# On a Characterization of the Tensor Product of the Selfdual Cones Associated to the Standard von Neumann Algebras

By
Yasuhide MIURA and Jun TOMIYAMA

(Received April 27, 1983)

## § 1. Introduction

The notion of the selfdual cones in a Hilbert space was introduced by Araki [1] and Connes [2], and it is highly instrumental in determining the algebraic structure of the standard von Neumann algebra. Our purpose in the present paper is to discuss the characterization of the tensor product of the selfdual cones associated to the standard von Neumann algebras.

Let $(M_1, H_1, J_1, P_1)$ and $(M_2, H_2, J_2, P_2)$ be two standard von Neumann algebras defined by Haagerup [3] where $M_i$ is a von Neumann algebra on a Hilbert space $H_i$ and $J_i$ is an isometric involution on $H_i$ and $P_i$ is a selfdual cone in $H_i$ for $i=1, 2$. Then the closure of the algebraic tensor product of two selfdual cones $P_1$ and $P_2$, i. e.,

$$P_1 \otimes P_2 = \overline{\mathrm{co}}\,\{\xi \otimes \eta \mid \xi \in P_1,\ \eta \in P_2\}$$

is not always seldual in $H_1 \otimes H_2$ where $\overline{\mathrm{co}}$ denotes the closed convex hull.

In § 2 we shall characterize the selfdual cone associated to the tensor product of two standard von Neumann algebras modifying the idea of completely positive maps. With this characterization, we shall investigate some properties of the abelian standard von Neumann algebras in § 3.

We refer mainly [6] and [7] for stnandard results in the theory of the operator algebras, and also refer [8] for the discussion of completely positive maps.

Before going into the discussion, the authors wish to express their hearty thank to Dr. Katayama for his many valuable suggestions.

## § 2. Characterizations of the tensor product of the selfdual cones

Let $M$ be a von Neumann algebra on a Hilbert space $H$. Let $J$ be an isometric involution on $H$, and $P$ be a selfdual cone in $H$, i. e., $P$ coincides with the dual cone $P' = \{\xi \in H \mid (\xi, \eta) \geqq 0 \text{ for all } \eta \in P\}$.

DEFINITION 2. 1. ([3; Definition 2. 1]). The quadruple $(M, H, J, P)$ is called the standard form of a von Neumann algebra $M$ if it satisfies the following conditions:

i) $JMJ=M'$,

ii) $JcJ=c^*$, $c\in M\cap M'$,

iii) $J\xi=\xi$, $\xi\in P$,

iv) if $x$ belongs to $M$, then $xJxJ(P)\subset P$.

DEFINITION 2. 2. Let $(M, H, J, P)$ be a standard von Neumann algebra. A matrix $[\xi_{ij}]_{i,j=1}^{n}\in M_n(H)$ $(\xi_{ij}\in H)$ is said to be J-positive with respect to $P$ if

$$\sum_{i,j=1}^{n} a_i J a_j J \xi_{ij}\in P$$

for these elements $\{a_i\}$ of $M$.

Let $M$ be a von Neumann algebra on a Hilbert space $H$ with a cyclic and separating vector $\xi_0$ and let $J_{\xi_0}$ be the modular conjugation of the left Hilbert algebra $M\xi_0$. We put

$$P_{\xi_0}=\overline{\{xJ_{\xi_0}\, xJ_{\xi_0}\xi_0 \mid x\in M\}}.$$

Then, $P_{\xi_0}$ is the selfdual cone in $H$ and $(M, H, J_{\xi_0}, P_{\xi_0})$ is of standard form. In particular, we put $\widetilde{M}=\mathrm{B}(H_n)\otimes I_n$, $\widetilde{H}=H_n\otimes H_n$ where $H_n$ is an n-dimensional Hilbert space with a complete orthonormal basis $\{e_1, e_2, \dots, e_n\}$. Then, $\eta_0=e_1\otimes e_1+e_2\otimes e_2+\dots+e_n\otimes e_n$ is the cyclic and separating vector for $\widetilde{M}$, and we have

$$J_{\eta_0}(\xi\otimes\eta)=\eta\otimes\xi,\quad \xi,\ \eta\in H_n,$$

$$P_{\eta_0}=[(\mathrm{B}(H_n)\otimes I_n)^{+}\eta_0].$$

LEMMA 2. 3. *Keep the notations as above. We identify $\mathcal{M}_n(H)$ with $H\otimes(H_n\otimes H_n)$ by the linear map: $[\xi_{ij}]_{i,j=1}^{n}\rightarrow\sum_{i,j=1}^{n}\xi_{ij}\otimes(e_i\otimes e_j)$ of $\mathcal{M}_n(H)$ onto $H\otimes(H_n\otimes H_n)$. The canonical cone $P_{\xi_0\otimes\eta_0}$ with respect to the cyclic and separating vector $\xi_0\otimes\eta_0$ then coincides with the set of all $J_{\xi_0}$-positive elements with respect to $P_{\xi_0}$, and also coincides with the closure of the convex hull of the elements of matrices $[a_iJ_{\xi_0}a_jJ_{\xi_0}\xi_0]_{i,j=1}^{n}$ where each $a_i$ is an element of $M$.*

PROOF. In this proof the modular conjugations $J_{\xi_0}$ and $J_{\eta_0}$ are simply denoted by $J$ and $\widetilde{J}$ respectively, and we put $\widetilde{M}=B(H_n)\otimes I_n$. Let $x$ be an arbitrary element of the weakly dense part of $M\otimes\widetilde{M}$ such that $x=\sum_{i=1}^{m}a_i\otimes b_i$, $a_i\in M$, $b_i\in\widetilde{M}$. We have then,

$$x=(J\otimes\widetilde{J})x(J\otimes\widetilde{J})(\xi_0\otimes\eta_0)=\sum_{i,j=1}^{m}a_iJa_jJ\xi_0\otimes b_i\widetilde{J}b_j\widetilde{J}\eta_0.$$

