Nihonkai Math. J.
Vol.20(2009), 99–108

STRONGLY GENERALIZED DIFFERENCE $\left[V^{\lambda}, \Delta^{M}, P\right]$-SUMMABLE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

AYHAN ESİ

Abstract. We introduce the strongly generalized difference $\left[V^{\lambda}, \Delta^{m}, p\right]$-summable sequences and give the relation between the spaces of strongly generalized difference $\left[V^{\lambda}, \Delta^{m}, p\right]$-summable sequences and strongly generalized difference $\left[V^{\lambda}, \Delta^{m}, p\right]$-summable sequences with respect to a sequence of moduli. Also we give natural relationship between strongly generalized difference $\left[V^{\lambda}, \Delta^{m}, p\right]$-convergence with respect to a sequence of moduli and strongly generalized difference $S^{\lambda}\left(\Delta^{m}\right)$-statistical convergence.

1. Introduction

Let l_{∞}, c and c_o denote the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, normed by $\|x\|=\sup_k |x_k|$, respectively.

Let $\lambda=(\lambda_r)$ be a non-decreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \leq \lambda_r + 1$, $\lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by $t_r(x) = \lambda_r^{-1}\sum_{k\in I_r} x_k$, $I_r = [r-\lambda_r+1, r]$. A sequence $x=(x_k)$ is said to be (V,λ)-summable to a number L if $t_r(x) \to L$ as $r\to\infty$, [10]. If $\lambda_r = r$, then the (V,λ)-summability is reduced to $(C,1)$-summability. We write $[V,\lambda] = \left\{x=(x_k) : \lim_{r\to\infty} \lambda_r^{-1}\sum_{k\in I_r} |x_k - L| = 0, \text{for some} L\right\}$ for set of sequences $x=(x_k)$ which are strongly (V,λ)-summable to L.

The notion of modulus function was introduced by Nakano [15]. The notion was further investigated by Ruckle [13] and many others. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that (i) $f(x)=0$ if and only if $x=0$, (ii) $f(x+y) \leq f(x) + f(y)$ for $x,y \geq 0$, (iii) f is increasing, (iv) f is continuous from the right at 0. It is immediate from (ii) and (iv) that f must be continuous on $[0,\infty)$. Also from condition (ii), we have $f(nx) \leq nf(x)$ for all $n \in N$. A modulus function may be bounded or unbounded. Ruckle [13], Connor [1], Maddox [12], Esi [2], Esi and Tripathy [3] and several authors used a modulus f to contruct some sequence spaces. For a sequence of moduli $F=(f_k)$ we give the following conditions: $(C1)$ $\sup_k f_k(t) < \infty$ for all $t>0$, $(C2)$ $\lim_{t\to 0} f_k(t) = 0$, uniformly in $k \geq 1$. We remark

---

2000 *Mathematics Subject Classification.* 40A05, 40C05, 46A45.

Key words and phrases. De la Vallee-Poussin mean, modulus function, difference sequence, statistical convergence.

that in case $f_k = f$ $(k \geq 1)$, where f is a modulus, the conditions $(C1)$ and $(C2)$ are automatically fulfilled.

The difference sequence space $X(\Delta)$ was introduced by Kizmaz [8] as follows: $X(\Delta) = \{x = (x_k) : \ (\Delta x_k) \in X\}$, for $X = l_\infty, c$ and c_o, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$. Later, these difference sequence spaces were generalized by Et and Çolak [6] as follows: Let $n \in N$ be fixed, then $X(\Delta^n) = \{x = (x_k) : \ (\Delta^n x_k) \in X\}$, for $X = l_\infty$, c and c_o, where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ and $\Delta^0 x_k = x_k$ for all $k \in N$. The generalized difference has the following binomial representation: $\Delta^n x_k = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x_{k+i}$ for each $k \in N$.

Let X be a sequence space. Then X is called solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in N$. A sequence space X is called monotone if X contains preimages of all its step spaces. If X is normal, then it is monotone.

