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Nevanlinna-type spaces on the upper half plane

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Abstract

In this paper, we shall define the Smirnov class $\mathfrak{N}_*$ and its subspace $\mathfrak{N}^p$, $p > 1$, on the upper half plane and show some properties of $\mathfrak{N}_*$ and a canonical factorization theorem for $\mathfrak{N}^p$.

0. Introduction.

Let U be the unit disk in the complex plane and T the unit circle. The class N^p, $p > 1$, is the class of all of holomorphic functions f on U which satisfy

$$\sup_{0<r<1} \int_T \left(\log^+ |f(r\zeta)|\right)^p d\sigma(\zeta) < +\infty,$$

where $d\sigma$ denotes the normalized Lebesgue measure on T. Letting $p = 1$, we have the Nevanlinna class N. It is well-known that each function f in N has the nontangential limit $f^*(\zeta) = \lim_{r \to 1^-} f(r\zeta)$ (a.e. $\zeta \in T$).

We let the Smirnov class, N_*, consist of all holomorphic functions f on U such that $\log(1 + |f(z)|) \leqq Q[\phi](z)$ $(z \in U)$ for some $\phi \in L^1(T)$, $\phi \geqq 0$, where the right side means the Poisson integral in U.

The class N^p, $p > 1$, lies between the Hardy spaces H^q $(0 < q \leqq \infty)$ and N_*, i.e., we have $H^q \subset N^p \subset N_*$ $(0 < q \leqq \infty, p > 1)$. These including relations are proper. The notion of N^p was introduced by Stoll [7] and has been explored by several authors. N and its subspaces (N_*, N^p and H^q) are called *Nevanlinna-type spaces.*

Let $D := \{z \in \mathbf{C} \mid \mathrm{Im} z > 0\}$. Krylov [3] introduced the Nevanlinna class $\mathfrak{N}$ on D as follows:

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A holomorphic function f on D is said to belong to the class $\mathfrak{N}$ if there holds

$$\sup_{y>0} \int_{\mathbf{R}} \log^+ |f(x+iy)| \, dx < +\infty.$$

In this paper, we shall define the Smirnov class $\mathfrak{N}_*$ and the class $\mathfrak{N}^p$ $(p > 1)$ on D, analogous to the definitions of N_* and N^p $(p > 1)$, that is, we let $\mathfrak{N}_*$ consist of all holomorphic functions f on D such that $\log^+ |f(z)| \leqq P[\phi](z)$ $(z \in D)$ for some $\phi \in L^1(\mathbf{R})$, $\phi \geqq 0$, where the right side means the Poisson integral on D, and we let $\mathfrak{N}^p$, $p > 1$, consist of all holomorphic functions f on D such that

$$\sup_{y>0} \int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p dx < +\infty.$$

First we obtain some properties of the class $\mathfrak{N}_*$. Moreover, a factorization theorem for $\mathfrak{N}^p$ is also given.

1. Preliminaries.

Let ν be a real measure on T and $\Psi(z) = (z-i)/(z+i)$ $(z \in \overline{D})$. Then there corresponds a finite real measure μ on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} h(t) \, d\mu(t) = \int_{T^*} (h \circ \Psi^{-1})(\eta) \, d\nu(\eta) \quad (h \in C_c(\mathbf{R})),$$

where $T^* = T \setminus \{1\}$. Let $H(w, \eta) = (\eta + w)/(\eta - w)$ $((w, \eta) \in U \times T)$. There holds

$$\text{(1.1)} \qquad \frac{1}{i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \, d\mu(t) = \int_{T^*} H(\Psi(z), \eta) \, d\nu(\eta)$$

$$= \int_{T} H(\Psi(z), \eta) \, d\nu(\eta) - i\alpha z \quad (z \in D),$$

where $\alpha = -\nu(\{1\})$. We write Poisson integrals as follows:

$$P[\mu](z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{y}{(x-t)^2 + y^2} \, d\mu(t) \quad (z = x + iy \in D),$$

$$Q[\nu](w) = \int_T \frac{1-|w|^2}{|\eta - w|^2}\, d\nu(\eta) \quad (w \in U).$$

Taking the real parts in (1.1), we have

$$P[\pi(1+t^2)d\mu(t)](z) = Q[\nu](\Psi(z)) + \alpha \cdot \mathrm{Im} z \quad (z \in D). \tag{1.2}$$

2. Some properties on N, N_* and N^p.

In this section, we collect some properties on Nevanlinna-type spaces on the unit disk. For the following results, the reader is referred to [1], [5] and [7].

