.

$$\text{(RDS)} \qquad \left\{\begin{aligned} u_t &= d_1\Delta u - e_1\mathbb{E}\cdot\nabla u + \gamma wu - \beta vu - c_1 u \\ v_t &= d_2\Delta v - e_2\mathbb{E}\cdot\nabla v + a_2\nabla u\cdot\nabla v + \varepsilon_2 uv - c_2 v \\ w_t &= d_3\Delta w + e_3\mathbb{E}\cdot\nabla w - a_3\nabla u\cdot\nabla w - \varepsilon_3 uw + c_3 w \end{aligned}\right.$$

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under the initial conditions $u(0,\cdot) = u_0$, $v(0,\cdot) = v_0$ and $w(0,\cdot) = w_0$. We impose homogeneous Neumann boundary conditions on u, v, and w, namely that

$$\text{(BC)} \qquad u_\nu = v_\nu = w_\nu = 0$$

on the boundary $\partial\Omega$, where ν represents the outward normal vector.

Firstly, $u = u(t,x)$ represents the concentration of calcium at location $x \in \overline{\Omega}$ and time $t > 0$. The cell densities of osteoblasts and osteoclasts are given by $v = v(t,x)$ and $w = w(t,x)$, respectively. The $\mathbb{R}^3$-valued function $\mathbb{E} \in C^1(\overline{\Omega})$ represents electric fields produced by stress-strain distribution through the bone. Diffusion effects on each of these quantities are given by $d_1\Delta u$, $d_2\Delta v$ and $d_3\Delta w$, where Δ represents the Laplace operator under 0-Neumann boundary conditions. Advection effects along the negative and positive directions of physical and chemical stimulation $\mathbb{E}$, are represented by $-e_1\mathbb{E}\cdot\nabla u$, $-e_2\mathbb{E}\cdot\nabla v$, and $e_3\mathbb{E}\cdot\nabla w$, for advection coefficients e_1, e_2 and e_3, where ∇ stands for the gradient operator. Similarly the terms $a_2\nabla v\cdot\nabla u$ and $a_3\nabla w\cdot\nabla u$ describe the advection effect on osteoblasts and osteoclasts along the gradient of the concentration of u.

The effect on the release of calcium by osteoclasts and the mineralization of calcium by osteoblasts may be expressed as $\gamma wu - \beta vu$, for some positive coefficients β and γ. The decrease and increase of calcium, osteoblasts and osteoclasts are represented as $-c_1u$, $-c_2v$ and c_3w.

This paper is organized as follows. In the first section we give some preliminary results such that appropriate convective diffusion operators can be formulated. In Section 2 we construct a new type of descrete scheme consistent with (RDS). The third section is concerned with obtaining some essential estimates, and finally in the fourth section we show the convergence of the scheme and thereby construct strong solutions.

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1 Formulation of convective diffusion and reaction operators

In preparation for constructing the scheme we recall the following well-known results. For more details, the reader is refered to [1], [6], [2] and [3].

1 Definition. In this section let $\mathbf{b} = (b_1,\ldots,b_n)$ be a function in $C^1(\overline{\Omega};\mathbb{R}^n)$, $d > 0$. Ω is assumed to be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. For $p > n$, Λ_p shall represent an operator of the form

$$\text{(1.1)} \qquad \Lambda_p z = d\Delta z + \mathbf{b}\cdot\nabla z \quad , \quad z \in D(\Lambda_p)$$

where

$$D(\Lambda_p) = \left\{ z \in W^{2,p} \mid z_\nu = 0 \text{ on } \partial\Omega \right\}. \tag{1.2}$$

Here ν represents the outward unit normal vector on the boundary $\partial\Omega$ of Ω. Note that if $p < q \leq L^\infty$ then, since we are considering bounded domains in $\mathbb{R}^n$, $L^p \supset L^q \supset L^\infty$, and $\Lambda_p \supset \Lambda_q \supset \Lambda_\infty$. Hence we shall often write just Λ when no confusion will occur.

2 Lemma. (Elliptic Estimates) *Let $v \in W^{2,p}$. Then there exists some constant C such that*

$$\|\nabla v\|_\infty \leq C(\|\Lambda v\|_p + \|v\|_p). \tag{1.3}$$

Here $C \equiv C(n, \Omega, d, K)$, where $|b_j| < K$.

Proof. This follows simply from well known estimates. Since

$$\|v\|_{W^{2,p}} \leq C'(\|\Lambda v\|_p + \|v\|_p),$$

we know that $\nabla v \in W^{1,p}$, and we have the embedding $W^{1,p} \hookrightarrow C^{1-n/p}(\overline{\Omega})$, the statement is obtained through the estimates

$$\|\nabla v\|_\infty + [\nabla v]_{1-n/p} \leq C'' \|\nabla v\|_{W^{1,p}} \leq C''' \|v\|_{W^{2,p}}.$$

Full details can be found in [6]. □

3 Lemma. *Consider the following parabolic initial boundary value problem in the space L^p.*

$$\begin{cases} u_t = \Lambda u \quad , \ t > 0, x \in \Omega \\ u_\nu = 0 \text{ on } \partial\Omega, \\ u(0,x) = v(x) \geq 0 \quad , \ x \in \Omega \end{cases} \tag{1.4}$$

where ν represents the outward normal. Here $v(\cdot)$ is assumed to belong to L^p. Then the following hold:

1. *There exists a strong solution $u \in W^{2,p}$ such that $\nabla u \in L^\infty$.*

2. *If $v(x) \geq 0$, $x \in \Omega$, then $u(t,x) \equiv u(t,x;v) \geq 0$ for all $t \geq 0$.*

3. *In fact $u \in C^2(\overline{\Omega})$, namely u is a* classical solution.

4 Proposition. *Let $1 < p < \infty$, Λ be as above, and let $\mathbf{F} : L^p \to (L^p)^*$ be the duality mapping from L^p into its dual. Then, the following estimate holds:*

$$\langle \Lambda z, \mathbf{F}(z) \rangle \leq \left(\frac{\|\mathbf{b}\|_\infty^2}{4d(p-1)} \right) \|z\|_p^2 \tag{1.5}$$

for $z \in D(\Lambda) \cap L^p{}_+$.