In the present note we introduce the new definitions of strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-summable sequences and give the relation between the spaces of strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-summable sequences and strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-summable sequences with respect to a sequence of moduli. Also we give natural relationship between strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-convergence with respect to a sequence of moduli and strongly generalized difference $S^\lambda(\Delta^m)$-statistical convergence.

The following inequality will be used throughout the paper:

$$|x_k + y_k|^{p_k} \leq K\left(|x_k|^{p_k} + |y_k|^{p_k}\right) \tag{1.1}$$

where x_k and y_k are complex numbers, $K = \max\left(1, 2^{H-1}\right)$ and $H = \sup_k p_k < \infty$, [11].

2. Strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-summable sequences

Let $u = (u_k)$ is any sequence such that $u_k \neq 0$ $(k = 0, 1, 2, \ldots)$ and $p = (p_k)$ be a bounded sequence of positive real numbers $(0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty)$ and $F = (f_k)$ be a sequence of moduli and $m \geq 0$ be fixed integer then, we define

$$\left[V^\lambda, F, \Delta^m, p\right] = \left\{ x = (x_k) \,\middle|\, \begin{array}{r} \lim\limits_{r\to\infty} \lambda_r^{-1} \sum\limits_{k\in I_r} \left[f_k(|u_k \Delta^m x_{k+s} - L|)\right]^{p_k} = 0 \\ \text{uniformly in } s, \quad \text{for some } L \end{array} \right\},$$

$$\left[V^\lambda, F, \Delta^m, p\right]_0 = \left\{ x = (x_k) \,\middle|\, \begin{array}{r} \lim\limits_{r\to\infty} \lambda_r^{-1} \sum\limits_{k\in I_r} \left[f_k(|u_k \Delta^m x_{k+s}|)\right]^{p_k} = 0 \\ \text{uniformly in } s \end{array} \right\},$$

$$\left[V^\lambda, F, \Delta^m, p\right]_\infty = \left\{ x = (x_k) \,\middle|\, \sup_{r,s} \lambda_r^{-1} \sum_{k\in I_r} \left[f_k(|u_k \Delta^m x_{k+s}|)\right]^{p_k} < \infty \right\}.$$

If $u = e = (1,1,1,\ldots)$, $s = 0$, $\Delta^m x_k = x_k$, $f_k = f$ and $p_k = 1$ for all $k \in N$ then the sequence space $\left[V^\lambda, F, \Delta^m, p\right]$ reduce to well-known sequence space $[V, \lambda]$.

If $u = e = (1,1,1,\ldots)$, $s = 0$, $f_k = f$ for all $k \in N$ then the sequence spaces $\left[V^\lambda, F, \Delta^m, p\right]$, $\left[V^\lambda, F, \Delta^m, p\right]_0$ and $\left[V^\lambda, F, \Delta^m, p\right]_\infty$ reduce to

$$[V, \lambda, f, p]\left(\Delta^m\right), [V, \lambda, f, p]_0 \left(\Delta^m\right), \text{ and } [V, \lambda, f, p]_\infty \left(\Delta^m\right)$$

which were defined and studied by Et, Altin, and Altinok [5].

Theorem 2.1 *Let $F = (f_k)$ be a sequence of moduli then the sequence spaces $\left[V^\lambda, F, \Delta^m, p\right]$, $\left[V^\lambda, F, \Delta^m, p\right]_0$ and $\left[V^\lambda, F, \Delta^m, p\right]_\infty$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We give the proof only for $\left[V^\lambda, F, \Delta^m, p\right]_0$. Since the proof is analogous for the spaces $\left[V^\lambda, F, \Delta^m, p\right]$ and $\left[V^\lambda, F, \Delta^m, p\right]_\infty$, we omit the details. Let $x, y \in \left[V^\lambda, F, \Delta^m, p\right]_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist integers T_α and T_β such that $|\alpha| \le T_\alpha$ and $|\beta| \le T_\beta$. We therefore have

$$\begin{aligned}
&\lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m\left(\alpha x_{k+s} + \beta y_{k+s}\right)\right|\right)\right]^{p_k} \\
&= \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|\alpha u_k \Delta^m x_{k+s} + \beta u_k \Delta^m y_{k+s}\right|\right)\right]^{p_k} \\
&\le K\left[T_\alpha\right]^H \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m x_{k+s}\right|\right)\right]^{p_k} + K\left[T_\beta\right]^H \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m y_{k+s}\right|\right)\right]^{p_k} \\
&\quad \to 0 \text{ as } r \to \infty, \text{uniformly in } s.
\end{aligned}$$

