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LOCAL TIME DECAY FOR A QUASILINEAR SCHRÖDINGER EQUATION*

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Abstract. We study the solutions of a quasilinear Schrödinger equation which has been derived in many areas of physical modeling. Using the Morawetz Radial Identity, we show that the local energy of a solution is integrable in time and the local L^2 norm of the solution approaches zero as time approaches the infinity.

Key words. Time decay, quasilinear Schrödinger equation.

AMS subject classifications. 35Q55

1. Introduction.

Consider the equation

$$iu_t + a\Delta u + q(|u|^2)u - b(\Delta(h(|u|^2)))h'(|u|^2)u = 0 \tag{1}$$

where $u = u(x,t)$ is a complex-valued function, $x = (x_1, x_2, \dots, x_n)$ is in the n-dimensional Euclidean space R^n, $t > 0$, $i = (-1)^{1/2}$, Δ is the n-dimensional Laplacian in x, q and h are real-valued functions, and a and b are real constants.

The equation (1) has appeared in several areas of physical modeling including plasma physics, Heisenberg ferromagnets and magnons, dissipative quantum mechanics, nanotubes and fullerenes, and condensed matter theory [2, 5, 6, 13, 16-21, 23, 26, 30, 35-37, 39].

Recently, there have been many studies about the local and the global well-posedness problems for equation (1), see, for example, [3, 4, 9 -11, 12, 14, 15, 22, 32-34, 40] as well as its localized solutions, see, for example, [1, 7, 8, 24, 25, 27, 31, 38] . In this article, we shall show the following property for the equation (1).

THEOREM. Assume the following conditions for the equation (1):
(A1) the solution u is a global smooth function that vanishes sufficiently fast at the spatial infinity,
(A2) the spatial dimension $n \geq 3$,
(A3) $a < 0$, $b > 0$,
(A4) q satisfies the relation $q(s)s \geq (1 + c_0/(n-1))Q(s) \geq 0$, for some constant $c_0 \geq 1/3$,
where $Q'(s) = q(s)$ and $Q(0) = 0$, and
(A5) $h'(s)h''(s) \geq 0$ for all $s \geq 0$.
Then, the local energy, which is defined as

$$E_R(t) = \int_{|x|\leq R} \left[-a|\nabla u|^2 + Q(|u|^2) + (b/2)|\nabla(h(|u|^2))|^2\right](x,t)dx \text{ for } R > 0,$$

is integrable in time from 0 to ∞ and the local L^2 norm of the solution goes to zero as t approaches the infinity.

As usual, ∇u denotes the gradient of u, $\nabla \cdot u$ denotes the divergence of u, and $r = |x|$. Also the subscript denotes the partial derivative, thus $u_t = \partial u/\partial t$, etc.. We also use the notation $u_r = (x/r) \cdot \nabla u$. The complex conjugate of u is denoted by u^*.

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2. Conservation laws. Multiplying the equation (1) by u^*, taking the imaginary part, and integrating over the whole space R^n, we get a conservation law,

$$\int_{R^n} |u|^2(x,t)dx = \text{constant} = L,$$

where $L = \int_{R^n} |u|^2(x,0)dx$.

Multiplying the equation by u_t^*, taking the real part, and integrating over the whole space R^n, we get another conservation law,

$$E(t) = \int_{R^n} \left[-a|\nabla u|^2 + Q(|u|^2) + (b/2)|\nabla(h(|u|^2))|^2\right](x,t)dx = \text{constant}.$$

$E(t)$ will be called the energy. By the assumptions (A3) and (A4), $Q \geq 0$, $a < 0$, and $b > 0$. Thus $E(t)$ is a nonnegative constant. Let $E = E(0)$.

3. Morawetz' Radial Identity. Let $\zeta = \zeta(|x|) = \zeta(r)$ be a smooth real-valued function that depends only on the spatial variables. Following [28], we multiply the equation (1) by $\zeta(u_r^* + ((n-1)/(2r))u^*)$, take the real part, and get

