JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 5, Number 3, 249–280, 2007

QUANTIZATION OF THE SERRE SPECTRAL SEQUENCE

Jean-Francois Barraud and Octav Cornea

The present paper is a continuation of our earlier work [Lagrangian intersections and the Serre spectral sequence, Ann. of Math. **166** (2007), 657–722.]. It explores how the spectral sequence introduced there interacts with the presence of bubbling. As consequences are obtained some relations between binary Gromov–Witten invariants and relative Ganea–Hopf invariants, a criterion for detecting the monodromy of bubbling as well as algebraic criteria for the detection of periodic orbits.

Dedicated to Dusa McDuff

Contents

1. Introduction

In [**1**] has been introduced an algebraic way to encode the properties of high-dimensional moduli spaces of trajectories in Morse–Floer type theories. The basic idea is that, by making use of a "representation" theory of the relevant moduli spaces

$$\mathcal{M}(x,y) \xrightarrow{l_{x,y}} G$$

into some sufficiently large topological monoid G, one can define a "rich" Morse type chain complex whose differential is of the usual form

$$dx = \sum_y a_{x,y} y,$$

but $a_{x,y}$, the coefficient "measuring" the moduli space $\mathcal{M}(x,y)$, belongs to a graded ring (for example, the ring of cubical chains of G) and is, in general, not zero when $\dim(\mathcal{M}(x,y)) > 0$. By representation theory, it is meant here not only that the maps $l_{x,y}$ are continuous but also that they are compatible in the obvious way with compactification and with the crucial boundary formula:

$$\partial\overline{\mathcal{M}}(x,y) = \bigcup_z \overline{\mathcal{M}}(x,z) \times \overline{\mathcal{M}}(z,y). \tag{1.1}$$

The complex constructed this way comes with a natural filtration induced by the grading of the generators $x, y, \ldots$. The pages of order greater than 1 of the associated spectral sequence are invariant with respect to the various choices made in the construction and their differentials encode algebraically the properties of the $\mathcal{M}(x,y)$'s.

This construction is described in the absence of bubbling in [**1**] and, in [**2**], it is shown to be easily extendable to cases when pseudo-holomorphic spheres and disks exist as long as we work under the threshold of bubbling.

The present paper explores what happens when bubbling does occur.

It is obvious that, to study this case, it is natural to start with the Hamiltonian version of Floer homology and this is indeed the setting of this paper. In particular, the moduli spaces $\mathcal{M}(x,y)$ consist of Floer tubes and the monoid G is the space of pointed Moore loops on M, ΩM, with (M^{2n},ω) our underlying symplectic manifold. We will also restrict to the monotone case even if the machinery described here appears to extend to the general case. The reason for this is that the main phenomena we have identified are already present in this case and, at the same time, in this way we avoid to deal with the well-know transversality issues which are present in full generality.

Here is a short summary of our findings. First, it is not surprising that when bubbling is possible, only some of the pages of the spectral sequence mentioned before exist. It is also expectable that the number of pages that are defined should roughly be the minimal Chern class, $c_{\min}$, and that, moreover, some of these pages should again be independent of the choices made in the construction.

What is remarkable is that, in general, these pages do not coincide with those associated to a Morse function: a quantum deformation is generally present. Given that in the Morse case the resulting spectral sequence is, as shown in [**1**], the Serre spectral sequence of the path-loop fibration over M, we see that this construction provides a new symplectic invariant which consists of the first $c_{\min}$ pages (together with their differential) of a spectral sequence which is a quantum deformation of the Serre spectral sequence. One additional important point is that, on the last defined page, the presumptive differential, d^r, is still defined and invariant but might not verify $(d^r)^2 = 0$.

Of course, the next stage is to understand — at least in part — this quantum deformation in terms of classical Gromov–Witten invariants. In this respect, we obtain that the quantum part of the first interesting differential, d^2, can be expressed in terms of binary Gromov–Witten invariants (these are those associated to spheres with two marked points) and, often times, the classical part of the differential can be expressed in terms of Ganea–Hopf invariants. In this case, the relation $(d^2)^2 = 0$ becomes a relation between these two types of invariants which takes place in the Pontryagin ring $H_*(\Omega M)$. Undoubtedly, this is just a first step towards understanding the deeper relationships between the combinatorics of Gromov–Witten invariants and classical algebraic topology invariants encoded in the ring structure of $H_*(\Omega M)$. In a different direction, in the case of a Hamiltonian fibration over S^2, the components of the differentials involving curves of degree 1 over the base can be thought of as extensions of the Seidel morphism [**11**].