On the other hand, if $b_i=y_i\otimes 1$ for $y_i=[\lambda_{st}^{(i)}]\in\mathrm{B}(H_n)$, we have

$$\begin{aligned} b_i\widetilde{J}b_j\widetilde{J}\eta_0&=(y_i\otimes 1)\widetilde{J}(y_j\otimes 1)\widetilde{J}\eta_0\\ &=(y_i\otimes 1)(1\otimes\overline{y_j})\Big(\sum_{p=1}^{n}e_p\otimes e_p\Big)\end{aligned}$$

$$=\sum_{p=1}^{n} y_i e_p \otimes \overline{y_j e_p}$$

$$=\sum_{p,q=1}^{n} \sum_{t=1}^{n} \lambda_{pt}^{(i)} \overline{\lambda_{qt}^{(j)}} e_p \otimes e_q, \quad \text{where } \overline{y_i}=[\overline{\lambda_{st}^{(i)}}].$$

Therefore,

$$\sum_{i,j=1}^{m} a_i J a_j J \xi_0 \otimes b_j \tilde{J} b_i \tilde{J} \eta_0 = \sum_{i,j=1}^{m} \sum_{p,q=1}^{n} \sum_{t=1}^{n} \lambda_{pt}^{(i)} a_i J \lambda_{qt}^{(j)} a_j J \xi_0 \otimes (e_p \otimes e_q)$$

$$=\sum_{t=1}^{n} \sum_{p,q=1}^{n} (\sum_{i=1}^{m} \lambda_{pt}^{(i)} a_i) J (\sum_{i=1}^{m} \lambda_{qt}^{(i)} a_i) J \xi_0 \otimes (e_p \otimes e_q).$$

Hence,

$$x(J \otimes \tilde{J}) x (J \otimes \tilde{J})(\xi_0 \otimes \eta_0) = \sum_{t=1}^{n} \sum_{p,q=1}^{n} A_p^{(t)} J A_q^{(t)} J \xi_0 \otimes (e_p \otimes e_q),$$

where $A_p^{(t)}=\sum_{i=1}^{m} \lambda_{pt}^{(i)} a_i \in M$. It follows that $P_{\xi_0} \otimes_{\eta_0} \subset \overline{\mathrm{co}}\,\{[a_i J a_j J \xi_0] \mid a_i \in M\}$.

Now, if $a_i$ is an arbitrary element of $M$, then one sees that

$$\sum_{i,j=1}^{n} b_i J b_j J a_i J a_j J \xi_0 = \sum_{i,j=1}^{n} b_i a_i J b_j a_j J \xi_0$$

$$=(\sum_{i=1}^{n} b_i a_i) J (\sum_{i=1}^{n} b_i a_i) J \xi_0,$$

for all elements $b_i$ of $M$. Hence we have

$$\sum_{i,j=1}^{n} b_i J b_j J a_i J a_j J \xi_0 \in P_{\xi_0}.$$

Therefore, the matrix $[a_i J a_j J \xi_0]$ is a $J$-positive element with respect to $P_{\xi_0}$. Note that the set of all J-positive elements with respect to $P_{\xi_0}$ is the closed cone. It follows that $\overline{\mathrm{co}}\,\{[a_i J a_j J \xi_0] \mid a_i \in M\} \subset Q_0$ where $Q_0$ denotes the set of all $J$-positive elements with respect to $P_{\xi_0}$.

On the other hand, if $[\xi_{ij}]$ is a $J$-positive element of $\mathcal{M}_n(H)$ with respect to $P_{\xi_0}$, then

$$([\xi_{ij}], [a_i J a_j J \xi_0]) = \sum_{i,j=1}^{n} (\xi_{ij},\ a_i J a_j J \xi_0)$$

$$=\sum_{i,j=1}^{n} (a_i^* J a_j^* J \xi_{ij},\ \xi_0) \geqq 0$$

for all elements $a_i$ of $M$. It follows that $\overline{\mathrm{co}}\,\{[a_i J a_j J \xi_0] \mid a_i \in M\} \subset Q'_0$.

Therefore, we obtain that $P_{\xi_0 \otimes \eta_0} \subset Q_0$ and $P_{\xi_0 \otimes \eta_0} \subset Q'_0$. It follows that $P_{\xi_0 \otimes \eta_0} = Q_0$ because of the selfduality of $P_{\xi_0 \otimes \eta_0}$. Hence we obtain the required results. This completes the proof. Q.E.D.

Now we characterize the set of all $J$-positive elements of order $n$ with respect to $P$

as the selfdual cone associated to the standard form of the matrix von Neumann algebra $\mathcal{M}_n(M)$. Namely we have the following:

PROPOSITION 2. 4. *With* $(\widetilde{M}, \widetilde{H}, \widetilde{J}, \widetilde{P})$ *as before where* $\widetilde{J}=J_{\eta_0}$ *and* $\widetilde{P}=P_{\eta_0}$, *let* $(M, H, J, P)$ *be a standard von Neumann algebra. And let* $Q$ *be the set of all* $J$*-positive elements of* $\mathcal{M}_n(H)$ *(which is identified with* $H\otimes\widetilde{H}$*) with respect to* $P$. *Then,* $Q$ *contains* $P\otimes\widetilde{P}$ *and is the selfdual cone in* $H\otimes\widetilde{H}$ *such that* $(M\otimes\widetilde{M}, H\otimes\widetilde{H}, J\otimes\widetilde{J}, Q)$ *is standard.*