This proves that the sequence space $\left[V^\lambda, F, \Delta^m, p\right]_0$ is linear. □

Theorem 2.2 *Let $F = (f_k)$ be a sequence of moduli then the inclusions*

$$\left[V^\lambda, F, \Delta^m, p\right]_0 \subset \left[V^\lambda, F, \Delta^m, p\right] \subset \left[V^\lambda, F, \Delta^m, p\right]_\infty$$

hold.

Proof. The inclusion $\left[V^\lambda, F, \Delta^m, p\right]_0 \subset \left[V^\lambda, F, \Delta^m, p\right]$ is obvious. Now let $x \in \left[V^\lambda, F, \Delta^m, p\right]$. By using (1.1), we have

$$\begin{aligned}
&\sup_{r,s} \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m x_{k+s}\right|\right)\right]^{p_k} \\
&= \sup_{r,s} \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m x_{k+s} - L + L\right|\right)\right]^{p_k} \\
&\le K \sup_{r,s} \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m x_{k+s} - L\right|\right)\right]^{p_k} + K \sup_{r,s} \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(|L|\right)\right]^{p_k} \\
&\le K \sup_{r,s} \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|u_k \Delta^m x_{k+s} - L\right|\right)\right]^{p_k} + K \max\left(f_k\left(|L|\right)^h, f_k\left(|L|\right)^H\right) \\
&< \infty.
\end{aligned}$$

Hence $x \in \left[V^\lambda, F, \Delta^m, p\right]_\infty$, which shows that $\left[V^\lambda, F, \Delta^m, p\right] \subset \left[V^\lambda, F, \Delta^m, p\right]_\infty$. This completes the proof. □

Theorem 2.3 *The sequence spaces*

$$\left[V^\lambda, F, \Delta^m, p\right], \left[V^\lambda, F, \Delta^m, p\right]_0, \text{ and } \left[V^\lambda, F, \Delta^m, p\right]_\infty$$

are solid and hence monotone.

Proof. Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$, for all $k \in N$. Since f_k is monotone for all $k \in N$, we get

$$\begin{aligned} \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(|u_k \Delta^m \alpha_{k+s} x_{k+s}|\right)\right]^{p_k} &\leq \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(\left|\sup_{k,s} \alpha_{k+s}\right| |u_k \Delta^m x_{k+s}|\right)\right]^{p_k} \\ &\leq \lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(|u_k \Delta^m x_{k+s}|\right)\right]^{p_k}, \end{aligned}$$

which leads us to the desired result. □

Now we give the relation between strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-convergence and strongly generalized difference $\left[V^\lambda, \Delta^m, p\right]$-convergence with respect to a sequence of moduli.

Theorem 2.4 *Let $F = (f_k)$ be a sequence of moduli then*

$$\left[V^\lambda, \Delta^m, p\right] \subset \left[V^\lambda, F, \Delta^m, p\right], \quad \left[V^\lambda, \Delta^m, p\right]_0 \subset \left[V^\lambda, F, \Delta^m, p\right]_0$$

and

$$\left[V^\lambda, \Delta^m, p\right]_\infty \subset \left[V^\lambda, F, \Delta^m, p\right]_\infty.$$

Proof. We consider only the case $\left[V^\lambda, \Delta^m, p\right]_0 \subset \left[V^\lambda, F, \Delta^m, p\right]_0$.