The next interesting point is to understand what happens for the first r for which $(d^r)^2 \neq 0$. Clearly, the culprit is bubbling but interestingly enough what this non-vanishing relation detects is monodromy — the fact that in the appropriate moduli space the attachment point of the bubbles turns non-trivially around Floer cylinders — which turns out to interfere with the representation maps $l_{x,y}$. The fact that d^r is invariant but, simultaneously, $(d^r)^2$ might not vanish is quite remarkable and, indeed, this morphism d^r by itself is already of some interest.

Finally, we also discuss an application of this structure to the detection of periodic orbits. This provides a sort of algebraic counterpart to the result of Hofer–Viterbo [**6**].

The paper is structured as follows. In the second section, we introduce the main notation and give the precise statements of our results. The third section contains the proofs. In the last section, we first shortly mention some possible extensions of the construction, we then provide some examples and, finally, we discuss the application to periodic orbits.

2. Notation and statement of results

2.1. Setting and recalls. Fix the symplectic manifold (M^{2n},ω) and we suppose for now that M is closed. We also assume that M is monotone in the sense that the two morphisms $\omega:\pi_2(M)\to\mathbb{R}$ and $c_1:\pi_2(M)\to\mathbb{Z}$ are proportional with a positive constant of proportionality ρ. We denote by $c_{\min}$ the minimal Chern class and by $\omega_{\min}$ the corresponding minimal positive symplectic area (so that we have $\omega_{\min}=\rho c_{\min}$).

2.1.1. Binary Gromov–Witten invariants. Fix on M a generic almost complex structure J which tames ω. The binary Gromov–Witten invariants we are interested in can be described as follows: pick a generic Morse function f and a metric on M. Denote by $i(x)=\mathrm{ind}_f(x)$ for each $x\in\mathrm{Crit}(f)$. For two critical points x and y and a class $\alpha\in\pi_2(M)$ such that $i(x)-i(y)+2c_1(\alpha)-2=0$, we define $\mathrm{GW}_\alpha(x,y)$ as the number of elements in the moduli space $\mathcal{M}(J,\alpha;x,y)$ which consists of J-holomorphic spheres in the homology class α with two marked points, one lying on the unstable manifold of x and the other on the stable manifold of y, modulo reparametrization. As such $\mathrm{GW}_\alpha(x,y)$ is not an invariant (because x,y might not be Morse cycles). However, if for two Morse homology classes $[x]=[\sum\lambda_i x_i]$ and $[y]=[\sum\mu_i y_i]$, we define $\mathrm{GW}_\alpha([x],[y])=\sum\lambda_i\mu_j\mathrm{GW}_\alpha(x_i,y_j)$ then we obtain an invariant. For $\alpha\in\pi_2$, let $[\alpha]$ be its image by the morphism $\pi_2(M)\to H_1(\Omega M)$.

2.1.2. The Novikov ring. Let $\mathcal{L}(M)$ be the space of contractible loops in M.

Let Γ be the image of the Hurewicz morphism $\pi_2(M)\to H_2(M,\mathbb{Z}/2)$. The two forms ω and c_1 define morphisms $\Gamma:\xrightarrow{\omega,c_1}\mathbb{R},\mathbb{Z}$ which under our monotonicity assumption are proportional. Let $\Gamma_0=\Gamma/\ker(\omega)$. We let Λ be the associated Novikov ring which is defined as follows

$$\Lambda=\left\{\sum_{\alpha\in\Gamma_0}\lambda_\alpha e^\alpha\right\},$$

where the coefficients λ_α belong to $\mathbb{Z}/2$ such that

$$\forall c>0,\quad \sharp\{\alpha,\lambda_\alpha\neq 0,\omega(\alpha)\le c\}<+\infty.$$

The grading of the elements in Λ is given by $|e^\lambda|=-2c_1(\lambda)$.