PROOF. We first assume that $M$ is $\sigma$-finite. Then we can find a cyclic and separating vector $\xi_0$ in $P$, and $(M, H, J_0, P_0)$ is standard where $J_0$ and $P_0$ denote the modular conjugation $J_{\xi_0}$ and the canonical cone $P_{\xi_0}$ with respect to $\xi_0$ respectively. Since $(M, H, J, P)$ is standard, by [3; Theorem 2. 18] there exists a unitary $u$ on $H$ such that

$$x=uxu^{-1}\ (x\in M),\quad J=uJ_0u^{-1},\quad P=uP_0.$$

The operator $u$ belongs to $M'$. Suppose that $[\xi_{ij}]$ is $J_0$-positive with respect to $P_0$, then

$$\sum_{i,j=1}^{n} a_iJa_jJu\xi_{ij}=\sum_{i,j=1}^{n} ua_iJ_0a_jJ_0\xi_{ij}\in uP_0=P,$$

for each element $a_i$ of $M$. Therefore $[u\xi_{ij}]$ belongs to $Q$. By the symmetric argument, we see that

$$Q=\{[u\xi_{ij}]\in\mathcal{M}_n(H)\,[\xi_{ij}]\ \textit{is}\ J_0\textit{-positive w. r. t.}\ P_0\}.$$

Thus, by Lemma 2. 3, we have that $Q=(u\otimes 1)P_{\xi_0\otimes\eta_0}$. Therefore one easily sees that $Q$ is selfdual and contains $P\otimes\widetilde{P}$. Since $(M\otimes\widetilde{M}, H\otimes\widetilde{H}, J_0\otimes\widetilde{J}, P_{\xi_0\otimes\eta_0})$ is standatd, we see that $(M\otimes\widetilde{M}, H\otimes\widetilde{H}, J\otimes\widetilde{J}, Q)$ is also standard without dufficulty.

In the general case, choose an increasing net $\{p_\alpha\}$ $(\alpha\in\mathbf{I})$ of $\sigma$-finite projections in $M$ which converges strongly to 1. If we put $q_\alpha=p_\alpha Jp_\alpha J$, the family $\{q_\alpha\}$ is also an increasing net which converges strongly to 1. Cosider the reduced standard von Neumann algebra $(q_\alpha Mq_\alpha, q_\alpha H, J_\alpha, q_\alpha P)$ where $J_\alpha$ means $q_\alpha Jq_\alpha$. Let $Q_\alpha$ be the set of all $J_\alpha$-positive elements of $\mathcal{M}_n(q_\alpha H)$ with respect to $q_\alpha P$. By the first part of the proof, $Q_\alpha$ is selfdual in $q_\alpha H\otimes\widetilde{H}$. We shall show that $\{Q_\alpha\}$ is an increasing family. If $\alpha_1\leqq\alpha_2$, then $q_{\alpha_1}\leqq q_{\alpha_2}$. By Lemma 2. 3, $Q_\alpha$ coincides with the closure of the convex hull of the elements $[q_\alpha a_i q_\alpha Jq_\alpha\xi a_j q_\alpha Jq_\alpha\xi]$ for $a_i\in M$ and $\xi\in q_\alpha P$. Since

$$q_{\alpha_1}a_iq_{\alpha_1}Jq_{\alpha_1}a_jq_{\alpha_1}Jq_{\alpha_1}\xi=q_{\alpha_2}p_{\alpha_1}a_ip_{\alpha_1}q_{\alpha_2}Jq_{\alpha_2}p_{\alpha_1}a_jp_{\alpha_1}q_{\alpha_2}Jq_{\alpha_2}\xi$$

for $a_i\in M$ and $\xi\in q_{\alpha_1}P\subset q_{\alpha_2}P$, we obtain the inclusion $Q_{\alpha_1}\subset Q_{\alpha_2}$.

On the other hand, if $[\xi_{ij}]$ is a $J$-positive element with respect to $P$, then

$$\sum_{i,j=1}^{n} q_\alpha a_iJ_\alpha a_jJ_\alpha q_\alpha\xi_{ij}=q_\alpha\Big(\sum_{i,j=1}^{n} p_\alpha a_ip_\alpha Jp_\alpha a_jp_\alpha J\xi_{ij}\Big)\in q_\alpha P$$

for $a_i\in M$. Therefore $[q_\alpha\xi_{ij}]$ belongs to $Q_\alpha$. Hence we have $(q_\alpha\otimes 1)Q\subset Q_\alpha$. Furthermore, the equality

$$q_\alpha a_iJ_\alpha a_jJ_\alpha q_\alpha\xi=p_\alpha a_ip_\alpha Jp_\alpha a_jp_\alpha J\xi,\quad a_i\in M,\ \xi\in P$$

implies that $Q_\alpha \subset Q$. It follows that

$$Q \subset \overline{\bigcup_\alpha (q_\alpha \otimes 1)Q} \subset \overline{\bigcup_\alpha Q_\alpha} \subset Q,$$

and then $Q = \overline{\bigcup_\alpha Q_\alpha}$. Since $\{Q_\alpha\}$ is an increasing family of selfdual cones in $q_\alpha H \otimes \widetilde{H}$, $Q$ is selfdual in $H \otimes \widetilde{H}$. We easily see that $Q$ contains $P \otimes \widetilde{P}$.

Finally, we shall show that $(M \otimes \widetilde{M}, H \otimes \widetilde{H}, J \otimes \widetilde{J}, Q)$ is standard. It is easy to see the conditions i) to iii) in Definition 2. 1. For the condition iv) take an element $x \in M \otimes \widetilde{M}$, then there exists a bounded net $\{x_\alpha\}$ of the elements of $q_\alpha M q_\alpha \otimes \widetilde{M}$ which converges strongly to $x$. Hence, for a vector $\xi$ of $Q$ we have

$$x(J \otimes \widetilde{J})x(J \otimes \widetilde{J})\xi = \lim_\alpha x_\alpha (J_\alpha \otimes \widetilde{J}) x_\alpha (J_\alpha \otimes \widetilde{J})(q_\alpha \otimes 1)\xi \subset Q$$

by the first part of the proof because $(q_\alpha \otimes 1)\xi \in Q_\alpha$. This completes the proof. Q.E.D.