Let $x \in \left[V^\lambda, \Delta^m, p\right]_0$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that $f_k(t) < \varepsilon$ for every t with $0 \leq t \leq \delta$. We can write

$$\begin{aligned} &\lambda_r^{-1} \sum_{k \in I_r} \left[f_k\left(|u_k \Delta^m x_{k+s}|\right)\right]^{p_k} \\ &= \lambda_r^{-1} \sum_1 \left[f_k\left(|u_k \Delta^m x_{k+s}|\right)\right]^{p_k} + \lambda_r^{-1} \sum_2 \left[f_k\left(|u_k \Delta^m x_{k+s}|\right)\right]^{p_k} \\ &\leq \max\left(\varepsilon^h, \varepsilon^H\right) + \max\left(1, (2 f_k(1) \delta^{-1})^H\right) \lambda_r^{-1} \sum_2 |u_k \Delta^m x_{k+s}|^{p_k} \end{aligned}$$

where the summation $\sum_1$ is over $|u_k \Delta^m x_{k+s}| \leq \delta$ and the summation $\sum_2$ is over $|u_k \Delta^m x_{k+s}| > \delta$. Hence we obtain $x \in \left[V^\lambda, F, \Delta^m, p\right]_0$. □

Theorem 2.5 *Let $F=(f_k)$ be a sequence of moduli. If* $\lim_{t\to\infty}\frac{f_k(t)}{t}=\beta>0$, *for all* $k\in N$, *then* $\left[V^\lambda,\Delta^m,p\right]=\left[V^\lambda,F,\Delta^m,p\right]$, $\left[V^\lambda,\Delta^m,p\right]_0=\left[V^\lambda,F,\Delta^m,p\right]_0$ *and* $\left[V^\lambda,\Delta^m,p\right]_\infty=\left[V^\lambda,F,\Delta^m,p\right]_\infty$.

Proof. For any modulus function, the existence of a positive limit given with β was introduced by Maddox [2]. Let $\beta>0$ and $x\in\left[V^\lambda,F,\Delta^m,p\right]_0$. Since $\beta>0$, we have $\frac{f_k(t)}{t}\geq\beta$ for all $t>0$ and all $k\in N$. From this inequality, it is easy to see that $x\in\left[V^\lambda,F,\Delta^m,p\right]_0$. In Theorem 2.4, it was shown that $\left[V^\lambda,\Delta^m,p\right]_0\subset\left[V^\lambda,F,\Delta^m,p\right]_0$. This completes the proof. □

Theorem 2.6 *If $m\geq1$, then the inclusions* $\left[V^\lambda,F,\Delta^{m-1},p\right]\subset\left[V^\lambda,F,\Delta^m,p\right]$, $\left[V^\lambda,F,\Delta^{m-1},p\right]_0\subset\left[V^\lambda,F,\Delta^m,p\right]_0$ *and* $\left[V^\lambda,F,\Delta^{m-1},p\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$ *are strict. In general,* $\left[V^\lambda,F,\Delta^i,p\right]\subset\left[V^\lambda,F,\Delta^m,p\right]$, $\left[V^\lambda,F,\Delta^i,p\right]_0\subset\left[V^\lambda,F,\Delta^m,p\right]_0$ *and* $\left[V^\lambda,F,\Delta^i,p\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$ *for all $i=1,2,3,\ldots,m-1$ and the inclusions are strict.*

Proof. We give the proof for $\left[V^\lambda,F,\Delta^{m-1},p\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$. The others can be proved in a similar way. Let $x\in\left[V^\lambda,F,\Delta^{m-1},p\right]_\infty$. Then we have $\sup_{r,s}\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|u_k\Delta^{m-1}x_{k+s}|\right)\right]^{p_k}<\infty$. By definition of f_k for all $k\in N$, from (1.1) we have

$$\begin{aligned}
&\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|u_k\Delta^m x_{k+s}|\right)\right]^{p_k}\\
&\leq\ K\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(\left|u_k\Delta^{m-1}x_{k+s}\right|\right)\right]^{p_k}+K\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(\left|u_k\Delta^{m-1}x_{k+s+1}\right|\right)\right]^{p_k}\\
&<\ \infty
\end{aligned}$$

for all $s\in N$. Thus $\left[V^\lambda,F,\Delta^{m-1},p\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$.Proceeding in this way one will have $\left[V^\lambda,F,\Delta^i,p\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$ for all $i=1,2,3,\ldots,m-1$. Now let $\lambda_n=n$ for each $n\in N$. Then the sequence $x=(k^m)$ ($\Delta^m x_k=(-1)^m m!$ and $\Delta^{m-1}x_k=(-1)^{m+1}m!\left(k+\frac{m-1}{2}\right)$) for example, belongs to $\left[V^\lambda,F,\Delta^m,p\right]_\infty$, but it does not belong to $\left[V^\lambda,F,\Delta^{m-1},p\right]_\infty$ for $f_k=id$, $p_k=1$ for all $k\in N$ and $u=e$. □