Proof. We begin by recalling that $(L^p)^* \equiv L^q$ for p and q such that $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, and that $\mathbf{F}(z)(x) = \|z\|_p^{2-p} |z(x)|^{p-2} z(x)$ for $x \in \Omega$, and

$$\langle \Lambda z, \mathbf{F}(z) \rangle = \int_\Omega \Lambda z . \mathbf{F}(z) dx. \tag{1.6}$$

Write $\langle \Lambda z, \mathbf{F}(z) \rangle = \langle d\Delta z, \mathbf{F}(z0\rangle + \langle \mathbf{b} \cdot \nabla z, \mathbf{F}(z) \rangle$. The contraction semigroup $\mathrm{e}^{t\Delta}$ is analytic and positive, preserving all L^p-norms for general p with $1 \leq p \leq \infty$, so that for $\eta > 0$, $\mathrm{e}^{\eta\Delta} z \in D(\Lambda) \cap C^\infty$. Thus $\mathbf{F}(z) \nabla \mathrm{e}^{\eta\Delta} z \in C^1$, and hence we may apply the divergence theorem to obtain

$$\begin{aligned}
\int_\Omega (\Delta \mathrm{e}^{\eta\Delta} z)(\mathbf{F}(z)) dx &= \int_\Omega \operatorname{div} \left(\mathbf{F}(z) \nabla \mathrm{e}^{\eta\Delta} z \right) dx - \int_\Omega (\nabla \mathrm{e}^{t\Delta} z) \cdot (\nabla \mathbf{F}(z)) dx \\
&= \int_{\partial\Omega} \mathbf{F}(z) \left(\nabla \mathrm{e}^{\eta\Delta} z \cdot \nu \right) dS - \int_\Omega (\nabla \mathrm{e}^{\eta\Delta} z) \cdot (\nabla \mathbf{F}(z)) dx \\
&= - \int_\Omega (\nabla \mathrm{e}^{\eta\Delta} z) \cdot (\nabla \mathbf{F}(z)) dx,
\end{aligned}$$

where $\int_{\partial\Omega} dS$ represents the integral over the surface of Ω, and as usual ν is the outward normal. Let η tend to 0. Then $\Delta \mathrm{e}^{\eta\Delta} z$ tends to Δz, and by the elliptic estimate of Lemma 2 we see that $\nabla \mathrm{e}^{t\Delta} z$ tends to ∇z. In other words, we have

$$\langle \Delta z, \mathbf{F}(z) \rangle = - \int_\Omega \nabla z \cdot \nabla \mathbf{F}(z) dx.$$

Noting that $\nabla \mathbf{F}(z) = (p-1)\left\|z\right\|_p^{2-p} |z|^{p-2}\nabla z$, we obtain

$$\begin{aligned}
\langle \Lambda z, \mathbf{F}(z)\rangle &= -d(p-1)\left\|z\right\|_p^{2-p} \int_\Omega |z(x)|^{p-2}|\nabla z(x)|^2 dx \\
&\quad + \left\|z\right\|_p^{2-p} \int_\Omega z(x)|z(x)|^{p-2}\mathbf{b}\cdot\nabla z(x) dx \\
&\leq -d(p-1)\left\|z\right\|_p^{2-p} \int_\Omega |z(x)|^{p-2}|\nabla z(x)|^2 dx \\
&\quad + \left\|z\right\|_p^{2-p} \int_\Omega |z(x)|^{p-2}|\mathbf{b}|.|\nabla z(x)|.|z(x)| dx \\
&\leq -d(p-1)\left\|z\right\|_p^{2-p} \int_\Omega |z(x)|^{p-2}\Big\{|\nabla z(x)|^2 \\
&\quad -d^{-1}(p-1)^{-1}|\mathbf{b}|.|\nabla z(x)|.|z(x)| + \frac{1}{4d^2(p-1)^2}|z(x)|^2|\mathbf{b}|^2\Big\} dx \\
&\quad + (4d(p-1))^{-1}\left\|\mathbf{b}\right\|_\infty^2 \left\|z\right\|_p^{2-p}\int_\Omega |z(x)|^p dx,
\end{aligned}$$

where the last inequality is obtained by simply adding and subtracting the term $(2d(p-1))^{-2}|z(x)|^2|\mathbf{b}|^2$ in the integrand. We then rewrite this in the following way to obtain the inequality in the statement of the proposition.

$$\begin{aligned}
\langle \Lambda z, \mathbf{F}(z)\rangle &\leq -d(p-1)\left\|z\right\|_p^{2-p} \int_\Omega |z(x)|^{p-2}\left\{|\nabla z(x)| - \frac{|\mathbf{b}|}{2d(p-1)}|z(x)|\right\}^2 dx \\
&\quad +(4d(p-1))^{-1}\left\|\mathbf{b}\right\|_\infty^2 \left\|z\right\|_p^{2-p}\int_\Omega |z(x)|^p dx \\
&\leq \left(\frac{\left\|\mathbf{b}\right\|_\infty^2}{4d(p-1)}\right)\left\|z\right\|_p^2 .
\end{aligned}$$

□

5 Lemma. *Suppose the assumptions of Proposition 4 hold, and let p be a positive real number such that $n < p < \infty$. Then Λ generates a positive analytic semigroup $e^{t\Lambda}$ in L^p. Futhermore,*

$$\left\|\Lambda e^{t\Lambda} z\right\|_p \leq \frac{M_p}{t}\left\|z\right\|_p \quad \text{for } z \in L^p. \tag{1.7}$$

Namely, for $v \in L^p$, we have $u(t) = e^{t\Lambda}v \in D(\Lambda)$. Moreover, $e^{t\Lambda}$ is a contraction semigroup in L^∞.