The above result can easily be generalized to the case where $\widetilde{H}$ is an infinite dimensional separable Hilbert space in the following way. Put $\widetilde{M} = B(K) \otimes I$, $\widetilde{H} = K \otimes K$, $\widetilde{J} = J_{\eta_0}$, $\widetilde{P} = P_{\eta_0}$ where $\eta_0 = \sum_{n=1}^{\infty} \frac{1}{n} e_n \otimes e_n$ isa cyclic and separating vector in $\widetilde{H}$ for $\widetilde{M}$ and $\{e_n\}$ is a countable orthonormal basis in $K$. Let $p_n$ be the projection on $K$ such that $p_n e_i = e_i$ $(i \leqq n)$, $p_n e_i = 0 (i < n)$. put $q_n = (p_n \otimes 1)\widetilde{J}(p_n \otimes 1)\widetilde{J}$ which is equal to $q_n \otimes p_n$. Since $\{q_n\}$ is an increasing sequence which converges strongly to 1, we have the following proposition using Proposition 2. 4.

PROPOSITION 2. 5. *With $(\widetilde{M}, \widetilde{H}, \widetilde{J}, \widetilde{P})$ as above, let $(M, H, J, P)$ be the standard form. Put*

$$Q = \overline{\bigcup_{n=1}^{\infty} \{[\xi_{ij}]_{i,j=1}^{n} \in H \otimes q_n \widetilde{H} \mid [\xi_{ij}] \text{ is } J\text{-positive w. r. t. } P\}}.$$

*Then $Q$ is selfdual in $H \otimes \widetilde{H}$ which contains $P \otimes \widetilde{P}$, and $(M \otimes \widetilde{M}, H \otimes \widetilde{H}, J \otimes \widetilde{J}, Q)$ is standard.*

Before going into the discussion of the general case, we need the following lemma.

LEMMA 2. 6. *Let $M$ and $N$ be two von Neumann algebras on $H$ and $K$ both of which have cyclic and separating vectors $\xi_0$ and $\eta_0$ respectively. Then the closure of the union with respect to $n$ of all elements $\sum_{i,j=1}^{n} \xi_{ij} \otimes \eta_{ij}$ such that $[\xi_{ij}]_{i,j=1}^{n}$ and $[\eta_{ij}]_{i,j=1}^{n}$ are $J_{\xi_0}$ and $J_{\eta_0}$-positive elements with respect to $P_{\xi_0}$ and $P_{\eta_0}$ respectively coincides with $P_{\xi_0 \otimes \eta_0}$, and therefore it is selfdual in $H \otimes K$.*

PROOF. Let $x$ be an arbitrary element of the strongly dense part of $M \otimes N$ such that $x = \sum_{n=1}^{n} a_i \otimes b_i$, $a_i \in M$, $b_i \in N$. Then we have

$$x(J_1 \otimes J_2)x(J_1 \otimes J_2)\xi_0 \otimes \eta_0 = \sum_{i,j=1}^{n} a_i J_1 a_j J_1 \xi_0 \otimes b_i J_2 b_j J_2 \eta_0,$$

where $J_1$ and $J_2$ denote the modular conjugations $J_{\xi_0}$ and $J_{\eta_0}$ respectively. Using Lemma

2. 3, we obtain the required result. Q.E.D.

Let $(M_1, H_1, J_1, P_1)$ and $(M_2, H_2, J_2, P_2)$ be two standard von Neumann algebras. For any element $\xi$ of $H_1$, let $R_\xi$ be the right slice map of $H_1 \otimes H_2$ into $H_2$ with respect to $\xi$ such that $R_\xi(\xi' \otimes \eta') = (\xi', \xi)\eta'$, $\xi' \in H_1$, $\eta' \in H_2$. For any element $x$ of $H_1 \otimes H_2$, we put

$$r(x)(\xi) = R_\xi(x),\ \xi \in H_1.$$

Then, $r(x)$ is a bounded conjugate linear map of $H_1$ into $H_2$.

Definition 2. 7. Keep the notations as above. For each natural number $n$ we shall call that $r(x)$ is an $n$-$J$-positive map of $H_1$ into $H_2$ if for any $J_1$-positive element $[\xi_{ij}]_{i,j=1}^n$ of $\mathcal{M}_n(H_1)(\xi_{ij} \in H_1)$ with respect to $P_1$, $[r(x)(\xi_{ij})]_{i,j=1}^n (\in \mathcal{M}_n(H_2))$ is $J_2$-positive with respect to $P_2$. If $r(x)$ is $n$-$J$-positive for all natural number $n$, it is said to be completely $J$-positive. The set of all elements $x$ of $H_1 \otimes H_2$ such that $r(x)$ is a completely $J$-positive map of $H_1$ into $H_2$ is denoted by $P_1 \widehat{\otimes} P_2$.

With this definition we can characterize the selfdual cone associated to the tensor product of standard von Neumann algebras.

Theorem 2. 8. *Let $(M_1, H_1, J_1, P_1)$ and $(M_2, H_2, J_2, P_2)$ be two standard von Neumann algebras. Then the cone $P_1 \widehat{\otimes} P_2$ contains $P_1 \otimes P_2$ and is the selfdual cone in $H_1 \otimes H_2$ such that $(M_1 \otimes M_2, H_1 \otimes H_2, J_1 \otimes J_2, P_1 \widehat{\otimes} P_2)$ is standard.*

Proof. We fist assume that $M_1$ and $M_2$ are $\sigma$-finite. Then both $M_1$ and $M_2$ have cyclic and separating vectors $\xi_0$ and $\eta_0$ in $P_1$ and $P_2$ respectively. We shall show that $P_{\xi_0} \widehat{\otimes} P_{\eta_0} = P_{\xi_0 \otimes \eta_0}$. If $x = \sum_{k=1}^m \xi_k \otimes \eta_k$ is an arbitrary element of the dense part of $P_{\xi_0 \otimes \eta_0}$, and if $[\xi_{ij}]_{i,j=1}^n$ and $[\eta_{ij}]_{i,j=1}^n$ are $J_{\xi_0}$ and $J_{\eta_0}$-positive with respect to $P_{\xi_0}$ and $P_{\eta_0}$ respectively, then we have