We consider that (p_k) and (q_k) are any bounded sequences of strictly positive real numbers. We are able to prove $\left[V^\lambda,\Delta^m,q\right]\subset\left[V^\lambda,F,\Delta^m,p\right]$, $\left[V^\lambda,\Delta^m,q\right]_0\subset\left[V^\lambda,F,\Delta^m,p\right]_0$ and $\left[V^\lambda,\Delta^m,q\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$ only under additional conditions.

Theorem 2.7 *Let $0<p_k\leq q_k$ for all $k\in N$ and let $\left(\frac{q_k}{p_k}\right)$ be bounded. Then* $\left[V^\lambda,\Delta^m,q\right]\subset\left[V^\lambda,F,\Delta^m,p\right]$, $\left[V^\lambda,\Delta^m,q\right]_0\subset\left[V^\lambda,F,\Delta^m,p\right]_0$ *and* $\left[V^\lambda,\Delta^m,q\right]_\infty\subset\left[V^\lambda,F,\Delta^m,p\right]_\infty$.

Proof. If we take $t_{k,s} = [f_k(|u_k \Delta^m x_{k+s}|)]^{q_k}$ for all $k, s \in N$, then using the same technique of Theorem 2 of Nanda [16], the proof is easy. $\square$

Corollary 2.1 *The following statements are valid:*

(i) *If* $0 < \inf_k p_k \leq 1$ *for all* $k \in N$, *then*

$$\left[V^\lambda, \Delta^m\right] \subset \left[V^\lambda, F, \Delta^m, p\right], \quad \left[V^\lambda, \Delta^m\right]_0 \subset \left[V^\lambda, F, \Delta^m, p\right]_0$$

and

$$\left[V^\lambda, \Delta^m\right]_\infty \subset \left[V^\lambda, F, \Delta^m, p\right]_\infty.$$

(ii) *If* $1 \leq p_i \leq \sup_i p_i = H < \infty$, *then*

$$\left[V^\lambda, \Delta^m, p\right] \subset \left[V^\lambda, F, \Delta^m\right], \quad \left[V^\lambda, \Delta^m, p\right]_0 \subset \left[V^\lambda, F, \Delta^m\right]_0$$

and

$$\left[V^\lambda, \Delta^m, p\right]_\infty \subset \left[V^\lambda, F, \Delta^m\right]_\infty.$$

Proof. (i) follows from Theorem 2.7 with $q_k = 1$ for all $k \in N$, and (ii) follows from the same theorem with $p_k = 1$ for all $k \in N$. $\square$

Theorem 2.8 $\left[V^\lambda, F, \Delta^m, p\right]_0$ *is a paranormed space with*

$$h_{\Delta^m}(x) = \sup_{r,s} \left(\lambda_r^{-1} \sum_{k \in I_r} f_k\left(|u_k \Delta^m x_{k+s}|\right)^{p_k} \right)^{\frac{1}{M}},$$

where $M = \max\left(1, \sup_k p_k\right) < \infty$.

Proof. Clearly $h_{\Delta^m}(x) = h_{\Delta^m}(-x)$. It is trivial that $\Delta^m x_k = 0$ for $x = 0$.Since $f_k(0) = 0$ for all $k \in N$, we get $h_{\Delta^m}(x) = 0$ for $x = 0$. Since $\frac{p_k}{M} \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of f_k, for each $r, s \geq 1$, we have