Proof. The only part of the statement which is not well known is the final remark concerning the preservation of the L^∞-norm. We refer the reader to [10], and in particular Chapter 2, to justify the following explanation:

Consider the parabolic domain $\Omega \times (0, \infty)$, and assume that the initial function $v(\cdot)$ belongs to L^∞. Firstly, there cannot exist maxima on the boundary of Ω taking values greater than $\|v\|_\infty$, since u would have derivative in any inward direction (including the direction along the inward normal of Ω) less than zero, contradicting the 0-Neumann boundary conditions. If at some time $t > 0$ the solution $e^{t\Lambda}v$ takes a maximal value over the set $\Omega \times (0, t^0)$ greater than $\|v\|_\infty$ at some $(x^0, t^0) \in D$, then u must take the same value on all of $\Omega \times (0, t^0)$, implying that v must also attain this value. This contradicts our original assumption on $\|v\|_\infty$. □

Finally, in preparation for construction of the scheme, we need the following.

6 Definition. Define the following operators on our space L^p, for $z \in W^{2,p}(\overline{\Omega})$.

$$\begin{aligned}
\Lambda_1 \quad &= d_1\Delta - e_1\mathbb{E} \cdot \nabla \\
\Lambda_2(z) \quad &= d_2\Delta - e_2\mathbb{E} \cdot \nabla + a_2\nabla z \cdot \nabla \\
\Lambda_3(z) \quad &= d_3\Delta + e_3\mathbb{E} \cdot \nabla - a_3\nabla z \cdot \nabla
\end{aligned}$$

2 Construction of descrete scheme

Essentially, our argument shall be carried out in the space L^∞, although with respect to the L^p norms, using the estimates obtained in the previous section. Convergence in the L^p topologies also implies convergence in L^1, however, and it is this space we are interested in, since the L^1 norm is most meaningful in a physical sense when we consider what the functions in our system actually represent.

We are now in a position to construct the appropriate descrete scheme for (RDS). Given a sufficiently small time spacing $h > 0$ we construct sequences $\{u^i\}$, $\{v^i\}$ and $\{w^i\}$ of elements in $L^1 \cap L^\infty$, such that $ih \leq \tau$ using the following scheme.

Let the functions $u^{k-1} \in C^2(\overline{\Omega})$, $v^{k-1}, w^{k-1} \in L^1 \cap L^\infty$ be given. By Lemma 3, we know that the operator Λ_1 generates a positive analytic quasi-contractive semigroup $T_1(\cdot)$ on L^p and that $T_1(t)u \in C^2(\overline{\Omega}) \subset L^\infty(\Omega)$ for $u \in L^p$. Furthermore, the resolvent $(I - h\Lambda_1)^{-1}$ exists on the whole space L^p. Let u^k be given by

$$u^k = [I - h\Lambda_1]^{-1} \left[1 + h\gamma w^{k-1} - h\beta v^{k-1} - hc_1\right] u^{k-1}. \tag{2.1}$$

At first glance it would seem natural to use the resolvents $[1 - h\Lambda_2(u^k)]^{-1}$ in the next step, however since we cannot guarantee that u^k is C^2, and hence $\nabla u^k \in C^1$, this resolvent may not exist. Applying the operator $T_1(h)$ to u^k allows us to generate a resolvent of an approximate operator, by Lemma 3, and continue generating the scheme in the following way:

$$v^k \;= \left[1 - h\Lambda_2(T_1(h)u^k)\right]^{-1} \left[I + h\varepsilon_2 u^{k-1} - hc_2\right] v^{k-1} \tag{2.2}$$

$$w^k \;= \left[1 - h\Lambda_3(T_1(h)u^k)\right]^{-1} \left[I - h\varepsilon_3 u^{k-1} + hc_3\right] w^{k-1} \tag{2.3}$$

7 Remark. Note that the operators $T_1(h)$, $(I - h\Lambda_1)^{-1}$, $\left[I - h\Lambda_2(T_1(h)u^k)\right]^{-1}$ and $\left[I - h\Lambda_3(T_1(h)u^k)\right]^{-1}$ are all positivity preserving. We claim that they also preserve the L^∞ norm. This is clear for $T_1(h)$ by the maximum principle arguments of Lemma 5, and we now show that this holds for the remaining operators. $\Lambda_2(T_1(h)u^k)$ and $\Lambda_3(T_1(h)u^k)$ also generate analytic semigroups, $T_2(\cdot)$ and $T_3(\cdot)$ which again preserve the L^∞ norm. Considering for the moment $\Lambda_2(T_1(h)u^k)$, its resolvent is given by the Laplace transform

$$\left[I - h\Lambda_2(T_1(h)u^k)\right]^{-1} v = h^{-1} \int_0^\infty \mathrm{e}^{-t/h} T_2(t) v dt,$$

where the integral is taken in L^p in the sense of Bochner. The mapping $\|\cdot\|_\infty$: $L^p \to [0, \infty]$ is a convex, lower semicontinuous functional, and therefore since $T_2(\cdot)v$ is continuous in L^p we have that $\|T_2(t)v\|_\infty$ is integrable with respect to t, and obtain

$$\begin{aligned}\left\| \left[I - h\Lambda_2(T_1(h)u^k)\right]^{-1} v \right\|_\infty &= \left\| h^{-1} \int_0^\infty \mathrm{e}^{-t/h} T_2(t) v dt \right\|_\infty \\ &\le h^{-1} \int_0^\infty \mathrm{e}^{-t/h} \|T_2(t)v\|_\infty \, dt \\ &\le \|v\|_\infty .\end{aligned}$$

We may proceed similarly for $\Lambda_3(u^k)$, and it is trivial to use this type of argument for $(I - h\Lambda_1)^{-1}$. •