$$\begin{aligned} ([r(x)(\xi_{ij})], [\eta_{ij}]) &= \sum_{i,j=1}^n (r(x)(\xi_{ij}), \eta_{ij}) \\ &= \sum_{i,j=1}^n \sum_{k=1}^m (\xi_k, \xi_{ij})(\eta_k, \eta_{ij}) \\ &= (x, \sum_{i,j=1}^n \xi_{ij} \otimes \eta_{ij}) \geqq 0 \end{aligned}$$

by Lemma 2. 6. Hence, by Theorem 2. 4, $[r(x)(\xi_{ij})]$ is $J_{\eta_0}$-positive with respect to $P_{\eta_0}$. It follows that $r(x)$ is a completely $J$-positive map and $P_{\xi_0 \otimes \eta_0} \subset P_{\xi_0} \widehat{\otimes} P_{\eta_0}$ because of the closedness of $P_{\xi_0} \widehat{\otimes} P_{\eta_0}$. Similarly we obtain the converse inclusion using the above equalities.

Next, we assert that $P_1 \widehat{\otimes} P_2 = (u_1 \otimes u_2)(P_{\xi_0} \widehat{\otimes} P_{\eta_0})$ for some unitary elements $u_1$ and $u_2$ of $M'_1$ and $M'_2$ respectively. By [3; Theorem 2. 18], there exists unitaries $u_1$ and $u_2$ in $M'_1$ and $M'_2$ respectively such that $P_1 = u_1 P_{\xi_0}$ and $P_2 = u_2 P_{\eta_0}$. Take an element $x$ of $P_{\xi_0} \widehat{\otimes} P_{\eta_0}$ and let $[\xi_{ij}]$ and $[\eta_{ij}]$ be $J_1$ and $J_2$-positive with respect to $P_1$ and $P_2$ respectively, then by the first equalities of the proof we have

$$([r((u_1\otimes u_2)x)(\xi_{ij})],\ [\eta_{ij}])=(x,\ \sum_{i,j=1}^{n} u_1^{-1}\xi_{ij}\otimes u_2^{-1}\eta_{ij}).$$

Note that $[u_1^{-1}\xi_{ij}]$ and $[u_2^{-1}\eta_{ij}]$ are $J_{\xi_0}$ and $J_{\eta_0}$-positive with respect to $P_{\xi_0}$ and $P_{\eta_0}$ respectively by the proof of Proposition 2. 4. It follows that $(u_1\otimes u_2)P_{\xi_0}\widehat{\otimes}P_{\eta_0}\subset P_1\widehat{\otimes}P_2$. We obtain the converse inclusion by the symmetric argument. Therefore, we see that $P_1\widehat{\otimes}P_2$ is the selfdual cone in $H_1\otimes H_2$ which contains $P_1\otimes P_2$, and $(M_1\otimes M_2, H_1\otimes H_2, J_1\otimes J_2, P_1\widehat{\otimes}P_2)$ is the standard form.

In the general case, let $P_1$ and $P_2$ be $\sigma$-finite projections of $M_1$ and $M_2$ respectively. Put $q_1=p_1J_1p_1J_1$ and $q_2=p_2J_2P_2J_2$. We assert that $(q_1\otimes q_2)(P_1\widehat{\otimes}P_2)\subset q_1P_1\widehat{\otimes}q_2P_2$. In fact if $x$ belongs to $P_1\widehat{\otimes}P_2$, we have

$$\begin{aligned}([r((q_1\otimes q_2)x)(q_1\xi_{ij})],\ [q_2\eta_{ij}])&=(x,\ \sum_{i,j=1}^{n} q_1\xi_{ij}\otimes q_2\eta_{ij})\\ &=([r(x)(q_1\xi_{ij})],\ [q_2\eta_{ij}])\geqq 0,\end{aligned}$$

where $[q_1\xi_{ij}]$ and $[q_2\eta_{ij}]$ are $q_1J_1q_1$ and $q_2J_2q_2$-positive with respect to $q_1P_1$ and $q_2P_2$ respectively because $[q_1\xi_{ij}]$ and $[q_2\eta_{ij}]$ are also $J_1$ and $J_2$-positive with respect to $P_1$ and $P_2$ respectively by the last half of the proof of Proposition 2. 4. Furthermore, We have another inclusion $q_1P_1\widehat{\otimes}q_2P_2\subset P_1\widehat{\otimes}P_2$. For, if $x=\sum_{s,t=1}^{m} q_1\xi_{st}\otimes q_2\eta_{st}$ is an arbitrary element of the dense part of $q_1P_1\widehat{\otimes}q_2P_2$ where $[q_1\xi_{st}]$ and $[q_2\eta_{st}]$ are $q_1J_1q_1$ and $q_2J_2q_2$-positive elements with respect to $q_1P_1$ and $q_2P_2$ respectively, and if $[\xi'_{ij}]$ and $[\eta'_{ij}]$ are $J_1$ and $J_2$-positive with respect to $P_1$ and $P_2$ respectively, then by the first part of the proof,

$$([r(x)(\xi'_{ij})],\ [\eta'_{ij}])=(x,\ \sum_{i,j=1}^{n} q_1\xi'_{ij}\otimes q_2\eta'_{ij})\geqq 0,$$

because of the selfduality of $q_1P_1\widehat{\otimes}q_2P_2$. Therefore $x$ belongs to $P_1\widehat{\otimes}P_2$.

Now, choose two increasing net $\{p_\alpha\}(\alpha\in \boldsymbol{I})$ and $\{p_\beta\}(\beta\in \boldsymbol{J})$ of $\sigma$-finite projections of $M_1$ and $M_2$ which converge strongly to 1 respectively. Put $q_\alpha=p_\alpha J_1p_\alpha J_1$ and $r_\beta=p_\beta J_2p_\beta J_2$. Then $\{q_\alpha\}$ and $\{r_\beta\}$ are also increasing nets which converge strongly to 1. By the above arguments, we have

$$P_1\widehat{\otimes}P_2\subset\overline{\bigcup_{\alpha,\beta}(q_\alpha\otimes r_\beta)(P_1\widehat{\otimes}P_2)}\subset\overline{\bigcup_{\alpha,\beta}q_\alpha P_1\widehat{\otimes}r_\beta P_2}\subset P_1\widehat{\otimes}P_2.$$

Therefore we have $P_1\widehat{\otimes}P_2=\overline{\bigcup_{\alpha,\beta}q_\alpha P_1\widehat{\otimes}r_\beta P_2}$.