We also denote by $\tilde{\mathcal{L}}(M)$ the covering of $\mathcal{L}(M)$ associated to Γ_0: it is the quotient of the space of couples (γ,Δ), where $\gamma\in\mathcal{L}(M)$ and Δ is a disk bounded by γ, under the equivalence relation $(\gamma,\Delta)\sim(\gamma',\Delta')$ if $\gamma=\gamma'$ and $\omega([\Delta-\Delta'])=c_1([\Delta-\Delta'])=0$.

Remark 2.1. Here and later in the paper we could also use, alternatively, rational coefficients as all the moduli spaces involved are orientable and the orientations are compatible with our formulae.

2.1.3. Moduli spaces of Floer tubes. Let $H : M \times S^1 \to \mathbb{R}$ be a Hamiltonian function. The Hamiltonian flow associated to H is the flow of the (time dependent) vector field X_H defined by:

$$\omega(X_{H_t}, \cdot) = -\, dH_t.$$

All along this paper, the periodic orbits of X_H will be supposed to be non-degenerate. We denote by $\mathcal{P}_H \subset \mathcal{L}(M)$ the set of all contractible periodic orbits of the Hamiltonian flow associated to H and we let $\tilde{\mathcal{P}}_H$ be the covering of $\mathcal{P}_H$ which is induced from $\tilde{\mathcal{L}}(M)$.

For each periodic orbit $x \in \mathcal{P}_H$, we fix a lift $(x, \Delta_x) \in \tilde{\mathcal{P}}_H$. For a generic pair (H, J) and $x, y \in \mathcal{P}_H$, $\lambda \in \Gamma_0$, we now consider the moduli spaces

$$\mathcal{M}'(x, y; \lambda) = \{u : \mathbb{R} \times S^1 : u \text{ verifies } (2.1)\},$$

so that the pasted sphere $\Delta_x \cup u \cup (-\Delta_y)$ is of class λ and

(2.1)
$$\partial_s u + J(u)\partial_t u - J(u)X_H(u) = 0, \quad \lim_{s\to-\infty} u(s,t) = x(t), \quad \lim_{s\to+\infty} u(s,t) = y(t).$$

Of course, these moduli spaces are quite well known in the field and we refer to [**10**] for their properties. In particular, they have natural orientations and, when $(x, \Delta_x) \neq (y, \Delta_y)$ they admit a free $\mathbb{R}$-action. We denote the quotient by this action by $\mathcal{M}(x, y; \lambda)$ and we have

$$\dim \mathcal{M}(x, y; \lambda) = \mu\left((x, \Delta_x)\right) - \mu\left((y, \Delta_y)\right) + 2c_1(\lambda) - 1,$$

where $\mu\left((x, \Delta_x)\right)$ is the Conley–Zehnder index of the orbit x computed with respect to the capping disk Δ_x.

2.1.4. Monodromy of bubbling. Among the standard properties of the moduli spaces above we recall that they admit a natural topology as well as natural compactifications, $\overline{\mathcal{M}}(x, y; \lambda)$, so that the following formula is valid:

$$\partial\overline{\mathcal{M}}(x, y; \lambda) = \bigcup_{z, \lambda' + \lambda'' = \lambda} \overline{\mathcal{M}}(x, z; \lambda') \times \overline{\mathcal{M}}(z, y; \lambda'') \;\cup\; \Sigma_{x,y,\lambda}. \tag{2.2}$$

Here $\Sigma_{x,y,\lambda}$ is a set of codimension 2 which consists of Floer tubes with at least one attached bubble.

We will say that (H, J) has *bubbling monodromy* if there exist $x, y \in \mathcal{P}_H$ and $\lambda \in \Gamma_0$ such that:

$$H^1(\Sigma_{x,y,\lambda}; \mathbb{Z}) \neq 0.$$

This means, in particular, that $\pi_1(\Sigma_{x,y,\lambda}) \neq 0$ so that there are non-contractible loops in the space of Floer tubes with bubbles.

2.1.5. Truncated differentials and spectral sequences. The following algebraic notions will be useful in the formulation of our results.