3 Essential estimates

In this section we shall give some more estimates on various elements of the scheme described above, with a view to showing the convergence to a generalized solution of the equation (RDS). This will be done by evaluating error terms $\mathbf{f}^k$ in the difference approximation

$$\frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{h} - \mathbf{f}^k = (\mathscr{A} + \mathscr{B})\mathbf{u}^k, \tag{3.1}$$

where $\mathbf{u}^k$ denotes the element $\left(u^k\ ,\ v^k\ ,\ w^k\right) \in (L^p)^3$, for terms u^k, v^k and w^k generated in the scheme above. Operators $\mathscr{A}$ and $\mathscr{B}$ are defined by

$$\mathscr{A} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \Lambda_1 u \\ \Lambda_2(u)v \\ \Lambda_3(u)w \end{pmatrix} \quad , \quad \mathscr{B} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \gamma wu - \beta vu - c_1 u \\ \varepsilon_2 uv - c_2 v \\ -\varepsilon_3 uw + c_3 w \end{pmatrix} . \tag{3.2}$$

for $(u\ ,\ v\ ,\ w) \in D(\mathscr{A})$ or $D(\mathscr{B})$. Note that we define $D(\mathscr{A})$ as simply $D(\Lambda)^3$ or $D(\Lambda_1) \times D(\Lambda_2) \times D(\Lambda_3)$. Using these operators we can now rewrite the system (RDS) in the form

$$\begin{cases} \mathbf{u}'(t) = (\mathscr{A} + \mathscr{B})\mathbf{u} \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases} \tag{3.3}$$

Consistent with our scheme, define the following approximate operators to $\mathscr{A}$, for $h > 0$.

$$\mathscr{A}_h \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \Lambda_1 u \\ \Lambda_2 \left(T_1(h)u\right) v \\ \Lambda_3 \left(T_1(h)u\right) w \end{pmatrix} \quad \text{for} \quad D(\mathscr{A}_h) = D(\mathscr{A}). \tag{3.4}$$

The norms $\|\cdot\|_p$, for $1 \le p < \infty$, and $\|\cdot\|_\infty$ defined on the product spaces $(L^p)^3$ and $(L^\infty)^3$, respectively, shall be defined

$$\begin{aligned} \|(u\ ,\ v\ ,\ w)\|_p &= \left(\|u\|_p^p + \|v\|_p^p + \|w\|_p^p\right)^{\frac{1}{p}} \\ \|(u\ ,\ v\ ,\ w)\|_\infty &= \max\left\{\|u\|_\infty, \|v\|_\infty, \|w\|_\infty\right\} \end{aligned} \tag{3.5}$$

for elements $(u\ ,\ v\ ,\ w)$ in the appropriate product spaces. Then for $1 < p < \infty$, the dual spaces of $(L^p)^3$ are simply $(L^q)^3$ with similar norms, where $p^{-1} + q^{-1} = 1$. In a similarly natural way, the dual of $(L^1)^3$ is $(L^\infty)^3$, and the dual of $(L^\infty)^3$ is $(ba(\Omega))^3$. If it can be shown that

$$\lim_{h \downarrow 0} h \sum_{i=1}^{[\tau/h]} \|\mathbf{f}^i\|_p = 0 \tag{3.6}$$

then we can use an argument similar to that given in [5] to prove convergence of the scheme to a generalized solution to the problem (RDS).

8 Proposition. *Let u_0, v_0 and w_0 be functions in $L^1 \cap L^\infty$, such that $u_0(x)$, $v_0(x)$ and $w_0(x) \ge 0$ for almost every $x \in \Omega$. Then u^k, v^k and w^k generated as above, are all positive in the same sense, and are uniformly L^∞-bounded above.*

Proof. Noting Remark 7, at the end of the previous section, concerning the positivity preserving properties of the operators $(I - h\Lambda_1)^{-1}$, $\left[I - h\Lambda_2(T_1(h)u^k)\right]^{-1}$ and $\left[I - h\Lambda_3(T_1(h)u^k)\right]^{-1}$, we shall show the statement by means of an induction argument. Assume that for $i = 0, \ldots, k-1$ all elements u^i, v^i and w^i are all positive and belong to L^∞. Firstly,

$$\begin{aligned} w^k &\le \prod_{i=1}^{k-1} \left(1 - h\varepsilon_3 u^i + hc_3\right) w_0 \\ &\le \left(1 + hc_3\right)^k w_0 \\ &\le e^{hkc_3} w_0 \le e^{\tau c_3} w_0 \equiv \alpha_3. \end{aligned} \tag{3.7}$$

Note that α_3 is in fact a bound for all w^i, $i = 0, \ldots, k$. Next,

$$(3.8)\qquad \begin{aligned}(1 + h\gamma w^{k-1} - h\beta v^{k-1} - hc_1)u^{k-1} &\leq (1 + h\gamma w^{k-1})u^{k-1} \\ &\leq (1 + h\gamma\alpha_3)^k u_0 \leq \mathrm{e}^{\tau\gamma\alpha_3} \equiv \alpha_1.\end{aligned}$$

Again, the estimate holds for u^i, $i = 0, \ldots, k$. Now consider v^k,

$$(3.9)\qquad \begin{aligned}(1 + h\varepsilon_2 u^{k-1} - hc_2)v^{k-1} &\leq (1 + h\varepsilon_2\alpha_1)v^{k-1} \\ &\leq (1 + h\varepsilon_2\alpha_1)^k v_0 \leq \mathrm{e}^{\tau\varepsilon_2\alpha_1} v_0 \equiv \alpha_2.\end{aligned}$$

So we have three constants bounding the L^∞ norms of elements u^i, v^i and w^i for $i = 0, \ldots, k-1$ and bounding the values of the functions u^k, v^k and w^k above. It remains to show positivity of u^k, v^k and w^k. Define

$$(3.10)\qquad h_0 \leq \min\left\{\frac{1}{(\beta\alpha_2 + c_1)}, \frac{1}{c_2}, \frac{1}{\varepsilon_3\alpha_1}\right\},$$

noting that this is greater than zero, and let $0 < h < h_0$. Then

$$\begin{aligned}(1 + h\gamma w^{k-1} - h\beta v^{k-1} - hc_1)u^{k-1} &\geq \left(1 - h(\beta v^{k-1} + c_1)\right) u^{k-1} \\ &\geq (1 - h(\beta\alpha_2 + c_1)) \geq 0 \text{ for } 0 < h \leq h_0.\end{aligned}$$

$$(1 + h\varepsilon_2 u^{k-1} - hc_2)v^{k-1} \geq (1 - hc_2)v^{k-1} \geq 0 \text{ for } 0 < h \leq h_0.$$

$$\begin{aligned}(1 - h\varepsilon_3 u^{k-1} + hc_3)w^{k-1} &\geq (1 - h\varepsilon_3 u^{k-1})w^{k-1} \\ &\geq (1 - h\varepsilon_3\alpha_1)w^{k-1} \geq 0 \text{ for } 0 < h \leq h_0.\end{aligned}$$