$$\begin{aligned}
&\left(\lambda_r^{-1} \sum_{k \in I_r} [f_k(|u_k(\Delta^m x_{k+s} + \Delta^m y_{k+s})|)]^{p_k} \right)^{\frac{1}{M}} \\
&\leq \left(\lambda_r^{-1} \sum_{k \in I_r} [(f_k(|u_k \Delta^m x_{k+s}|) + f_k(|u_k(\Delta^m x_{k+s})|))]^{p_k} \right)^{\frac{1}{M}} \\
&\leq \left(\lambda_r^{-1} \sum_{k \in I_r} [f_k(|u_k \Delta^m x_{k+s}|)]^{p_k} \right)^{\frac{1}{M}} + \left(\lambda_r^{-1} \sum_{k \in I_r} [f_k(|u_k \Delta^m y_{k+s}|)]^{p_k} \right)^{\frac{1}{M}}.
\end{aligned}$$

Hence h_{Δ^m} is subadditive. Finally, to check the continuity of multiplication, let us take any complex number α. By definition of f_k , we have

$$h_{\Delta^m}(\alpha x) = \sup_{r,s}\left(\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|\alpha u_k \Delta^m x_{k+s}|\right)\right]^{p_k}\right)^{\frac{1}{M}} \leq T^{\frac{H}{M}} h_{\Delta^m}(x)$$

where T is a positive integer such that $|\alpha| \leq T$. Now, let $\alpha \to 0$ for any fixed x with $h_{\Delta^m}(x) \neq 0$. By definition of f_k for $|\alpha| < 1$, we have

$$\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|\alpha u_k \Delta^m x_{k+s}|\right)\right]^{p_k} < \varepsilon \quad \text{for} \quad r > r_o. \tag{2.2}$$

Also, for $1 \leq r \leq r_o$, taking α small enough, since f_k is continuous for all $k \in N$, we have

$$\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|\alpha u_k \Delta^m x_{k+s}|\right)\right]^{p_k} < \varepsilon. \tag{2.3}$$

Conditions (2.2) and (2.3) together imply that $h_{\Delta^m}(\alpha x) \to 0$ as $\alpha \to 0$. This completes the proof. □

3. Strongly generalized difference $S^{\lambda}(\Delta^m)$-statistical convergence

In this section, we introduce natural relationship between strongly generalized difference $\left[V^{\lambda}, \Delta^m, p\right]$-convergence with respect to a sequence of moduli and strongly generalized difference $S^{\lambda}(\Delta^m)$-statistical convergence. Fast [7] introduced the idea of statistical convergence. This idea was later studied by Connor [1], Salat [18], Savaş [19], Tripathy [17], Esi and Tripathy [4] and many others.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_n \left|\frac{A(\varepsilon)}{n}\right| = 0$, where $|A(\varepsilon)|$ denotes the number of elements in $A(\varepsilon) = \{k \in N : \ |x_k - L| \geq \varepsilon\}$.

In [9], Et and Nuray defined a sequence $x = (x_k)$ is Δ^m-statisticaly convergent to the number if for every $\varepsilon > 0$, $\lim_n \left|\frac{K(\varepsilon)}{n}\right| = 0$, where $|K(\varepsilon)|$ denotes the number of elements in $K(\varepsilon) = \{k \in N : \ |\Delta^m x_k - L| \geq \varepsilon\}$. The set of Δ^m-statisticaly convergent sequences is denoted by $S(\Delta^m)$.

Mursaleen [14] introduced the concept of λ-statistical convergence as follows: A sequence $x = (x_k)$ is said to be λ-statistically convergent to L if for every $\varepsilon > 0$, $\lim_r \lambda_r^{-1}|C(\varepsilon)| = 0$, where $|C(\varepsilon)|$ denotes the number of elements in $C(\varepsilon) = \{k \in I_r : \ |x_k - L| \geq \varepsilon\}$. The set of all λ-statistically convergent sequences is denoted by S^{λ}.

A sequence $x = (x_k)$ is said to be strongly generalized difference $S^{\lambda}(\Delta^m)$-statistically convergent to the number L if for any $\varepsilon > 0$, $\lim_r \lambda_r^{-1}|C(\varepsilon, s)| =$

0, uniformly in s, where $|C(\varepsilon,s)|$ denotes the number of elements in $C(\varepsilon,s)=\{k\in I_r:\ |u_k\Delta^m x_{k+s}-L|\geq\varepsilon\}$. The set of all strongly generalized difference generalized statistically convergent sequences is denoted by $S^\lambda(\Delta^m,s)$.