We say that the sequence of graded vector spaces (E^r, d^r), $0 \le r \le k$ is a *truncated spectral sequence of order* k if (E^r, d^r) is a chain complex for each $r \le k-1$ which verifies $H_*(E^r, d^r) = E^{r+1}$ and d^k is a linear map of degree -1. A truncated spectral sequence of ∞-order is a usual spectral sequence. A morphism of order k truncated spectral sequences is a sequence of chain maps $\phi_r : (E^r, d^r) \to (F^r, d^r)$, $0 \le r \le k$ such that $H_*(\phi_r) = \phi_{r+1}$ for $0 \le r \le k-1$. We say that two truncated spectral sequences are isomorphic starting from page s is they admit a morphism which is an isomorphism on page s (and hence on each later page).

The typical example of a truncated spectral sequence appears as follows. Assume that C_* is a graded vector space and that F^iC is an increasing filtration of C_*. We say that a linear map $d : C_* \to C_{*-1}$ is a *truncated differential of order* k compatible with the given filtration if $d(F^iC) \subset F^iC$ $\forall i$ and

$$(d \circ d)(F^rC) \subset F^{r-2k}C$$

for all $r \in \mathbb{Z}$. It is easy to see that a truncated differential of order k induces a truncated spectral sequence of the same order. Indeed, by using the standard descriptions of the r-cycles

$$Z_p^r = \{v \in F^pC : dv \in F^{p-r}C\} + F^{p-1}C$$

and r-boundaries

$$B_p^r = \{dF^{p+r-1}C \cap F^pC\} + F^{p-1}C,$$

it is immediate to see that $B_p^r \hookrightarrow Z_p^r$ for $0 \le r \le k$ which allows us to define the pages of the truncated spectral sequence by $E_p^r = Z_p^r / B_p^r$. Obviously, d induces differentials d^r on E^r when $r < k$ as well as a degree -1 linear map d^k on E^k.

2.2. Main statement. We will formulate our main statement in a simple case and we will discuss various extensions at the end of the paper. Therefore, we assume here that (M, ω) is closed, simply-connected and monotone with $c_{\min} \ge 2$.

Theorem 2.2. *There exists a truncated spectral sequence of order* $c_{\min}$*,* $(E^r(M), d^r)$*, whose isomorphism type starting from page 2 is a symplectic invariant of* (M, ω) *and which has the following additional properties.*

(i) *As a bi-graded vector space we have an isomorphism*

$$E^2 \cong H_*(M) \otimes H_*(\Omega M) \otimes \Lambda.$$

(ii) *The differential* d^2 *has the decomposition*

$$d^2 = d_0^2 + d_Q^2,$$

where d_0^2 is the differential appearing in the classical Serre spectral sequence of the path loop fibration $\Omega M \to PM \to M$ and d_Q^2 is a $H_*(\Omega M)\otimes\Lambda$ module map given on each $x \in H_*(M)$ by:

$$d_Q^2 x = \sum_{y,\alpha} \mathrm{GW}_\alpha(x,y) y[\alpha] e^\alpha.$$

(iii) *If $(d^{c_{\min}})^2 \neq 0$, then any regular pair (H, J) has bubbling monodromy.*

Remark 2.3. In certain cases, the equation $d^2 \circ d^2 = 0$ (which is verified, for example, if $c_{\min} \geq 3$), translates into relations between binary Gromov–Witten and certain relative Ganea–Hopf invariants which take place in the Pontryagin algebra $H_*(\Omega M)$. To make these relations explicit, we first recall the construction of the Ganea–Hopf invariants. They are defined in the presence of two cofibration sequences $S^{q-1} \xrightarrow{r} X \to X'$ and $S^{p-1} \xrightarrow{s} X' \to X'' \subset M$ (here X, X', X'' are finite type CW-complexes included in the ambient manifold M) by the following procedure. There is a co-action map associated to the first co-fibration sequence, $j : X' \to X' \vee S^q$, which we compose with the inclusion $X' \vee S^q \hookrightarrow M \vee S^q$ to get a map $\hat{j} : X' \to M \vee S^q$. We then consider the composition $\bar{j} = \hat{j} \circ s : S^{p-1} \to M \vee S^q$ and we see that its projection onto M is null-homotopic. This means that $\bar{j}$ lifts to the homotopy fiber of $M \vee S^q \to M$ which is homotopy equivalent to $\Sigma^q(\Omega M^+)$, where the $+$ indicates the addition of a disjoint base point and Σ represents the pointed suspension. The homotopy class of this lift is easily seen to be unique and thus we obtain a homotopy class $H(s,r) \in \pi_{p-1}(\Sigma^q(\Omega M^+))$ called the relative Ganea–Hopf invariant associated to s, r. In case the two cofibration sequences correspond to consecutive cell attachments associated to passage through two consecutive critical points of a Morse function f, there is a remarkable geometric interpretation of this invariant: it equals the framed bordism class of the moduli space of negative gradient flow lines of f which connect these two critical points [**5**]. Moreover, as shown in [**4**], under some restrictions these invariants can be interpreted as differentials in the stable homotopy Atiyah–Hirzebruch–Serre spectral sequence of the path loop fibration $\Omega M \to PM \to M$. The purely homological content of this, $[H(s,r)] \in H_{p-q-1}(\Omega M)$, is therefore identified with the corresponding differential — d_0^2 with the notation of the theorem when $p - q = 2$ — in the Serre spectral sequence of the same fibration.