This completes the proof of the statement. □

9 Remark. We have shown in Proposition 8 that there exists some $r > 0$ such that, for $0 < h < h_0$ all terms u^k, v^k and w^k are non-zero and have L^∞-norm bounded by r. It is also easily verified that each of the terms making up the operator $\mathscr{B}$ (namely the reaction terms in the equation itself) is at least bilinear and Lipschitz continuous on such L^∞-bounded sets. Hence, given $r > 0$ there exists some Lipschitz constant, which we denote by $m(r) > 0$, such that

$$(3.11)\qquad \|\mathscr{B}\mathbf{u} - \mathscr{B}\mathbf{v}\|_p \leq m(r)\,\|\mathbf{u} - \mathbf{v}\|_p$$

for all $\mathbf{u}$, $\mathbf{v} \in (L^p)^3$ with $\|\mathbf{u}\|_\infty$ and $\|\mathbf{v}\|_\infty \leq r$.

10 Lemma. *Let $\mathbf{u} \in D(\mathscr{A}) \cap (L^\infty)^3$. Then we have*

$$(3.12)\qquad \lim_{h\downarrow 0} \|\mathscr{A}_h\mathbf{u} - \mathscr{A}\mathbf{u}\|_p \to 0.$$

Proof. Let $\mathbf{u} = (u\ ,\ v\ ,\ w) \in D(\mathscr{A}) \cap (L^\infty)^3$. Then

$$\begin{aligned}\|\mathscr{A}_h \mathbf{u} - \mathscr{A}\mathbf{u}\|_p^p &= \|a_2 \nabla (T_1(h)u - u) \cdot \nabla v\|_p^p \\ &\quad + \|a_3 \nabla (T_1(h)u - u) \cdot \nabla w\|_p^p\end{aligned}$$

Lemma 2 implies that, since $\|\Lambda_2(u)v\|_p < \infty$,

$$\|\nabla v\|_\infty \leq C' \left(\|\Lambda_2(u)v\|_p + \|v\|_p \right) < \infty \tag{3.13}$$

for $q > n$. Note also that $\nabla v \in L^p$, and that u and $T_1(h)u$ are bounded uniformly in L^∞ for $0 < h < h_0$. It is sufficient to prove that

$$\|\nabla (T_1(h)u - u)\|_p \to 0$$

as h tends to zero. Employing the well known theory of fractional powers of closed linear operators (see, among others, [9] for a detailed treatment), we have

$$\|\nabla (T_1(h)u - u)\|_p \leq C_p'' \left\|(-\Lambda_1)^\theta (T_1(h)u - u)\right\|_p \text{ for } \tfrac{1}{2} < \theta < 1. \tag{3.14}$$

and

$$\left|(-\Lambda_1)^\theta T_1(h)\right| \leq \frac{M_{\theta,\tau,p}}{h^\theta}, \tag{3.15}$$

where $|\cdot|$ here represents the operator norm. Furthermore,

$$T_1(h)u - u = \int_0^h T_1(\xi)\Lambda_1 u d\xi$$

and so

$$\begin{aligned}\left\|(-\Lambda_1)^\theta (T_1(h)u - u)\right\|_p &= \left\| \int_0^h (-\Lambda_1)^\theta T_1(\xi)\Lambda_1 u d\xi \right\|_p \\ &\leq \int_0^h \left|(-\Lambda_1)^\theta T_1(\xi)\right| . \|\Lambda_1 u\|_p d\xi \\ &\leq \int_0^h \frac{M_{\theta,\tau}}{\xi^\theta} . \|\Lambda_1 u\|_p d\xi \\ &\leq M_{\theta,\tau,p} h^{1-\theta} . \|\Lambda_1 u\|_p \to 0 \quad \text{as } h \downarrow 0.\end{aligned}$$

□

The next step in our argument will be to show the boundedness of the terms $\|\mathscr{A}_h \mathbf{u}^k\|_p$ uniformly over h and k for $hk \leq \tau$.

11 Lemma. *Let h be such that $0 < h < h_0$, for h_0 as in equation (3.10), and let the elements $\mathbf{u}_h^k = (u_h^k, v_h^k, w_h^k)$ be generated through the scheme described previously, for $\mathbf{u}^0$ satisfying the conditions of the previous lemma. Then $\left\|\mathscr{A}_h \mathbf{u}_h^k\right\|_p$ is bounded uniformly over h and k, for $hk \leq \tau$.*

Proof. We have

$$\begin{aligned}
\left\|\mathbf{u}_h^k - \mathbf{u}_h^{k-1} - h(\mathscr{A}_h + \mathscr{B})\mathbf{u}_h^k\right\|_p &= \left\|(I - h\mathscr{A}_h)\mathbf{u}_h^k - \mathbf{u}_h^{k-1} - h\mathscr{B}\mathbf{u}_h^k\right\|_p \\
&= \left\|(I + h\mathscr{B})\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-1} - h\mathscr{B}\mathbf{u}_h^k\right\|_p \\
&= h\left\|\mathscr{B}\mathbf{u}_h^{k-1} - \mathscr{B}\mathbf{u}_h^k\right\|_p \\
&\leq hm(r). \left\|\mathbf{u}_h^{k-1} - \mathbf{u}_h^k\right\|_p
\end{aligned}$$

