On the abstract semi-linear differential equation

By

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The present paper is concerned with the abstract semi-linear differential equation

(1) $$du/dt+A(t)u=F(t, u),\ 0\leqq t\leqq T,$$

in a Banach space X, where the unknown $u(t)$ and the given function $F(t, u)$ in $[0, T]\times X$ take values in X and where $\{A(t),\ 0\leqq t\leqq T]\}$ is a family of not necessarily bounded operator acting in X. This equation was treated by K. Asano [1] when $A(t)$ does not depend on t: $A(t)=A$. The main object of the present article is to show that his method can be applied to this equation when $-A(t)$, $0\leqq t\leqq T$ are infinitesimal generators of analytic semi-groups $\exp(-sA(t))$ of bounded linear operators on X which have the properties (I), (II) and (III) stated below.

We are also interested in finding sufficient conditions on $F(t, u)$ under which the solution of (1) exists in some sense or other. In order to construct a strict solution of (1) we had to assume among other things the strong Hölder continuity of $F(t, A(t)^{-\alpha}p)$ in $t\in[0, T]$ for $p\in X$ with some positive α, which seems to be rather restrictive. It is possible, however, to construct approximate solutions to (1) replacing this assumption with weaker one.

1. Preliminaries.

We first state the assumptions to be made throughout this paper. By $D(A)$ and $R(A)$ we denote the domain and the range of an operator A.

(I) For each $t\in[0, T]$, $A(t)$ is a densely defined closed linear operator in X. The resolvent set of $A(t)$ contains a fixed closed sector $\sum=\{\lambda: arg\ \lambda\notin(-\theta, \theta)\}$, $0<\theta<\pi/2$ and the resolvent of $A(t)$ satisfies $\|(\lambda-A(t))^{-1}\|\leqq M/|\lambda|$ for any $t\in\sum$, where θ and M are constants independent of t and λ;

(II) $A(t)^{-1}$ is continuously differentiable in t in the uniform operator topology;

(III) There exists a positive number $\rho\geqq 1$ such that $R(dA(t)^{-1}/dt)\subset D(A(t)^{\rho})$

and $A(t)^\rho dA(t)^{-1}/dt$ is strongly cotinuous in $t\in[0, T]$. Hence with some positive constant N independent of t we have $\|A(t)^\rho dA(t)^{-1}/dt\| \leq N$.

Under these assumptions the fundamental solution $U(t, s)$, $0\leqq s\leqq t\leqq T$ of the equation $du/dt+A(t)u=0$, $0\leqq t\leqq T$ is constructed as follows:

$$U(t, s)=\exp(-(t-s)A(t))+W(t, s),\ W(t, s)=\int_s^t \exp(-(t-\sigma)A(t))R(\sigma, s)d\sigma,$$

$$R(t, s)=\sum_{m=1}^{\infty}R_m(t, s),\ R_m(t, s)=\int_s^t R_1(t, \sigma)R_{m-1}(\sigma, s)d\sigma,\ m=2, 3,\cdots\cdots,$$

$$R_1(t, s)=-(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(t))=\frac{-1}{2\pi i}\int_\Gamma e^{-\lambda(t-s)}(\partial/\partial t)(\lambda-A(t))^{-1}d\lambda,$$

$$\exp(-sA(t))=\frac{-1}{2\pi i}\int_\Gamma e^{-\lambda s}(\lambda-A(t))^{-1}d\lambda,\ s>0,$$

where Γ is a smooth contour running in Σ from $\infty e^{-\theta i}$ is $\infty e^{\theta i}$. For the details, see [2] and [3].

As to fractional powers of $A(t)$, by (I), $A(t)^\beta$, $0<\beta<1$ is well defined by

$$A(t)^\beta=(A(t)^{-\beta})^{-1},\ A(t)^{-\beta}=\frac{\sin\pi\beta}{\pi}\int_0^\infty \lambda^{-\beta}(\lambda+A(t))^{-1}d\lambda.$$

Next we assume on $F(t, u)$ the following condition:

(IV) $F(t, v(t))$ is a function defined on

$$\{(t, v(t)):\ v(t)\in D(A(t)^\alpha),\ t\in[0, T]\}$$

into X for some constant α with $0<\alpha<\rho$ and satisfies

$$\|E(t, A(t)^{-\alpha}p)\|\leqq f(\|p\|)$$

$$\text{and } \|F(t,A(t)^{-\alpha}p)-F(t, A(t)^{-\alpha}q)\|\leqq g(\|p\|+\|v\|)\|p-q\|$$

for p, $q\in X$ and $t\in[0, T]$, where f and g are non-decreasing continuous functions on $[0, \infty)$ to $[0, \infty)$.