$$\partial X/\partial t + \nabla \cdot Y + Z = 0 \tag{2}$$

where

$$
\begin{aligned}
X &= -\zeta w v_r - \zeta((n-1)/(2r))wv - (1/2)\zeta' wv \\
Y &= \text{ a function which depends on } u,\ u_r,\ \nabla u,\ u_t,\ Q(|u|^2),\ \zeta,\ \zeta',\ \nabla(h(|u|^2)),\ a, \text{ and } b. \\
Z &= -a\left\{(1/2)(\zeta'' + \zeta'((n-1)/r))(ww_r + vv_r) + (\zeta/r - \zeta')(|\nabla u|^2 - |u_r|^2) + \zeta'|\nabla u|^2\right. \\
&\quad \left. + (\zeta(n-1)(n-3)/(4r^3) + \zeta'(n-1)/(4r^2))|u|^2\right\} \\
&\quad - (1/2)\zeta' Q(|u|^2) + \zeta((n-1)/(2r))(q(|u|^2)|u|^2 - Q(|u|^2)) \\
&\quad + (b/2)\left\{(\zeta/r - \zeta')(|\nabla(h(|u|^2))|^2 - ((h(|u|^2))_r)^2)\right. \\
&\quad + (1/2)((n-1)\zeta/r + \zeta')|\nabla(h(|u|^2))|^2 \\
&\quad - (1/r)((n-1)\zeta'' + (n-1)(n-3)(\zeta'/r - \zeta/r^2))G(|u|^2) \\
&\quad \left. + (n-1)(\zeta/r)h'(|u|^2)h''(|u|^2)|u|^2|\nabla(|u|^2)|^2\right\}
\end{aligned}
$$

where v and w are the real part and the imaginary part of u, respectively, and $G'(s) = (h'(s))^2 s$ with $G(0) = 0$.

Integrating both sides with respect to x in R^n and t from 0 to T, we get

$$\left|\int_0^T \int_{R^n} Z dx dt\right| \leq \left|\int_{R^n} X(x,0)dx\right| + \left|\int_{R^n} X(x,T)dx\right|.$$

Assuming $|\zeta|$ and $|\zeta'|$ are ≤ 1, we get

$$
\begin{aligned}
|X| &= |-\zeta w v_r - \zeta((n-1)/(2r))wv - (1/2)\zeta' wv| \\
&\leq |w(v_r + (n-1)v/(2r))| + (1/2)|wv| \\
&\leq (1/2)[w^2 + (v_r + (n-1)v/(2r))^2] + (1/4)(v^2 + w^2) \\
&= (1/2)[w^2 + (v_r)^2 + ((n-1)/r)vv_r + ((n-1)^2/(4r^2))v^2] + (1/4)|u|^2.
\end{aligned}
$$

Since $((n-1)/r)vv_r = \nabla \cdot [((n-1)x/(2r^2))v^2] - ((n-1)(n-2)/(2r^2))v^2$,

$$
\begin{aligned}
|X| &\leq (3/4)|u|^2 + (1/2)(v_r)^2 - ((n-1)(n-3)/(8r^2))v^2 + \nabla \cdot [((n-1)x/(4r^2))v^2] \\
&\leq (3/4)|u|^2 + (1/2)(v_r)^2 + \nabla \cdot [((n-1)x/(4r^2))v^2].
\end{aligned}
$$

Thus

$$\left|\int_{R^n} X(x,0)dx\right| + \left|\int_{R^n} X(x,T)dx\right| \le \int_{R^n} \left[(3/2)|u|^2 + (|u_r|)^2\right] dx$$
$$\le \int_{R^n} \left[(3/2)|u|^2 + (|\nabla u|)^2\right] dx$$

and we have

$$\left|\int_{R^n} X(x,0)dx\right| + \left|\int_{R^n} X(x,T)dx\right| \le c_1, \text{ where } c_1 \text{ depends on } E,\ |a|,\ b, \text{ and } L.$$

Hence, for all $T > 0$,

$$\left|\int_0^T \int_{R^n} Z dxdt\right| \le c_1.$$

Let $\zeta(r) = 1$. We get

$$\int_0^T \int_{R^n} \big\{-a[(1/r)(|\nabla u|^2 - |u_r|^2)$$
$$+((n-1)(n-3)/(4r^3))|u|^2] + ((n-1)/(2r))(q(|u|^2)|u|^2 - Q(|u|^2))$$
$$+(b/2)[(1/r)(|\nabla(h(|u|^2))|^2 - ((h(|u|^2))_r)^2) + ((n-1)/(2r))(|\nabla(h(|u|^2))|^2)$$
$$+((n-1)(n-3)/r^3)G(|u|^2) + ((n-1)/r)h'(|u|^2)h''(|u|^2)|u|^2|\nabla(|u|^2)|^2]\big\}\, dxdt \le c_1,$$

for all $T > 0$.

Note that all the terms in the integrand on the left-hand side are nonnegative by the assumptions (A3), (A4), and (A5).