If $u_k=e$ for all $k\in N$, $s=0$, and $\lambda_r=r$ for $r\geq 1$, then the space $S^\lambda(\Delta^m,s)$ reduces to the space $S(\Delta^m)$, which was defined and studied by Et and Nuray [9]. If $u_k=e$ for all $k\in N$, $s=0$, $m=0$ and $\lambda_r=r$ for $r\geq 1$, then the space $S^\lambda(\Delta^m,s)$ reduces to the space of ordinary statistical convergence. If $u_k=e$ for all $k\in N$, $s=0$, $m=0$ and then the space $S^\lambda(\Delta^m,s)$ reduces to the space of λ-statistical convergence which was defined and studied by Mursaleen [14].

Now we give the relation between strongly generalized difference $S^\lambda(\Delta^m)$-statistical convergence and strongly generalized difference $\left[V^\lambda,\Delta^m,p\right]$-convergence with respect to a sequence of moduli.

Theorem 3.1 *Let $F=(f_k)$ be a sequence of moduli then $\left[V^\lambda,F,\Delta^m,p\right]\subset S^\lambda(\Delta^m,s)$.*

Proof. Let $x\in\left[V^\lambda,F,\Delta^m,p\right]$. Then

$$\begin{aligned}
&\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|u_k\Delta^m x_{k+s}-L|\right)\right]^{p_k}\\
&\quad\geq\ \lambda_r^{-1}\sum_{1}\left[f_k\left(|u_k\Delta^m x_{k+s}-L|\right)\right]^{p_k}\\
&\quad\geq\ \lambda_r^{-1}\sum_{1}\left[f_k(\varepsilon)\right]^{p_k}\\
&\quad\geq\ \lambda_r^{-1}\sum_{1}\min\left(f_k(\varepsilon)^h,f_k(\varepsilon)^H\right)\\
&\quad\geq\ \lambda_r^{-1}\left|\{k\in I_r:|u_k\Delta^m x_{k+s}-L|\geq\varepsilon\}\right|\min\left(f_k(\varepsilon)^h,f_k(\varepsilon)^H\right),
\end{aligned}$$

where the summation $\sum_1$ is over $|u_k\Delta^m x_{k+s}-L|\geq\varepsilon$. Hence we obtain $x\in S^\lambda(\Delta^m,s)$. This completes the proof. □

Theorem 3.2 *Let $F=(f_k)$ be a uniformly bounded sequence of moduli on the interval $[0,\infty)$. Then $\left[V^\lambda,F,\Delta^m,p\right]=S^\lambda(\Delta^m,s)$.*

Proof. By Theorem 3.1, it is sufficient to show that $\left[V^\lambda,F,\Delta^m,p\right]\supset S^\lambda(\Delta^m,s)$. Let $x\in S^\lambda(\Delta^m)$. Since $F=(f_k)$ is uniformly bounded on the interval $[0,\infty)$, so there exists an integer $B>0$ such that $f_k\left(|u_k\Delta^m x_{k+s}-L|\right)\leq B$ for all $k\in N$. Then for a given $\varepsilon>0$, we have

$$\begin{aligned}
&\lambda_r^{-1}\sum_{k\in I_r}\left[f_k\left(|u_k\Delta^m x_{k+s}-L|\right)\right]^{p_k}\\
&\quad=\ \lambda_r^{-1}\sum_{1}\left[f_k\left(|u_k\Delta^m x_{k+s}-L|\right)\right]^{p_k}+\lambda_r^{-1}\sum_{2}\left[f_k\left(|u_k\Delta^m x_{k+s}-L|\right)\right]^{p_k}\\
&\quad\geq\ B^H\lambda_r^{-1}\left|\{k\in I_r:|u_k\Delta^m x_{k+s}-L|\geq\varepsilon\}\right|+\max\left(f_k(\varepsilon)^h,f_k(\varepsilon)^H\right),
\end{aligned}$$

where the summation $\sum_1$ is over $|u_k\Delta^m x_{k+s}| \leq \delta$ and the summation $\sum_2$ is over $|u_k\Delta^m x_{k+s}| > \delta$. Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, uniformly in s, we get $x \in \left[V^\lambda, F, \Delta^m, p\right]$. This completes the proof. $\square$