But we don't know whether the condition $\alpha<\rho$ is essential or not.

2. Existence and uniqueness of the weak solution.

In this section we consider the following abstract integral equation associated (1):

$$u(t)=U(t, 0)\varphi+\int_0^t U(t, s)F(s, u(s))ds,\ 0\leqq t\leqq T. \tag{2}$$

To solve this equation by successive approximation we assume

(V) $F(t, v(t))$ is strongly measurable in $t\in[0, T]$ if $v(t)\in D(A(t)^\alpha)$ and if $A(t)^\alpha v(t)$ is strongly continuous in $t\in[0, T]$.

We first prove the following

THOREME 1. *Under the assumptions (I)-(V) there exists, for every* $\varphi\in D(A(0)^\alpha)$, *one and only one solution $u(t)$ of (2) in $[0, T_0]$ and*

(i) *$u(t)$ is strongly continous in $[0, T_0]$,*

(ii) *$u(t) \in D(A(t)^\alpha)$ for each $t \in [0, T_0]$ and $A(t)^\alpha u(t)$ is strongly continuous in $[0, T_0]$,*

whbre T_0 is a constant with $0 < T_0 \leqq T$ depending only on $\|A(0)^\alpha \varphi\| + \|\varphi\|$.

We call $u(t)$ a mild soulution of (2) in $[0, T_0]$.

Proof. We put $u_0(t) = U(t, 0)\varphi$. Then $u_0(t)$ belongs to $D(A(t)^\alpha)$ and

$$A(t)^\alpha u_0(t) = \frac{-1}{2\pi i}\int_\Gamma \lambda^\alpha e^{-t}\{(\lambda - A(t))^{-1} - (\lambda - A(0))^{-1}\}\varphi d\lambda$$

$$+\exp(-tA(0))A(0)^\alpha \varphi + \int_0^t A(t)^\alpha \exp(-(t-s)A(t))R(s, 0)\varphi ds.$$

Noting $\alpha < \rho$ we can see that $A(t)^\alpha u_0(t)$ is strongly continuous in $[0, T]$ and

$$\|A(t)^\alpha u_0(t)\| \leqq C(\|A(0)^\alpha \varphi\| + \|\varphi\|) = a_0.$$

In what follows, various constants depending only on T, θ, M, ρ, N and α are denoted by C.

By the assumption (V), $u_k(t)$, $k=0, 1, \cdots\cdots$ can be defined for $t \in [0, T]$ step by step as follows:

$$(3) \qquad \begin{cases} u_0(t) = U(t, 0)\varphi, \\ u_k(t) = u_0(t) + \int_0^t U(t, s)F(s, u_{k-1}(s))ds, \ k=1, 2, \cdots\cdots. \end{cases}$$

As is easily seen,

$$\begin{cases} \|A(t)^\alpha u_0(t)\| \leqq a_0, \\ \|A(t)^\alpha u_k(t)\| \leqq a_0 + C\int_0^t (t-s)^{-\alpha} f(\|A(s)^\alpha u_{k-1}(s)\|)ds, \ k=1, 2, \cdots\cdots. \end{cases}$$

Hence there exist positive numbers a and T_0 with $0 < T_0 \leqq T$ depending only on $\|A(0)^\alpha \varphi\| + \|\varphi\|$ such that

$$\|A(t)^\alpha u(t)\| \leqq a \text{ for } t \in [0, T_0] \text{ and } k=0, 1, \cdots\cdots.$$