Thus

$$\int_0^\infty \int_{R^n} ((n-1)(n-3)/r^3)|u|^2 dxdt \le c_2,$$
$$\int_0^\infty \int_{R^n} ((n-1)/r)(q(|u|^2)|u|^2 - Q(|u|^2))dxdt \le c_2,$$
$$\int_0^\infty \int_{R^n} ((n-1)/r)|\nabla(h(|u|^2))|^2 dxdt \le c_2,$$
$$\int_0^\infty \int_{R^n} ((n-1)(n-3)/r^3)G(|u|^2)dxdt \le c_2,$$
$$\int_0^\infty \int_{R^n} ((n-1)/r)h'(|u|^2)h''(|u|^2)|u|^2|\nabla(|u|^2)|^2 dxdt \le c_2,$$

where c_2 depends only on E, $|a|$, b, and L.

4. Integrability of the local energy. In what follows, we let $\zeta(r) = 1 - (1/(2(1+r)))$. Thus,

$$0 < \zeta < 1,\ 0 < \zeta' < 1, \text{ and } \zeta' < \zeta/r.$$

Let us consider the case $n = 3$ first. From (2), using the assumption (A4) and the inequality

$$(\zeta'' + \zeta'(2/r))(ww_r + vv_r) = (r\zeta'' + 2\zeta')((w/r)w_r + (v/r)v_r)$$
$$\ge -(1/2)\,|r\zeta'' + 2\zeta'|\,(|u|^2/r^2 + |u_r|^2)$$

we get

$$
\begin{aligned}
Z \geq & -a\left\{(-1/4)\,|r\zeta''+2\zeta'|\,(|u|^2/r^2+|\nabla u|^2)+(\zeta/r-\zeta')(|\nabla u|^2-|u_r|^2)\right.\\
&\left.+\zeta'|\nabla u|^2+(\zeta'/(2r^2))|u|^2\right\}+((c_0/2)(\zeta/r)-\zeta'/2)Q(|u|^2)\\
&+(b/2)\left\{(\zeta/r-\zeta')(\left|\nabla(h(|u|^2))\right|^2-((h(|u|^2))_r)^2)+(1/2)(2\zeta/r+\zeta')\left|\nabla(h(|u|^2))\right|^2\right.\\
&\left.-(2\zeta''/r)G(|u|^2)+2(\zeta/r)h'(|u|^2)h''(|u|^2)|u|^2\left|\nabla(|u|^2)\right|^2\right\}\\
= & -a\left\{(\zeta'/2-(1/4)|r\zeta''+2\zeta'|)(1/r^2)|u|^2+(\zeta'/2-(1/4)|r\zeta''+2\zeta'|)|\nabla u|^2\right.\\
&\left.+(\zeta'/2)\,|\nabla u|^2+(\zeta/r-\zeta')(|\nabla u|^2-|u_r|^2)\right\}\\
&+((c_0/2)(\zeta/r)-\zeta'/2)Q(|u|^2)\\
&+(b/2)\left\{(\zeta/r-\zeta')(|\nabla(h(|u|^2))|^2-((h(|u|^2))_r)^2)+(1/2)(2\zeta/r+\zeta')|\nabla(h(|u|^2))|^2\right.\\
&\left.-(2\zeta''/r)G(|u|^2)+2(\zeta/r)h'(|u|^2)h''(|u|^2)|u|^2|\nabla(|u|^2)|^2\right\}.
\end{aligned}
$$

Using $\zeta(r)=1-(1/(2(1+r)))$, we get

$$
\begin{aligned}
Z \geq & -a\left\{(1/(4r(1+r)^3))|u|^2+(r/(4(1+r)^3))|\nabla u|^2+(1/(4(1+r)^2))|\nabla u|^2\right\}\\
&+(r/(6(1+r)^2))Q(|u|^2)\\
&+(b/2)\left\{(1/(1+r))|\nabla(h(|u|^2))|^2+(2/(r(1+r)^3))G(|u|^2)\right.\\
&\left.+((2r+1)/(r(1+r)))h'(|u|^2)h''(|u|^2)|u|^2|\nabla(|u|^2)|^2\right\}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_0^\infty\int_{R_n}(1/(r(1+r)^3)|u|^2dxdt \leq c_3\\
&\int_0^\infty\int_{R_n}(1/(1+r)^2)|\nabla u|^2dxdt \leq c_3\\
&\int_0^\infty\int_{R_n}(r/(1+r)^2)Q(|u|^2)dxdt \leq c_3\\
&\int_0^\infty\int_{R_n}(1/(1+r))|\nabla(h(|u|^2))|^2dxdt \leq c_3\\
&\int_0^\infty\int_{R_n}(1/(r(1+r)^3))G(|u|^2)dxdt \leq c_3\\
&\int_0^\infty\int_{R_n}(1/(1+r))h'(|u|^2)h''(|u|^2)|u|^2\left|\nabla(|u|^2)\right|^2dxdt \leq c_3
\end{aligned}
$$

where c_3 depends on $|a|$, b, E, and L.