By the last half of the proof of Proposition 2. 4 and the first half of the proof of this theorem, $\{q_\alpha P_1\widehat{\otimes}r_\beta P_2\}$ is an increasing family of selfdual cones. Therefore, we see that $P_1\widehat{\otimes}P_2$ is also selfdual in $H_1\otimes H_2$ and contains $P_1\otimes P_2$. It is now easy to see that $(M_1\otimes M_2,$

$H_1 \otimes H_2, J_1 \otimes J_2, P_1 \widehat{\otimes} P_2)$ is standard using the same argument of the last half of the proof of Proposition 2. 4. This completes the proof. Q.E.D.

As an immediate consequence of the above discussion we have the following corollary, which is the extension of Lemma 2. 6.

COROLLARY 2. 9. *With standard forms* $(M_1, H_1, J_1, P_1)$ *and* $(M_2, H_2, J_2, P_2)$ *as before, the cone* $P_1 \widehat{\otimes} P_2$ *conincides with the closure of the union with respect to* $n$ *of all elements* $\sum_{i,j=1}^{n} \xi_{ij} \otimes \eta_{ij}$ *where* $[\xi_{ij}]$ *and* $[\eta_{ij}]$ *are* $J_1$ *and* $J_2$*-positive elements with respect to* $P_1$ *and* $P_2$ *respectively.*

## § 3. Some properties of the abelian standard von Neumann algebras

In this section we shall investigate some properties of the abelian standard von Neumann algebras from the point of view of the tensor product of the selfdual cones.

PROPOSITION 3. 1. *Let* $(M, H, J, P)$ *be a standard form for an infinite dimensional separable Hilbert space* $H$. *Then,* $M$ *is isomorphic to the algebra* $\ell^\infty$ *of all bounded sequences if and only if* $P$ *contains a complete orthonormal basis of* $H$.

PROOF. Consider the von Neumann algebra $N = \ell^\infty$ on the Hilbert space $K = \ell^2$. Let $P_0$ be the set of positive $\ell^2$-sequences. One then easily sees that $P_0$ is a selfdual cone in $K$ and contains a complete orthonormal basis of $K$. Let $J_0$ be the isometric involution on $K$ such that $J_0 \xi = \xi$, $\xi \in P_0$. Without difficulty, one can show that $(N, K, J_0, P_0)$ is standard. If $(M, H, J, P)$ is standard and $M$ is isomorphic to $N$, then there exists an isometsy $u$ of $K$ onto $H$ such that $P = uP_0$ by [3; Theorem 2. 18]. Therefore $P$ contains a complete orthonormal basis of $H$.

Conversely, let $(M, H, J, P)$ be a standard form and suppose $P$ contains a countalbe orthonormal basis $\{e_i\}$ of $H$. Let $\widetilde{M}$ be the algebra of all operators $x$ on $H$ such that $xe_i = \lambda_i e_i$ and $\{\lambda_i\}$ is a bounded sequence. If we note that $P$ is generated by $\{e_i\}$, we see that $(\widetilde{M}, H, J, P)$ is the standard form by the first part of the proof. Since $\widetilde{M}$ is commutative, we have $M = \widetilde{M}$ by [3; Corollary 5. 11]. Therefore $M$ is isomorphic to the algebra $\ell^\infty$. This completes the proof. Q.E.D.

THEOREM 3. 2. *Let* $(M_1, H_1, J_1, P_1)$ *and* $(M_2, H_2, J_2, P_2)$ *be two standard forms. If either* $M_1$ *or* $M_2$ *is abelian, then* $P_1 \otimes P_2$ *is selfdual in* $H_1 \otimes H_2$, *and* $(M_1 \otimes M_2, H_1 \otimes H_2, J_1 \otimes J_2, P_1 \otimes P_2)$ *is the standard form.*

PROOF. Suppose that both $M_1$ and $M_2$ are $\sigma$-finite. We can then find cyclic and separating vectors $\xi_0$ and $\eta_0$ in $P_1$ and $P_2$ respectively. If either $M_1$ or $M_2$ is abelian, The convex cone of the algebraic tensor product $M_1^+ \otimes M_2^+$ is $\sigma$-weakly dense in $(M_1 \otimes M_2)^+$.

In fact, let $x_0$ be an element of $(M_1 \otimes M_2)^+$ which does not belong to the $\sigma$-weak closure of $M_1^+ \otimes M_2^+$. By the Hahn-Banach theorem, there exists a $\sigma$-weakly continuous linear functional $\phi_0$ on $M_1 \otimes M_2$ such that $\phi_0(x_0) < 0$ and $\phi_0(x) \geqq 0$ for $x \in M_1^+ \otimes M_2^+$. However, if either $M_1$ or $M_2$ is abelian the functional $\phi_0$ must be a positive functional on $M_1 \otimes M_2$ by [5; Theorem 3. 4], a contradiction.