From

$$\begin{cases} \|A(t)^\alpha (u_1(t) - u_0(t))\| \leqq C \cdot f(a)(1-\alpha)^{-1} t^{1-\alpha}, \\ \|A(t)^\alpha (u_{k+1}(t) - u_k(t))\| \leqq C \cdot g(2a)\int_0^t (t-s)^{-\alpha}\|A(s)^\alpha (u_k(s) - u_{k-1}(s))\| ds, \end{cases}$$

it follows immediately that

$$\|A(t)^\alpha (u_{k+1}(t) - u_k(t))\| \leqq \frac{f(a) \cdot (C \cdot g(2a)\Gamma(1-\alpha)t^{1-\alpha})^{k+1}}{g(2a) \cdot \Gamma((k+1)(1-\alpha)+1)}, \quad k=0, 1, \cdots\cdots$$

Thus $A(t)^\alpha u_n(t)$ converges uniformly on $[0, T_0]$ in the strong topology as $n \to \infty$ and so does $u_n(t)$ because of the uniform boundedness of $A(t)^{-\alpha}$.

Putting

$$s-\lim_{n \to \infty} u_n(t) = u(t)$$

and passing to the limit in (3), we can conclude without difficulty that $u(t)$ is a mild solution of (2) in $[0, T_0]$ with the desired properties.

In order to complete the proof it remains to show the uniqueness of the solution.

Let $u(t)$ and $v(t)$ be mild solutions of (2) in $[0, T'_0]$ $(0<T'_0\leqq T)$.

Putting

$$b(t)=\sup_{0<s<t}\|A(s)^{\alpha}(u(s)-v(s))\|,$$

$$K=C\cdot g(\sup_{0<s<T'_0}\|A(s)^{\alpha}u(s)\|+\sup_{0<s<T'_0}\|A(s)^{\alpha}v(s)\|),$$

we get

$$b(t)\leqq K\int_0^t(t-s)^{-\alpha}b(s)ds\leqq\frac{(K\Gamma(1-\alpha)t^{1-\alpha})^{k+1}}{\Gamma((k+1)(1-\alpha)+1)},\quad k=0,\ 1,\ \cdots\cdots,$$

which implies $b(t)=0$ on $[0, T'_0]$.

3. Approximate solutions.

In this section we investigate the behaviour of the solution $u_n(t)$ of the equation

$$(4)_n \qquad du/dt+A(t)u=(I+n^{-1}A(t)^{\gamma})^{-1}F(t,\ u),\quad 0\leqq t\leqq T$$

with the initial value $u_n(0)=\varphi\in D(A(0)^{\alpha})$ as $n\to\infty$. Here n and γ are arbitrary natural number and a positive constant with $\gamma\leqq 1$.

$F_n(t,\ u)=(I+n^{-1}A(t)^{\gamma})^{-1}F(t,\ u)$ satisfies

$$\|F_n(t,\ A(t)^{-\alpha}p\|\leqq M\cdot f(\|p\|)$$

and

$$\|F_n(t,\ A(t)^{-\alpha}p)-F_n(t,\ A(t)^{-\alpha}q)\|\leqq M\cdot g(\|p\|+\|q\|)\|p-q\|$$

for p, $q\in X$ and $t\in[0, T]$ with the aid of $\|(I+n^{-1}A(t)^{\gamma})^{-1}\|\leqq M$.

Now we assume

(VI) $F(t,\ A(t)^{-\alpha}p)$ is strongly continuous in $t\in[0, T]$ for $p\in X$.

Obviously the assumptions (IV) and (VI) imply that if $v(t)\in D(A(t)^{\alpha})$ and if $A(t)^{\alpha}v(t)$ be strongly continuous in $t\in[0, T]$, then $F(t,\ v(t))$ is strongly continuous in $t\in[0, T]$.

By Theorem 1, for every natural number n and $\varphi\in D(A(0)^{\alpha})$ there exists a unique mild solution $u_n(t)$ of the equation

$$u(t)=U(t,\ 0)\varphi+\int_0^t U(t,\ s)F_n(s,\ u(s))ds,\quad 0\leqq t\leqq T$$

in $[0, T_1]$ satisfying $\|A(t)^{\alpha}u_n(t)\|\leqq b$, where T_1 and b are constants with $0<T_1\leqq T$ and $0<b$ depending only on $\|A(0)^{\alpha}\varphi\|+\|\varphi\|$ but not on n.