Theorem 3.3 *If* $\liminf_r \frac{\lambda_r}{r} > 0$*, then* $S(\Delta^m, s) \subset S^\lambda(\Delta^m, s)$*, where*

$$S(\Delta^m, s) = \left\{ x = (x_k) \;\middle|\; \begin{array}{r} \lim\limits_r \dfrac{1}{r}\left|\{k \leq r : |u_k\Delta^m x_{k+s} - L| \geq \varepsilon\}\right| = 0, \\ \text{uniformly in } s, \text{ for some } L \end{array} \right\}.$$

Proof. Let $x \in S(\Delta^m, s)$. For given $\varepsilon > 0$, we get

$$\{k \leq r : |u_k\Delta^m x_{k+s} - L| \geq \varepsilon\} \supset C(\varepsilon, s).$$

Thus

$$\frac{1}{r}\left|\{k \leq r : |u_k\Delta^m x_{k+s} - L| \geq \varepsilon\}\right| \geq \frac{1}{r}\left|C(\varepsilon, s)\right| = \frac{\lambda_r}{r}\frac{1}{\lambda_r}\left|C(\varepsilon, s)\right|.$$

Taking limit as $r \to \infty$ and using $\liminf_r \frac{\lambda_r}{r} > 0$, we get $x \in S^\lambda(\Delta^m, s)$. This completes the proof. $\square$

References

[1] J. S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32**(2) (1989), 194–198.

[2] A. Esi, *Some new sequence spaces defined by a sequence of moduli*, Turkish J. of Math., **21** (1997), 61–68.

[3] A. Esi and B. C. Tripathy, *On some generalized new type difference sequence spaces defined by a modulus function in a seminormed space*, Fasciculi Mathematici, Fasc. Math., **40** (2008), 15–24.

[4] A. Esi and B. C. Tripathy, *Strongly almost convergent generalized difference sequences associated with multiplier sequences*, Math. Slovaca, **57** (2007), 339–348.

[5] M. Et, Y. Altin, and H. Altinok, *On some generalized difference sequence spaces defined by a modulus function*, Filomat, **17** (2003), 23–33.

[6] M. Et and R. Çolak, *On some generalized difference sequence spaces*, Soochow J.Math., **21** (1995), 377–386.

[7] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.

[8] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., **24** (1981), 169–176.

[9] M. Et and F. Nuray, Δ^m-*statistical convergence*, Indian J. Pure and Appl. Math., **6**(32) (2001),961–969.

[10] L. Leindler, *Über die la Vallee-Pousinche summierbarkeit allgemeiner orthogonalchen*, Acta Math. Hung., **16** (1965), 375–378.

[11] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser.2, **18** (1967), 345–355.

[12] I. J. Maddox, *Sequence spaces defined by a modulus function*, Math. Proc. Camb. Phil. Soc., **100** (1986), 161–166.

[13] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., **25** (1973), 973–978.

[14] Mursaleen, λ-*statistical convergence*, Math. Slovaca, **50** (2000), 111–115.

[15] H. Nakano, *Modular sequence spaces*, Proc. Japan Acad., **27** (1951), 508–512.

[16] S. Nanda, *Strongly almost summable and strongly almost convergent sequences*, Acta Math. Hung., **49**(1-2) (1987), 71–76.

[17] B. C. Tripathy, *Generalized difference paranormed statistically convergent sequences defined by Orlicz function in a locally convex space*, Soochow J. Math., **29**(3) (2003), 313–326.

[18] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca, **2** (1980), 139–150.

[19] E. Savaş, *Some sequence spaces and statistical convergence*, Int. J. Math. and Math. Sci., **29**(5) (2002), 303–306.

(Ayhan Esi) Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey

E-mail address: aesi23@hotmail.com

Received January 20, 2009
Revised August 3, 2009