It follows that $M_1^+ \otimes M_2^+$ is also strongly dense in $(M_1 \otimes M_2)^+$. Therefore the closure of the algebraic tensor product of two convex cones $M_1^+ \xi_0$ and $M_2^+ \eta_0$ in $H_1 \otimes H_2$ coincides with that of $(M_1 \otimes M_2)^+ (\xi_0 \otimes \eta_0)$. Put $\Delta = \Delta_1 \otimes \Delta_2$ where $\Delta_1$ and $\Delta_2$ are the modular operators with respect to $\xi_0$ and $\eta_0$ respectively. For an arbitrary element $\xi$ in $(M_1 \otimes M_2)^+ (\xi_0 \otimes \eta_0)$, there exists a sequence $\{\xi_n\}$ in the algebraic tensor product of $M_1^+ \xi_0$ and $M_2^+ \eta_0$ which is convergent to $\xi$. Since $\Delta^{1/2} \eta = \Delta^{1/2} S \eta = J \eta$ for $\eta \in (M_1 \otimes M_2)^+ (\xi_0 \otimes \eta_0)$ where $S$ and $J$ denote the $\sharp$-involution and the modular conjugation with respect to $\xi_0 \otimes \eta_0$ respectively, (cf. [6]). The sequence $\{\Delta^{1/2} \xi_n\}$ is convergent and therefore $\{\Delta^{1/4} \xi_n\}$ is also convergent. Thus we obtain

$$\overline{\Delta_1^{1/4} M_1^+ \xi_0 \otimes \Delta_2^{1/4} M_2^+ \eta_0} = \Delta^{1/4} (M_1 \otimes M_2)^+ (\xi_0 \otimes \eta_0),$$

that is, $P_{\xi_0} \otimes P_{\eta_0} = P_{\xi_0 \otimes \eta_0}$. New by [3; Theorem 2. 18], there exists two unitaries $u_1$ and $u_2$ in $M'_1$ and $M'_2$ suchthat $J_1 = u_1 J_{\xi_0} u_1^{-1}$, $J_2 = u_2 J_{\xi_0} u_2^{-1}$ and $P_1 = u_1 P_{\xi_0}$, $P_2 = u_2 P_{\xi_0}$. It follows that $P_1 \otimes P_2 = (u_1 \otimes u_2)(P_{\xi_0} \otimes P_{\eta_0}) = (u_1 \otimes u_2)\ P_{\xi_0 \otimes \eta_0}$ is a selfdual cone and satisfies required condition.

In the general case, considering increasing nets of $\sigma$-finite projections of $M_1$ and $M_2$ converging strongly to 1 and the reduced standard von Neumann algebras, we obtain the conclusion by the similar arguments of the proof of Proposition 2. 4. This completes the proof. Q.E.D.

PROPOSITION 3. 3. *Let* $(\tilde{M}, \tilde{H}, \tilde{J}, \tilde{P})$ *and* $(M, H, J, P)$ *are two standard von Neumann algebras where* $\tilde{M} = \mathrm{B}(K) \otimes I$ *and* $\tilde{H} = K \otimes K$ *for a seqarable Hilbert space* $K$ *and* $\dim K \geqq 2$. *If* $P \otimes \tilde{P}$ *is selfdual, then* $M$ *is abelian.*

PROOF. Let $\{e'_n\}$ be a countable orthonormal basis in $K$. By [3; Theorem 2. 18], there exists a unitary $u$ on $\tilde{H}$ such that $\tilde{J} = u J_{\eta_0'} u^{-1}$ and $\tilde{P} = u P_{\eta_0'}$ for a cyclic and separating vector $\eta'_0 = \sum_{n=1}^{\infty} \frac{1}{n} e'_n \otimes e'_n$ in $\tilde{H}$ for $\tilde{M}$. Let $p_n$ be an $n$-dimensional projection on $K$ such that $p_n e_i = e_i$ $(i \leqq n)$ and $p_n e_i = 0$ $(i < n)$ for a natural number $n$. Put $q_n = (p_n \otimes 1) J_{\eta'_0} (p_n \otimes 1) J_{\eta'_0}$, which is equal to $p_n \otimes p_n$. If $P \otimes \tilde{P}$ is selfdual, then $P \otimes P_{\eta'_0}$ is also selfdual in $H \otimes \tilde{H}$. Hence one easily sees that $P \otimes q_n P_{\xi'_0}$ is selfdual in $H \otimes q_n H$ for each $n$. Consequently, if we consider the reduced standard von Neumann algebra $(q_n \tilde{M} q_n,\ q_n \tilde{H},\ q_n J_{\eta'_0} q_n,\ q_n P_{\eta'_0})$, we may assume that $\tilde{M} = \mathrm{B}(H_n) \otimes I_n$, $\tilde{H} = H_n \otimes H_n$, $\tilde{J} = J_{\eta_0}$ and $\tilde{P} = P_{\eta_0}$ where $\eta_0$ is a canonical cyclic and separating vector if an $n^2$-dimensional Hilbert space $H_n \otimes H_n$ used in Lemma 2. 3 and $n \geqq 2$.

As usual, we first assume that $M$ is $\sigma$-finite and consider a cyclic and separating vector $\xi_0$ in $P$. Without loss of generality, we may then assume that $J = J_{\xi_0}$ and $P = P_{\xi_0}$.

Let $b=y\otimes 1$ be an element of $\widetilde{M}$ where $y=[\lambda_{ij}]\in B(H_n)$. Then we have

$$b\widetilde{J}b\widetilde{J}\eta_0=\sum_{i,j=1}^{n}\sum_{k=1}^{n}\lambda_{ik}\overline{\lambda_{jk}}\,e_i\otimes e_j.$$

If each $a_i$ and $a$ belong to $M$, then

$$\Big(\sum_{i,j=1}^{n}a_jJa_iJ\xi_0\otimes(e_i\otimes e_j),\ aJaJ\xi_0\otimes b\widetilde{J}b\widetilde{J}\eta_0\Big)$$

$$=\Big(\sum_{i,j=1}^{n}\sum_{k=1}^{n}\lambda_{jk}a^*a_jJ\lambda_{ik}a^*a_iJ\xi_0,\ \xi_0\Big)$$

$$=\sum_{k=1}^{n}\Big(\Big(\sum_{i=1}^{n}\lambda_{ik}a^*a_i\Big)J\Big(\sum_{i=1}^{n}\lambda_{ik}a^*a_i\Big)J\xi_0,\ \xi_0\Big)\geqq 0.$$