The equality

$$\int_0^t A(t)U(t, s)F_n(s, u_n(s))ds$$

$$=\int_0^t \{A(t)\exp(-(t-s)A(t))-A(s)\exp(-(t-s)A(s))\} F_n(s, u_n(s))ds$$

$$+\int_0^t A(s)^{1-r}\exp(-(t-s)A(s))A(s)^r(I+n^{-1}A(s)^r)F(s, u_n(s))ds$$

$$+\int_0^t A(t)W(t, s)F_n(s, u_n(s))ds$$

implies that $u_n(t)$ belongs to $D(A(t))$ and is continuously diffdrentiable in $t\in[0, T_1]$ in the strong topology. Furthermore $u_n(t)$ satisfies

$$u_n/dt+A(t)u_n(t)=F_n(t, u_n(t)) \text{ with } u_n(0)=\varphi.$$

We are now in a position to state

THEOREM 2. *Under the assumptions (I)-(IV) and (VII), there exists a unique solution $u_n(t)$ of $(4)_n$ in $[0, T_1]$ $(0<T_1\leqq T)$ with the initial value $u_n(0)=\varphi\in D(A(0)^\alpha)$.*

Moreover, if $(I+n^{-1}A(t)^r)^{-1}$ converges to I uniformly on $[0, T]$ in the strong topology as $n\to\infty$, then $u_n\to u(t)$ uniformly on $[0, T_0]\cap[0\ T_1]$, *where $u(t)$ is the unique solution of (2) in $[0, T_0]$.*

PROOF. We have only to prove the last half part. $u_n(t)$ may be given by

$$u_n(t)=\overset{K}{u_n}(t)+\sum_{k=K}^{\infty}(\overset{k+1}{u_n}(t)-\overset{k}{u_n}(t)),$$

where

$$\begin{cases} u_n{}^0(t)=U(t, 0)\varphi, \\ u_n{}^k(t)=U(t, 0)\varphi+\int_0^t U(t, s)F_n(s, u_n{}^{k-1}(s))ds, \quad k=1, 2, \cdots\cdots \end{cases}$$

On the other hand, from (3) $u(t)$ is expressed as

$$u(t)=u_K(t)+\sum_{k=K}^{\infty}(u_{k+1}(t)-u_k(t)).$$

Therefore we obtain

$$\|A(t)^\alpha(u_n(t)-u(t))\|\leqq\|A(t)^\alpha(u_n{}^K(t)-u_K(t))\|$$

$$+\sum_{k=K}^{\infty}\frac{f(b)\cdot(C\cdot g(2b)\Gamma(1-\alpha)t^{1-\alpha})^{k+1}}{g(2b)\cdot\Gamma)(k+1)(1-\alpha)+1)}+\sum_{k=K}^{\infty}\frac{f(a)\cdot(C\cdot g(2a)\Gamma(1-\alpha)t^{1-\alpha})^{k+1}}{g(2a)\cdot\Gamma((k+1)(1-\alpha)+1)}$$

for $t\in[0, T_0]\cap[0, T_1]$.

For any given $\varepsilon>0$ a natural number $K=K(\varepsilon)$ dependent only on ε can be chosen so that for any $t\in[0, T_0]\cap[0\ T_1]$ the second and third terms on the right hand side of the above inequality may be dominated by $\varepsilon/4$.
$F(t, u_{k-1}(t))$, $k=1, 2, \cdots\cdots, K$ are strongly continuous on $[0, T_0]\cap[0, T_1]$ and hence

$$C^k=\{F(t, u_{k-1}(t)) : t\in[0, T_0]\cap[0, T_1]\},\ k=1, 2, \cdots\cdots, K(\varepsilon)$$

are compact subsets of X depending only on ε.

Noting

$$\begin{cases} \|A(t)^{\alpha}(u^1{}_n(t)-u_1(t))\| \leqq \int_0^t C(t-s)^{-\alpha}\| \{(I+n^{-1}A(s)^{r})^{-1}-I\} F(s, U(s, 0)\varphi)\| ds, \\ \|A(t)^{\alpha}(u_n^k(t)-u_k(t))\| \leqq \int_0^t C(t-s)^{-\alpha} g(a+b) \|A(s)^{\alpha}(u_n^{k-1}(s)-u_{k-1}(s))\| ds \end{cases}$$

$$+\int_0^t C(t-s)^{-\alpha}\| \{(I+n^{-1}A(s)^{r})^{-1}-I\} F(s, u_{k-1}(s))\| ds,$$

we can show by induction that for any $k\in\{1, 2, \cdots\cdots, K\}$

$$A(t)^{\alpha}u^k{}_n(t) \longrightarrow A(t)^{\alpha}u_k(t)$$

uniformly on $[0, T_0]\cap[0, T_1]$ as $n\to\infty$.