Proposition 2.1 *Let $f \in N$, $f \not\equiv 0$. Then f can be factored as follows,*

$$f(z) = \frac{aB(z)F(z)S_1(z)}{S_2(z)},$$

where $a \in T$,

$B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$ *$(z \in U)$ is a Blaschke product determined by the zeros of f,*

$$F(z) = \exp\left(\int_T \frac{\zeta + z}{\zeta - z} \log|f^*(\zeta)|\, d\sigma(\zeta)\right),$$

$$S_j(z) = \exp\left(-\int_T \frac{\zeta + z}{\zeta - z} d\nu_j(\zeta)\right) \quad (j = 1, 2)$$

(ν_j are positive singular measures, and ν_1 and ν_2 are mutually singular). Except for the choice of the constant $a \in T$, the factorization is unique.

Remark that when $f \in N_*$, $\nu_2 = 0$, i.e., $S_2(z) \equiv 1$.

Proposition 2.2 *Let $f \in N^p$ $(p > 1)$, $f \not\equiv 0$. Then f can be expressed as follows,*

$$f(z) = aB(z)F(z)S(z),$$

where a, $B(z)$ and $F(z)$ are same as those in Proposition 2.1. Moreover, $S(z)$ corresponds to $S_1(z)$.

Proposition 2.3 *Let $f \in N$ and $p > 1$. Then $f \in N^p$ if and only if $(\log^+ |f|)^p$ has the harmonic majorant.*

Proposition 2.4 *Let $f \in N$. Then $\log^+ |f|$ has the least harmonic majorant $Q[\log^+ |f^*| + \nu_2]$.*

Proposition 2.5 *Let $f \in N^p$. Then $(\log^+ |f|)^p$ has the least harmonic majorant $Q[(\log^+ |f^*|)^p]$.*

3. The Nevanlinna class $\mathfrak{N}$.

In this section, we shall describe some properties of the class $\mathfrak{N}$. The following theorems are Krylov's results [3]:

Theorem 3.1 *Let $f \in \mathfrak{N}$, $f \neq 0$. Then f is factored, uniquely, in the form*

$$(3.1) \qquad f(z) = ae^{i\alpha z}b(z)d(z)g(z) \quad (z \in D),$$

where the factors above have the following properties:

(i) $a \in T$, $\alpha \geqq 0$.

(ii) *$b(z)$ is the Blaschke product with respect to the zeros of f.*

(iii) $d(z) = \exp\left(\frac{1}{\pi i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\frac{1}{1+t^2}\log h(t)\,dt\right)$,

where $h(t) \geqq 0$, $\log h \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$ and $\log^+ h \in L^1(\mathbf{R})$.

(vi) $g(z) = \exp\left(\frac{1}{i}\int_{\mathbf{R}} \frac{1+tz}{t-z}\,d\mu(t)\right)$, *where μ is a finite real measure on $\mathbf{R}$, singular with respect to the Lebesgue measure, and such that*

$$\int_{\mathbf{R}} (1+t^2)\,d\mu^+(t) < \infty.$$

If f is expressed in the form (3.1), then $f \in \mathfrak{N}$.

Theorem 3.2 *A function $f \in \mathfrak{N}$ has the following properties:*
(i) *The nontangential limit $f^*(x)$ exists for* a.e. $x \in \mathbf{R}$.
(ii) $$\sup_{y>0} \int_{\mathbf{R}} \log^+ |f(x+iy)|\, dx = \lim_{y\to 0+} \int_{\mathbf{R}} \log^+ |f(x+iy)|\, dx$$
$$= \int_{\mathbf{R}} \log^+ |f^*(x)|\, dx + \frac{1}{2}\int_{\mathbf{R}} \pi(1+x^2)\, d\mu^+(x).$$

Proofs of Theorems 3.1 and 3.2. See [3].

4. The Smirnov class $\mathfrak{N}_*$.

Recall that a convex function φ on $\mathbf{R}$ is *strongly convex* if φ is nonnegative, nondecreasing and $\lim_{t\to\infty} \dfrac{\varphi(t)}{t} = \infty$.

Theorem 4.1 *Let $f \in \mathfrak{N}$, $f \neq 0$. Then the following assertions are mutually equivalent:*
(i) $f \in \mathfrak{N}_*$.
(ii) $f \circ \Psi^{-1} \in N_*$.
(iii) $\mu \leqq 0$, *in the factorization (3.1).*
(iv) $$\lim_{y\to 0+} \int_{\mathbf{R}} \log^+ |f(x+iy)|\, dx = \int_{\mathbf{R}} \log^+ |f^*(x)|\, dx.$$
(v) *There exists a strongly convex function φ such that*
$$\sup_{y>0} \int_{\mathbf{R}} \varphi(\log^+ |f(x+iy)|)\, dx < \infty.$$
(vi) *The family $\{\log^+ |f(x+iy)|\}_{y>0}$ is uniformly integrable on* $\mathbf{R}$.