(Note also that this holds equally for the $\|\cdot\|_\infty$ norm). Assume for the moment that we can find an upper bound ω_p for all constants of quasi-dissipativity of the appropriate operators. Then,

$$\begin{aligned}
\left\|\mathbf{u}_h^k - \mathbf{u}_h^{k-1}\right\|_p &\leq \mathrm{e}^{h\omega_p}\left\|(I - h\mathscr{A}_h)\mathbf{u}_h^k - (I - h\mathscr{A}_h)\mathbf{u}_h^{k-1}\right\|_p \\
&= \mathrm{e}^{h\omega_p}\left\|(I + h\mathscr{B})\mathbf{u}_h^{k-1} - (I + h\mathscr{B})\mathbf{u}_h^{k-2}\right\|_p \\
&\leq \mathrm{e}^{h\omega_p}(1 + hm(r))\left\|\mathbf{u}_h^{k-1} - \mathbf{u}_h^{k-2}\right\|_p \\
&\leq \mathrm{e}^{kh(m(r)+\omega_p)}\left\|\mathbf{u}_h^1 - \mathbf{u}^0\right\|_p . \qquad (3.16)
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
\left\|\mathbf{u}_h^1 - \mathbf{u}^0\right\|_p &\leq \mathrm{e}^{h\omega_p}\left\|(I - h\mathscr{A}_h)\mathbf{u}_h^1 - (I - h\mathscr{A}_h)\mathbf{u}^0\right\|_p \\
&\leq \mathrm{e}^{h\omega_p}\left\|(I + h\mathscr{B})\mathbf{u}^0 - \mathbf{u}^0 + h\mathscr{A}_h\mathbf{u}^0\right\|_p \\
&\leq h\mathrm{e}^{h\omega_p}\left(\left\|\mathscr{B}\mathbf{u}^0\right\|_p + \left\|\mathscr{A}_h\mathbf{u}^0\right\|_p\right),
\end{aligned}$$

so that bringing all these things together,

(3.17)
$$\left\|\mathbf{u}_h^k - \mathbf{u}_h^{k-1} - h(\mathscr{A}_h + \mathscr{B})\mathbf{u}_h^k\right\|_p \leq h^2 m(r)\mathrm{e}^{(k+1)h(m(r)+\omega_p)}\left[\left\|\mathscr{B}\mathbf{u}^0\right\|_p + \left\|\mathscr{A}_h\mathbf{u}^0\right\|_p\right]$$

and this is uniformly bounded by the convergence of $\mathscr{A}_h$, as proven in Lemma 10. By Proposition 8 and Remark 9 we have also the uniform boundedness of $\mathbf{u}_h^k$ and $\mathscr{B}\mathbf{u}^k$, and so it remains only to show that the uniform constant ω_p exists.

Performing the estimates above for elements u_h^i, where $\mathbf{u}_h^k = (u_h^k, v_h^k, w_h^k)$, we see by Lemma 4 that since $\mathbb{E}$ is fixed, the constant of quasi-dissipativity for Λ_1 is fixed and elements $\Lambda_1 u_k^i$ are uniformly bounded. Thus terms $\|\nabla u_h^i\|_\infty$ are uniformly bounded by the elliptic estimate of Lemma 2. Since $\Lambda_1 T_1(h) u_h^i = T_1(h)\Lambda_1 u_h^i$ and $T_1(h)$ preserves the L^∞ norm, we can again use the elliptic estimate to show the boundedness of $\|\nabla T_1(h)u_h^i\|_\infty$ and hence we may choose a constant ω_h, as described above and the statment holds, by Lemma 4. $\square$

12 Corollary. *We have also obtained the following estimate, which shall be of importance later on.*

$$\left\|\mathbf{u}_h^k - \mathbf{u}_h^{k-1}\right\|_p \leq h e^{(k+1)h(m(\tau)+\omega_p)} \left[\|\mathscr{B}\mathbf{u}^0\|_p + \|\mathscr{A}_h\mathbf{u}^0\|_p\right]$$

Note that this estimate is also uniform over k and h for $0 < h < h_0$ and $hk \leq \tau$.

13 Lemma. *Let $\mathbf{u}^0 \in D(\mathscr{A}) \cap (L^p{}_+)^3 \cap (L^\infty)^3$. Then terms $\mathscr{A}_h\mathbf{u}_h^k$ converge uniformly to $\mathscr{A}\mathbf{u}_h^k$, for $0 < h < h_0$ and $hk \leq \tau$.*

Proof. Let $\mathbf{u}_i^h = (u_i^h, v_i^h, w_i^h)$. Recalling the elliptic estimate of Lemma 2 and Lemma 11 we obtain uniform boundedness of $\left\|\nabla v_h^k\right\|_\infty$, since we can find a uniform constant in the inequality. This is achieved by noting that $\|\Lambda_1 u_h^i\|_p \leq \|\mathscr{A}_h\mathbf{u}_h^i\|_p$, and hence $\left\|\nabla u_i^h\right\|_\infty$, is uniformly bounded. A similar argument holds for w_i^h. Retracing our steps through the proof of lemma 10 we see that $\left\|\nabla\left(T_1(h)u_h^k - u_h^k\right)\right\|_p$ converges to zero uniformly by noting that $\left\|\Lambda_1 u_h^k\right\|_p \leq \|\mathscr{A}_h\mathbf{u}_h^i\|_p$ is uniformly bounded. Hence the statement holds. $\square$

4 Convergence of the scheme

Armed with the above lemmas we can now deal with the main estimates for the error terms in (3.1). Namely, we wish to show that (3.6) holds for the scheme we have defined, and that the operator $\mathscr{A} + \mathscr{B}$ is quasi-dissipative over the set $\{\mathbf{u}_h^k\}$, for $0 < h < h_0$ and $kh \leq \tau$. In fact, local quasi-dissipativity of $\mathscr{A} + \mathscr{B}$ is immediate, since we have Lemma 4 and as we have already seen, $\mathscr{B}$ is Lipschitz continuous on L^∞-bounded sets.