In other words, there exists a natural number $N=N(\varepsilon, K(\varepsilon))$ depending only on ε such that $\|A(t)^{\alpha}(u_n{}^K(t)-u_K(t))\|<\varepsilon/2$
and hence $\|A(t)^{\alpha}(u_n(t)-u(t))\|<\varepsilon$ for any $n\geqq N$ and $t\in[0, T_0]\cap[0, T_1]$.
Thus the proof is completed in such a way that was often used in [4].

4. Existence of the strict solution.

We begin with the proof of a preparatory lemma.

LEMMA. (i)*For any $\beta>\alpha$ and $\Psi\in D(A(0)^{\beta})$, $A(t)^{\alpha}U(t, 0)\Psi$ is strongly Hölder continuous in* $[0, T]$,

(ii) *For a strongly measurable and bounded function $w(t)$ on* $[0, T]$ *to* X

$\int_0^t A(t)^{\alpha}U(t, \sigma)w(\sigma)\, d\sigma$ *is strongly Hölder continuous in* $[0, T]$.

PROOF. From

$$A(t)^{1+r}\, U(t, s)=A(t)^{1+r} \exp(-(t-s)A(t))$$

$$+\int_s^t \{A(t)^{1+r} \exp(-(t-\sigma)A(t))-A(\sigma)^{1+r} \exp(-(t-\sigma)A(\sigma))\} R(\sigma, s)\, d\sigma$$

$$+\int_s^t A(\sigma)^{1-(\rho-r)/2} \exp(-(t-\sigma)A(\sigma))A(\sigma)^{(\rho+r)/2}R(\sigma, s)\, d\sigma$$

and

$$(\partial/\partial t)\{A(t)^{\alpha}U(t, s)\}=\frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\lambda^{\alpha}A(t)^{1-\rho}(\lambda+A(t))^{-1}A(t)^{\rho}dA(t)^{-1}/dt\cdot(\lambda+A(t))^{-1}d\lambda$$

$$\times A(t)U(t, s)-\frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\lambda^{\alpha-1}A(t)^{1-r}(\lambda+A(t))^{-1}d\lambda\cdot A(t)^{1+r}U(t, s)$$

for r with $\alpha<r<\rho$, we have

$$\|(\partial/\partial t)\{A(t)^{\alpha}U(t, s)\}\|\leqq C\|A(t)^{1+r_0}U(t, s)\|\leqq C(t-s)^{-r_0-1},\ r_0=(\alpha+\rho)/2.$$

(ii) is a direct consequence of

$$\left\|\int_0^t A(t)^{\alpha}U(t, \sigma)\, w(\sigma)d\sigma-\int_0^s A(s)^{\alpha}U(s, \sigma)w(\sigma)d\sigma\right\|$$

$$\leqq\int_s^t\|A(t)^\alpha U(t,\ \sigma)w(\sigma)\|d\sigma+\int_0^s d\sigma\int_s^t\|(\partial/\partial\tau)\{A(\tau)^\alpha U(\tau,\ \sigma)\}w(\sigma)\|d\tau$$

and the above inequality.

Noting

$$A(t)^{1+\alpha}U(t,\ 0)\Psi=\{A(t)^{1+\gamma}\exp(-tA(t))-A(0)^{1+\gamma}\exp(-tA(0))\}\Psi$$
$$+A(0)^{1+\gamma-\beta}\exp(-tA(0))\cdot A(0)^\beta\Psi+A(t)^{1+\gamma}W(t,\ 0)\Psi$$

for γ with $\alpha<\gamma<min(\beta,\rho)$, we can prove (i) and complete the proof.