Proof. That (i), (ii), (iii) and (iv) are mutually equivalent is the same as [4, Corollary 2.4].

Suppose $f \in \mathfrak{N}_*$. Then Proposition 2.4 implies $\log^+ |(f \circ \Psi^{-1})(w)| \leqq Q[\log^+ |(f\circ\Psi^{-1})^*|](w)$, hence $\log^+ |f(z)| \leqq P[\log^+ |f^*|](z)$. We have, for a strongly convex function φ,
$$\varphi(\log^+ |f(z)|) \leqq \varphi(P[\log^+ |f^*|](z)) \leqq P[\varphi(\log^+ |f^*|)](z),$$
where we utilize Jensen's inequality [5, p.31]. Therefore $\int_{\mathbf{R}} \varphi(\log^+ |f(x+iy)|)\, dx < \infty$. Hence the family $\{\log^+ |f(x+iy)|\}_{y>0}$ is uniformly integrable

on $\mathbf{R}$ by [5, Theorem 3.10]. Conversely, [5, Theorem 3.10] shows that if the family $\{\log^+|f(x+iy)|\}_{y>0}$ is uniformly integrable on $\mathbf{R}$, there exists a strongly convex function φ such that $\sup_{y>0}\int_{\mathbf{R}}\varphi(\log^+|f(x+iy)|)\,dx<\infty$.

Remark. For each strongly convex function φ on $\mathbf{R}$, we define H_φ to be the class of all holomorphic functions f on D for which $\sup_{y>0}\int_{\mathbf{R}}\varphi(\log^+|f(x+iy)|)\,dx<\infty$. Then we note that

$$\mathfrak{N}_* = \bigcup\{H_\varphi \mid \varphi : \text{strongly convex}\}.$$

5. The class $\mathfrak{N}^p$, $p>1$.

Let $p>1$. We define $\varphi(t)$ on $\mathbf{R}$ by $\varphi(t)=t^p$ for $t\geqq 0$, and equal to zero for $t<0$. Note that this function φ is strongly convex. Then the class H_φ coincides with the class $\mathfrak{N}^p$. Therefore we have $\mathfrak{N}^p\subset\mathfrak{N}_*$ $(p>1)$.

We easily have the following proposition by [6, Chapter II, Theorem 4.6].

Proposition 5.1 *Let $p>1$ and $f\in\mathfrak{N}^p$. Then we have the following:*

(i) $(\log^+|f|)^p$ *has the least harmonic majorant* $P[\tau]$, *where τ is a finite real measure on* $\mathbf{R}$.

(ii) $\|\tau\|\leqq \sup_{y>0}\int_{\mathbf{R}}\left(\log^+|f(x+iy)|\right)^p dx.$

(iii) *Let* $D_\delta=\{z\in\mathbf{C}\mid \mathrm{Im}z>\delta\}$. *Then* $\log^+|f(z)|\to 0$ *as* $|z|\to+\infty$ $(z\in\overline{D_\delta})$, *for each* $\delta>0$.

Using the above proposition, we observe that $\mathfrak{N}^p$ has the following properties:

Theorem 5.2 *A function $f\in\mathfrak{N}^p$ has the following properties:*

(i) $f\circ\Psi^{-1}\in N^p$.

(ii)
$$\sup_{y>0}\int_{\mathbf{R}}\left(\log^+|f(x+iy)|\right)^p dx=\lim_{y\to 0+}\int_{\mathbf{R}}\left(\log^+|f(x+iy)|\right)^p dx$$
$$=\int_{\mathbf{R}}\left(\log^+|f^*(x)|\right)^p dx.$$

Proof. Suppose $f \in \mathfrak{N}^p$. Then $f \circ \Psi^{-1} \in N^p$ by Proposition 2.3 and part (i) in Proposition 5.1.

If $f \in \mathfrak{N}^p$, then

$$\sup_{y>0} \int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p dx = \lim_{y \to 0+} \int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p dx$$

by part (iii) in Proposition 5.1 and [2, Theorem 1]. Moreover, $f \circ \Psi^{-1} \in N^p$ and Proposition 2.5 show that $(\log^+ |f^*|)^p$ has the least harmonic majorant $P[(\log^+ |f^*|)^p]$, hence we have

$$\int_{\mathbf{R}} \left(\log^+ |f^*(x)|\right)^p dx \leqq \sup_{y>0} \int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p dx$$

by part (ii) in Proposition 5.1. Finally, $f \circ \Psi^{-1} \in N^p$ and Proposition 2.5 imply

$$\sup_{y>0} \int_{\mathbf{R}} \left(\log^+ |f(x+iy)|\right)^p dx \leqq \int_{\mathbf{R}} \left(\log^+ |f^*(x)|\right)^p dx.$$

Theorem 5.3 *Let $p > 1$. Then $f \in \mathfrak{N}^p$, $f \neq 0$, is expressed in the form*

$$(5.1) \qquad f(z) = a e^{i\alpha z} b(z) d(z) g(z) \quad (z \in D),$$

where the factors above have the following properties:

(i) $a \in T$, $\alpha \geqq 0$.