Note that the cone $P\otimes\widetilde{P}$ is generated by the elements $aJaJ\xi_0\otimes b\widetilde{J}b\widetilde{J}\eta_0$, $a\in M$, $b\in\widetilde{M}$. It follows that the transpose ${}^t[a_iJa_jJ\xi_0]$ belongs to $P\otimes\widetilde{P}$ if $P\otimes\widetilde{P}$ is selfdual. By Proposition 2.4 we see that ${}^t[a_iJa_jJ\xi_0]$ is a $J$-positive element with respect to $P$. Hence we have

$$0\leqq\Big(\sum_{i,j=1}^{n}x_iJx_jJa_jJa_iJ\xi_0,\ \xi_0\Big)=\sum_{i,j=1}^{n}(a_jJx_ja_iJ\xi_0,\ x_i^*\xi_0)$$

$$=\sum_{i,j=1}^{n}(a_j\Delta^{1/2}a_j^*\ x_i^*\xi_0,\ x_i^*\xi_0)$$

for all elements $a_i$ and $x_i$ of $M$ where $\Delta$ is the modular operator with respect to $\xi_0$. Let $A_0$ be the maximal Tomita algebra in the left Hilbert algebra $M\xi_0$. If we put $a=\pi(\Delta^{-1/4}a\xi_0)$, $a\in\pi(A_0)$, then

$$\sum_{i,j=1}^{n}(\widehat{a_j}\,\widehat{a_i^*}\Delta^{1/4}x_i^*\xi_0,\ \Delta^{1/4}x_i^*\xi_0)=\sum_{i,j=1}^{n}(a_j\Delta^{1/2}a_i^*\ x_j^*\xi_0,\ x_i^*\xi_0)\geqq 0$$

for all elements $a_i$ and $x_i$ of $\pi(A_0)$. Note that $\Delta^{1/4}A_0=A_0$ is dense in H, and we see that ${}^t[a_ia_j^*](\in\mathcal{M}_n(\pi(A_0)))$ is positive. Because of the strong $*$-density of $\pi(A_0)$ in $M$, ${}^t[a_ia_j^*]$ must be positive for all elements $a_i$ of $M$. However, this is a contradiction if $M$ is not abelian.

In fact, if $M$ is non-abelian, then there exist two orthogonal projections $p$ and $q$ of $M$ such that $p=u^*u$, $q=uu^*$, $u\in M$. Put $a_1=p$, $a_2=u$, $a_i=0$ $(3\leqq i\leqq n)$. We obtain

$$\left(\begin{bmatrix}p & up\\ pu^* & q\end{bmatrix}\begin{bmatrix}qu\xi\\ -p\xi\end{bmatrix},\ \begin{bmatrix}qu\xi\\ -p\xi\end{bmatrix}\right)=-2(p\xi,\ \xi)<0,$$

for non-zero vectors $\xi$ of $pH$. This implies that ${}^t[a_ia_j^*]$ is not positive.

In the general case, there exists an increasing net $\{p_i\}$ of $\sigma$-finite projections of $M$ which is strongly convergent to the identity of $M$. We put $q_i=p_iJp_iJ$. Considering the reduced standard von Neumann algebra $(q_iMq_i,\ q_iH,\ q_iJq_i,\ q_iP)$, one easily sees that $q_iP\otimes\widetilde{P}$ is selfdual in $q_iH\otimes\widetilde{H}$ if $P\otimes\widetilde{P}$ is selfdual. By the first part of the proof, we see that $q_iMq_i$ is abelian. Therefore, $M$ is abelian. This completes the proof. Q.E.D.

## References

[1] H. Araki, *Some properties of the modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule*, Pacific J. Math., 50 (1974), 309–354.
[2] A. Connes, *Caractérisation des espaces vectoriels ordonnées sousjacents aux algèbres de von Neumann*, Ann. Inst Fourier, 24 (1974), 121–155.
[3] U. Haagerup, *The standard form of von Neumann algebras*, preprint, Univ. of Copenhagen, 1973.
[4] U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand., 37 (1975), 271–283.
[5] T. Takasaki and J. Tomiyama, *Stinespring type theorem for various types of completely positive maps associated to operator algebras*, Math. Japonica, 27 No. 1 (1982), 129–139.
[6] M. Takesaki, Tomita's theory of modular Hilbert algebras and its applications, Lectures Notes in Math., No. 128, Springer-Verlag, 1970.
[7] M. Takesaki, Theory of operator algebras I, Springer-Verlag, 1979.
[8] J. Tomiyama, *Complete positivity in operator algebras*, Lecture Note No. 4 in RIMS Kyoto Univ., 1978. (in Japanese)

Yasuhide MIURA
Department of Mathematics
College of Humanities and Social Sciences
Iwate University
Morioka 020
Japan

Jun TOMIYAMA
Faculty of Science
Niigata University
Niigata 950–21
Japan

### Added in proof

After we had finished our manuscript we have learned from S. Watanabe about two papers by L. M. Schmitt and G. Wittstock: *Characterization of matrix-ordered standard forms of W*-algebras, preprint, Univ. of Saarland* (1981); *Kernel representations of completely positive Hilbert-Schmidt operators on standard forms, Arch. Math.*, 38 (1982), 453–458. We have found that parts of their results are deeply related to ours and their starting Lemma 1. 1 in their first paper happens to coincide essentially with the last half of our Lemma 2. 3. The first different point of our present argument from theirs is the introduction of the notion of $J$-positive matrices of order $n$ by which we have given an intrinsic characterization of the cone $\mathscr{H}_n^+$ (in their notation) and the further characterization of the cone $P_1\widehat{\otimes}P_2=(\mathscr{H}^{(1)}\otimes\mathscr{H}^{(2)})^+$. Thus with this notion and with the result (Proposition 2. 4) one can see that our Theorem 2. 8 is actually equivalent to their main theorem in the second paper. We should remark here that Theorem 2. 8 may be regarded as the natural counterpart of the Effros' theorem about the characterization of the positive portion of the tensor product of von Neumann algebras as a convex cone of certain completely positive maps from the predual of one von Neumann algebra into the other. The problems of §3 are not discussed in their papers.