14 Theorem. *Let $\mathbf{u}_h^k$ be generated as described above for $0 < h < h_0$ and $hk \leq \tau$, where the initial conditions $\mathbf{u}^0$ satisfy $\|\mathbf{u}^0\|_\infty < \infty$ and $\|\mathscr{A}\mathbf{u}^0\|_\infty < \infty$. Then the elements $\mathbf{f}_h^k$ in the equation*

$$\frac{\mathbf{u}_h^k - \mathbf{u}_h^{k-1}}{h} - \mathbf{f}_h^k = (\mathscr{A} + \mathscr{B})\mathbf{u}_h^k \tag{4.1}$$

satisfy

$$\lim_{h\downarrow 0} h \sum_{k=0}^{[\tau/h]} \left\|\mathbf{f}_h^k\right\|_p = 0 \tag{4.2}$$

Proof. Assume that the element $\mathbf{u}_h^k$ may be denoted by (u_h^k, v_h^k, w_h^k). For notational concision we define the operators in the respective components of the definition of $\mathscr{B}(u,v,w)$ to be $\mathscr{B}_1(v,w)u$, $\mathscr{B}_2(u)v$ and $\mathscr{B}_3(u)w$. The reader will verify the appropriate (in)dependencies and also that, by Remark 9, each of these is locally Lipschitz continuous in each of its variables.
We consider the components seperately, starting with the first. We have

$$\begin{aligned}\left\|\frac{u_h^k - u_h^{k-1}}{h} - (\Lambda_1 + \mathscr{B}_1(v_h^k, w_h^k))u_h^k\right\|_p \\ &= h^{-1}\left\|(I - h\Lambda_1)u_h^k - u_h^{k-1} - h\mathscr{B}_1(v_h^k, w_h^k)u_h^k\right\|_p \\ &= h^{-1}\left\|(I + h\mathscr{B}_1(v_h^{k-1}, w_h^{k-1}))u_h^{k-1} - u_h^{k-1} - h\mathscr{B}_1(v_h^k, w_h^k)u_h^k\right\|_p \\ &\leq \left\|\mathscr{B}_1(v_h^{k-1}, w_h^{k-1})u_h^{k-1} - \mathscr{B}_1(v_h^k, w_h^k)u_h^k\right\|_p\end{aligned}$$

The uniform boundedness of all elements, and the estimate in Remark 12 (i) imply that the above converges to zero, and in fact

$$\begin{aligned}\left\|\frac{u_h^k - u_h^{k-1}}{h} - (\Lambda_1 + \mathscr{B}_1(v_h^k, w_h^k))u_h^k\right\|_p \\ &\leq hm(r)\mathrm{e}^{(\tau+h)(m(r)+\omega_p)}\left(\|\mathscr{B}\mathbf{u}^0\|_p + \|\mathscr{A}_h\mathbf{u}^0\|_p\right)\end{aligned} \tag{4.3}$$

Turning now to the second component, we may argue as follows:

$$\begin{aligned}\left\|\frac{v_h^k - v_h^{k-1}}{h} - (\Lambda_2(u_h^k) + \mathscr{B}_2(u_h^k))v_h^k\right\|_p \\ &= h^{-1}\left\|(I - h\Lambda_2(u_h^k))v_h^k - v_h^{k-1} - h\mathscr{B}(u_h^k)v_h^k\right\|_p\end{aligned} \tag{4.4}$$

Writing $(I - h\Lambda_2(u_h^k))$ as $(I - h\Lambda_2(T_1(h)u_h^k)) - [(I - h\Lambda_2(T_1(h)u_h^k)) - (I - h\Lambda_2(u_h^k))]$ it follows that the above term is bounded by

$$\begin{aligned}&\left\|\mathscr{B}_2(u_h^{k-1})v_h^{k-1} - \mathscr{B}_2(u_h^k)v_h^k\right\|_p \\ &\qquad + h^{-1}\left\|\left[\left(I - h\Lambda_2\left(T_1(h)u_h^k\right)\right) - \left(I - h\Lambda_2\left(u_h^k\right)\right)\right] v_h^k\right\|_p \\ &\leq \left\|\mathscr{B}_2(u_h^{k-1})v_h^{k-1} - \mathscr{B}_2(u_h^k)v_h^k\right\|_p + a_2\left\|\nabla(T_1(h)u_h^k - u_h^k)\cdot\nabla v_h^k\right\|_p.\end{aligned}$$

Recalling the elliptic estimate of Lemma 2 and Lemma 11 we obtain uniform boundedness of $\|\nabla v_h^k\|_\infty$. We use the estimate

$$\|\nabla v_h^k\|_\infty \leq C'\left(\|\Lambda_2(T_1(h)u_h^k)v_h^k\|_p + \|v_h^k\|_p\right), \tag{4.5}$$

and the fact that $\left\|\mathscr{A}_h \mathbf{u}_h^k\right\|_p$ is uniformly bounded. However, to maintain rigour we must also note the dependence of the constant C' on the term u_h^k, which of course may vary. In order to find a uniform constant we appeal to the fact that $\left\|\nabla u_h^k\right\|_\infty$ is uniformly bounded and choose C' sufficiently large (see Lemma 2). As in the proof of lemma 10, we see that $\left\|\nabla\left(T_1(h)u_h^k - u_h^k\right)\right\|_p$ converges to zero by noting that $\left\|\Lambda_1 u_h^k\right\|_p$ is uniformly bounded. Thus so does the expression in equation (4.4). In fact we have

$$
\begin{aligned}
&\left\|\frac{v_h^k - v_h^{k-1}}{h} - (\Lambda_2(u_h^k) + \mathscr{B}_2(u_h^k))v_h^k\right\|_p \leq m(r)\left\|v_h^k - v_h^{k-1}\right\|_p && (4.6)\\
&\qquad + a_2.C\left\|\nabla\left(T_1(h)u_h^k - u_h^k\right)\right\|_p \\
&\qquad \leq C_1.h + C_2.h^{1-\theta}. && (4.7)
\end{aligned}
$$

Making a similar estimate for the third component, i.e. terms w_h^k, which can be done in almost exactly the same way, we see that equation (4.2) holds true. □

An argument resembling very closely that given in [7] or [4], can now be applied to show convergence of the scheme to a generalized solution to the system (RDS). We obtain, therefore, the following proposition.