Thus, for $R > 0$,

$$\int_0^\infty \int_{|x|\le R} |u|^2 dxdt \le c_4 \tag{3a}$$

$$\int_0^\infty \int_{|x|\le R} |\nabla u|^2 dxdt \le c_4 \tag{3b}$$

$$\int_0^\infty \int_{|x|\le R} Q(|u|^2) dxdt \le c_4 \tag{3c}$$

$$\int_0^\infty \int_{|x|\le R} \left|\nabla(h(|u|^2))\right|^2 dxdt \le c_4 \tag{3d}$$

$$\int_0^\infty \int_{|x|\le R} G(|u|^2) dxdt \le c_4 \tag{3e}$$

$$\int_0^\infty \int_{|x|\le R} h'(|u|^2)h''(|u|^2)|u|^2 \left|\nabla(|u|^2)\right|^2 dxdt \le c_4 \tag{3f}$$

where c_4 depends on $|a|$, b, R, E, and L.

For the case $n > 3$, we can get the same result as (3a) – (3f) by rewriting

$$\zeta"(ww_r + vv_r) = \nabla \cdot [(x/(2r))\zeta''|u|^2] - (1/2)\zeta'''|u|^2 - ((n-1)/(2r))\zeta''|u|^2$$

and using $\zeta'((n-1)/r)(ww_r + vv_r) \ge -\zeta'(((n-1)^2/(4r^2))|u|^2 + |u_r|^2)$.

The c_4 in this case would depend on n as well.

Thus, for $n \ge 3$ and $R > 0$,

$$\int_0^\infty E_R(t)dt = \int_0^\infty \int_{|x|\le R} \left[-a|\nabla u|^2 + Q(|u|^2) + (b/2)\left|\nabla(h(|u|^2))\right|^2\right](x,t)dx \le c_5,$$

where c_5 depends on $|a|$, b, R, n, E, and L.

Hence the local energy is integrable in time from 0 to ∞.

5. Local L^2-norm decay. Let $R > 0$, and ϕ be a C^∞ real-valued function such that $\phi(x) = 0$ for $|x| \ge 2R$, $\phi(x) = 1$ for $|x| \le R$, and $0 \le \phi(x) \le 1$ for all x in R^n. Multiplying equation (2) by ϕu^* and taking the imaginary part of it, we get

$$\phi(|u|^2)_t = ia\nabla \cdot (\phi((\nabla u)u^* - (\nabla u^*)u)) - ia\nabla\phi \cdot ((\nabla u)u^* - (\nabla u^*)u).$$

Hence,

$$\left|\int_{|x|\le 2R} \phi(x)(|u|^2)_t dx\right| \le M \int_{|x|\le 2R} (|u|^2 + |\nabla u|^2) dx$$

where M depends on $|a|$ and the maximum value of $|\nabla\phi|$.

Thus

$$
\begin{aligned}
&\int_{|x|\le R}|u|^2(x,t)dx\\
\le&\int_{|x|\le 2R}\phi(x)(|u|^2)(x,t)dx\\
=&\int_{t-1}^{t}(\tau-t+1)\left(\int_{|x|\le 2R}\phi(x)(|u|^2(x,\tau))_\tau dx\right)d\tau+\int_{t-1}^{t}\int_{|x|\le 2R}\phi(x)(|u|^2)dxd\tau\\
\le&\int_{t-1}^{t}\left|\int_{|x|\le 2R}\phi(x)(|u|^2)_\tau dx\right|d\tau+\int_{t-1}^{t}\int_{|x|\le 2R}\phi(x)(|u|^2)dxd\tau\\
\le&(M+1)\int_{t-1}^{t}\int_{|x|\le 2R}(|u|^2+|\nabla u|^2)dxd\tau
\end{aligned}
$$

Hence

$$\int_{|x|\le R}|u|^2(x,t)dx\to 0 \text{ as } t\to\infty$$

by (3a) and (3b).

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