(ii) *$b(z)$ is the Blaschke product with respect to the zeros of f.*

(iii) $d(z) = \exp\left(\frac{1}{\pi i} \int_{\mathbf{R}} \frac{1+tz}{t-z} \frac{1}{1+t^2} \log h(t)\, dt\right)$,

where $h(t) \geqq 0$, $\log h \in L^1(\mathbf{R}, (1+t^2)^{-1} dt)$ and $\log^+ h \in L^p(\mathbf{R})$.

(iv) $g(z) = \exp\left(\frac{1}{i} \int_{\mathbf{R}} \frac{1+tz}{t-z}\, d\mu(t)\right)$, *where μ is a finite real measure on $\mathbf{R}$, singular with respect to the Lebesgue measure, and*

$\int_{\mathbf{R}} (1+t^2)\, d\mu^+(t) < \infty.$

If f is expressed in the form (5.1), then $f \in \mathfrak{N}^p$.

Proof. Let $f \in \mathfrak{N}^p$, $f \neq 0$. Then we have

$$(f \circ \Psi^{-1})(w) = aB(w)F(w)S(w) \quad (w \in U)$$

by Proposition 2.2. In the factorization $f(z) = aB(\Psi(z))F(\Psi(z))S(\Psi(z))$ $(z \in D)$, $b(z) := B(\Psi(z))$ is the Blaschke product with respect to the zeros of f, and the change of the variables $\eta = \Psi(t)$ $(t \in \mathbf{R})$ shows that $d(z) := F(\Psi(z))$ is of the form (iii). Note that, since $\log|(f \circ \Psi^{-1})^*| \in L^1(T)$, we have $\log|f^*| \in L^1(\mathbf{R}, (1+t^2)^{-1}dt)$

Now in the proof of [4, Theorem 2.1], $\nu_2 = 0$ since $f \circ \Psi^{-1} \in N^p$, so $S(\Psi(z)) := S_1(\Psi(z)) = g(z)e^{i\alpha z}$, where g is of the form (iv). Since f belongs to $\mathfrak{N}$, we observe that $\alpha \geq 0$ and that $\int_{\mathbf{R}} (1+t^2)\, d\mu^+(t) < \infty$ by Theorem 3.1.

Suppose, conversely, that f is of the form (5.1). Then

$$|f(z)| = |e^{i\alpha z}||b(z)| \exp(P[\log h + \pi(1+t^2)d\mu(t)](z)).$$

Let ν_0 be the measure on T concentrated on $\{1\}$ and $\nu_0(\{1\}) = -\alpha$. Letting $z = \Psi^{-1}(w)$, we have $|\exp(i\alpha z)| = \exp(\mathrm{Re}(-\alpha(1+w)/(1-w))) = \exp(Q[\nu_0](w))$.

Moreover, μ determines a singular measure ν on $T \setminus \{1\}$. Hence we see that $P[\pi(1+t^2)d\mu(t)](z) = Q[\nu](w)$ by (1.2). Therefore

$$|(f \circ \Psi^{-1})(w)| = |B(w)| \exp(Q[\log(h \circ \Psi^{-1}) + \nu + \nu_0](w)) \quad (w \in U).$$

Since $\log^+|(f \circ \Psi^{-1})(w)| \leq Q[\log^+(h \circ \Psi^{-1}) + \nu^+](w)$, we have $f \circ \Psi^{-1} \in N^p$. Letting $y \to 0^+$ in $|f(x+iy)|$, we see that $|f^*(x)| = h(x)$ for a.e. $x \in \mathbf{R}$.

By the way, $(\log^+|f \circ \Psi^{-1}|)^p$ has the least harmonic majorant $v' = Q[\{\log^+|(f \circ \Psi^{-1})^*|\}^p]$ by Proposition 2.5, $v := v' \circ \Psi$ is the least harmonic majorant of $(\log^+|f|)^p$, that is, $(\log^+|f(z)|)^p \leq P[(\log^+|f^*|)^p](z)$. Integrating the both sides, we have $f \in \mathfrak{N}^p$.

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