15 Proposition. *Let $\mathbf{u}_\lambda^i$ and $\mathbf{u}_\mu^j$ be generated as described above, for $0 < \lambda, \mu < h_0$, where $\mathbf{u}^0 \in D(\mathscr{A})$. Let r and $m(r)$ be defined as in Remark 9, and let $\hat{\mathbf{u}} \in D(\mathscr{A})$. Then the following estimate holds*

$$
\begin{aligned}
&(1-\lambda m(r))^{-i}(1-\mu m(r))^{-j}\left\|\mathbf{u}_i^\lambda - \mathbf{u}_j^\mu\right\| \\
&\qquad \leq 2\left\|\mathbf{u}_0 - \hat{\mathbf{u}}\right\| + \left\{(\lambda i - \mu j)^2 + \lambda^2 i + \mu^2 j\right\}^{\frac{1}{2}} . \left\|(\mathscr{A} + \mathscr{B})\hat{\mathbf{u}}\right\| && (4.8)\\
&\qquad + \sum_{k=1}^{i} \lambda\left\|\mathbf{f}_k^\lambda\right\| + \sum_{k=1}^{j} \mu\left\|\mathbf{f}_k^\mu\right\|.
\end{aligned}
$$

Combining this with the estimates we have obtained for the difference terms $\mathbf{f}_\lambda^i$ and $\mathbf{f}_\mu^j$, we obtain a Cauchy sequence converging to a continuous function $\mathbf{u}(\cdot) : [0, \tau] \to D$, in the following sense

$$
\mathbf{u}(t) = \lim_{\substack{h \downarrow 0 \\ ih \to t}} \mathbf{u}_i^h
$$

It remains to show regularity, and this is done using the results of [8].

16 Proposition. *The solution $\mathbf{u}(t)$ obtained above is in fact C^1 in the space $(L^p)^3$. In other words, $\mathbf{u}(t) \in C^1([0,\infty); (L^p)^3)$.*

Proof. Let $\mathbf{u}_1 = (u_1\ ,\ v_1\ ,\ w_2)$ and $\mathbf{u}_2 = (u_2\ ,\ v_2\ ,\ w_2)$ be elements of the schemes described above, for possibly different time-spacings h. We shall use the notation

$$\|\mathbf{u}\|_\theta = \left\|\left((-\triangle)^\theta u\ ,\ (-\triangle)^\theta v\ ,\ (-\triangle)^\theta w\right)\right\|_p \quad ,\ \mathbf{u} = (u\ ,\ v\ ,\ w) \in (L^p)^3$$

to represent the graph-norm of $(-\triangle)^\theta$. Note that this *stronger* norm satisfies

$$\|\mathbf{u}\|_p \le C\,\|\mathbf{u}\|_\theta \tag{4.9}$$

for some constant C. We shall next divide up the operator $\mathscr{A} + \mathscr{B}$ in a different way, namely $A + \Psi$, for

(4.10)

$$A\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} d_1\Delta u \\ d_2\Delta v \\ d_3\Delta w \end{pmatrix} \quad \text{and} \quad \Psi\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -e_1\mathbb{E}\cdot\nabla u \\ -e_2\mathbb{E}\cdot\nabla v + a_2\nabla u\cdot\nabla v \\ e_3\mathbb{E}\cdot\nabla w - a_3\nabla u\cdot\nabla w \end{pmatrix} + \mathscr{B}\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

It follows that, since $\|\nabla u_i\|_\infty$, $\|\nabla v_i\|_\infty$ and $\|\nabla w_i\|_\infty$ are all uniformly bounded,

$$\begin{aligned}
\|\Psi\mathbf{u}_1 - \Psi\mathbf{u}_2\|_p \ &\le \|\mathscr{B}\mathbf{u}_1 - \mathscr{B}\mathbf{u}_2\|_p + \|\nabla u_1\cdot\nabla v_1 - \nabla u_2\cdot\nabla v_2\|_p \\
&\quad + \|\nabla u_1\cdot\nabla w_1 - \nabla u_2\cdot\nabla w_2\|_p \\
&\le m(r)\,\|\mathbf{u}_1 - \mathbf{u}_2\|_p \\
&\quad + \|\nabla u_1\cdot(\nabla v_1 - \nabla v_2)\|_p + \|(\nabla u_1 - \nabla u_2)\cdot\nabla v_2\|_p \\
&\quad + \|\nabla u_1\cdot(\nabla w_1 - \nabla w_2)\|_p + \|(\nabla u_1 - \nabla u_2)\cdot\nabla w_2\|_p \\
&\le m(r)\,\|\mathbf{u}_1 - \mathbf{u}_2\|_p + \\
&\quad + \|\nabla u_1\|_\infty\cdot\|\nabla v_1 - \nabla v_2\|_p + \|\nabla v_2\|_\infty\cdot\|\nabla u_1 - \nabla u_2\|_p \\
&\quad + \|\nabla u_1\|_\infty\cdot\|\nabla w_1 - \nabla w_2\|_p + \|\nabla w_2\|_\infty\cdot\|\nabla u_1 - \nabla u_2\|_p
\end{aligned}$$

Next we use (4.9) and the estimate (3.14) for fractional powers, as employed in the proof of Lemma 10, to obtain

$$\|\Psi\mathbf{u}_1 - \Psi\mathbf{u}_2\|_p \le C'.\,\|\mathbf{u}_1 - \mathbf{u}_2\|_\theta \tag{4.11}$$

for some constant C'. To satisfy the hypothesis of [8] it remains to show an exponential-type growth condition, but this is satisfied automatically since we have the estimates (3.7) through (3.9) for $t \in [0, \tau]$. Thereby the regularity theorem of [8] holds and we obtain the statement of the proposition. □

The differentiability with respect to time of the function $\mathbf{u}(\cdot)$, and the fact that it is a mild solution to the evolution equation (3.3) imply that the derivative is in fact equal to $(\mathscr{A} + \mathscr{B})\mathbf{u}(t)$ in the sense of L^p and hence L^1 (see [7]). In summary then, we have the following theorem.

17 Theorem. *Let* $\mathbf{u}_0 = (u_0\ ,\ v_0\ ,\ w_0) \in D(\mathscr{A}) \cap (L^\infty)^3$. *Then the equation* (RDS) *has a strong solution* $\mathbf{u}(t) = (u(t,\cdot)\ ,\ v(t,\cdot)\ ,\ w(t,\cdot))$, *which is the limit of the scheme defined in Section 2.*

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