By Theorem 1, there exists a unique mild solution $u(t)$ of (2) in $[0,\ T_0]$ for $\phi\in D(A(0)^\beta)$ $(\beta>\alpha)$. To prove that $u(t)$ is also a solution of (1) in $[0,\ T_0]$ we must assume

(VII) $F(t,\ A(t)^{-\alpha}p)$ is strongly Hölder continuous in $t\in[0,\ T]$:

$$\|F(t,\ A(t)^{-\alpha}p)-F(s,\ A(s)^{-\alpha}p)\|\leqq h(\|p\|)|t-s|^\delta$$

for $p\in X$ and $t,\ s\in[0,\ T]$ with some $\delta>0$, h being such a function as f and g.

Then it is easy to see

$$\|F(t,\ v(t))-F(s,\ v(s))\|\leqq g(\|A(t)^\alpha v(t)\|+\|A(s)^\alpha v(s)\|)\|A(s)^\alpha v(t)-A(s)^\alpha v(s)\|$$
$$+h(\|A(s)^\alpha v(s)\|)|t-s|^\delta$$

for $v(t)\in D(A(t)^\alpha)$ and $t,\ s\in[0,\ T]$.

By the above lemma, $A(t)^\alpha u(t)$ is strongly Hölder continuous in $[0,\ T_0]$ and hence so is $F(t,\ u(t))$.

Writing

$$\int_0^t A(t)U(t,\ s)F(s,\ u(s))ds$$
$$=\int_0^t A(t)U(t,\ s)\{F(s,\ u(s))-F(t,\ u(t))\}ds-\int_0^t(\partial/\partial t+\partial/\partial s)\exp(-(t-s)A(t))ds$$
$$\times F(t,\ u(t))+\{I-\exp(-tA(t))\}F(t,\ u(t))-\int_0^t(\partial/\partial t)W(t,\ s)ds\cdot F(t,\ u(t)),$$

we have established

THEOREM 3. *Under the assumptions (I)−(IV) and (VII), for every* $\varphi\in D(A(0)^\beta)$ $(\beta>\alpha)$ *there exists a unique solution* $u(t)$ *of (1) in* $[0,\ T_0]$ $(0<T_0\leqq T)$ *with the initial value* $u(0)=\varphi$ *and*

(*i*) $u(t)$ *is strongly continuous in* $[0,\ T_0]$ *and continuously differentiable in* $(0,\ T_0]$,

(*ii*) $u(t)\in D(A(t))$ *for each* $t\in(0,\ T_0]$ *and* $A(t)u(t)$ *is strongly continuous in* $(0,\ T_0]$.

$u(t)$ *is called a strict solution of (1) in* $[0,\ T_0]$.

REMARK. As is easily seen in the preceding section, if we make the following assumption instead of (VII):

For $t\in[0, T]$ and $p\in X$, $F(t, A(t)^{-\alpha}p)$ belongs to $D(A(t)^{\delta})$ with some $\delta>0$ and $A(t)^{\delta}F(t, A(t)^{-\alpha}p)$ is strongly continuous in $t\in[0, T]$,
then we can prove similarly that (1) admits a unique strict solution in $[0, T_0]$ with the initial value $u(0)=\phi\in D(A(0)^{\alpha})$.

Especially if $F(t, u)=-B(t)u$, where
$\{B(t), 0\leqq t\leqq T\}$ is a family of closed linear operators acting in X such that $D(B(t))\supset D(A(t)^{\alpha})$, $D(A(t)^{\delta})\supset R(B(t)A(t)^{-\alpha})$ for $t\in[0, T]$ with $\alpha\in[0, 1)$ and $\delta\in(0, 1)$ and $A(t)^{\delta}B(t)A(t)^{-\alpha}$ is strongly continuous in $t\in[0, T]$,
we can construct the fundamental solution $V(t, s)$, $0\leqq s\leqq t\leqq T$ to the perturbed equation

$$du/dt+A(t)u+B(t)u=0,\ 0\leqq t\leqq T$$

without difficulty in the following manner:

$$V(t, s)=\sum_{m=0}^{\infty}V_m(t, s),$$

$$\begin{cases} V_0(t, s)=U(t, s), \\ V_m(t, s)=-\int_s^t U(t, \sigma)B(\sigma)V_{m-1}(\sigma, s)d\sigma,\ m=1, 2, \cdots\cdots \end{cases}$$

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