To see a particular case of interest here, assume that the symplectic manifold M is simply-connected and has (integral) homology only in even degrees. This ensures the fact that M admits a perfect Morse function $f : M \to \mathbb{R}$ (that is a Morse function whose associated Morse complex has trivial differential). Given two critical points a, b such that $\mathrm{ind}_f(a) = \mathrm{ind}_f(b) + 2$, the discussion above shows that we may define a relative Ganea–Hopf invariant

$[H(a,b)] = [H(s,r)] \in H_1(\Omega M)$, where s and r are the attaching maps corresponding, respectively, to the critical points a and b. As the Morse function is perfect each critical point is a Morse cycle and the classical differential d_0^2 is the unique linear extension of the map given by

$$d_0^2(a) = \sum_{|b| = |a|-2} [H(a,b)]b.$$

Finally, the relation $d^2 \circ d^2 = 0$ translates into:

$$\begin{aligned} &\sum_{|y| = |x| + 2c_1(\alpha)-2,\ \alpha+\beta=\gamma} \mathrm{GW}_\alpha(x,y)\mathrm{GW}_\beta(y,z)\ [\alpha]\cdot[\beta] \\ &\quad + \sum_{|y'| = |z|+2} \mathrm{GW}_\gamma(x,y')\ [\gamma]\cdot[H(y',z)] \\ &\quad + \sum_{|y''| = |x|-2} \mathrm{GW}_\gamma(y'',z)\ [H(x,y'')]\cdot[\gamma] = 0 \end{aligned}$$

which is valid for any pair x, z and γ such that $|x| - |z| + 2c_1(\gamma) - 4 = 0$ (where $-\cdot-$ is the Pontryagin product $H_1(\Omega M)\times H_1(\Omega M)\to H_2(\Omega M)$).

The relation with the Seidel homomorphism is seen by considering the spectral sequence in the case of a symplectic fibration over $\mathbb{CP}^1$.

We also formulate here a very simple version of our application to the detection of periodic orbits. We specialize to the case when the manifold M admits a perfect Morse function. We also need the following notion. Let $x, y \in H_*(M)$ and $\lambda \in \Lambda$. We will say that x and ze^λ (which exist on the E^2 page of the spectral sequence in Theorem 2.2) are d^r-related if x survives to the rth level of the spectral sequence and there is some $\gamma \in C_*(\Omega M)$ so that the product $\gamma \otimes ze^\lambda$ also survives to the rth page of the spectral sequence and we have $d^r([x]) = [\gamma \otimes ze^\lambda] + \cdots$.

Corollary 2.4. *Assume that there are homology classes $x, z \in H_*(M)$, $|x| < |z|$, so that x is d^r-related to ze^λ and $H_k(M)\otimes\Lambda_q = 0$ for $|x| > k+q > |ze^\lambda|$). Then for any self-indexed perfect Morse function f on M the set of values v so that $f^{-1}(v)$ contains a closed characteristic is dense in $[f(x), f(z)]$.*

By a self-indexed Morse function f we mean here that the critical points of the same index have the same critical value and $\mathrm{ind}_f(x) > \mathrm{ind}_f(y)$ implies $f(x) > f(y)$. This implies that for any homology class $a \in H_*(M)$ the number $f(a)$ is well defined.

There are many ways in which this statement can be extended and some will be discussed at the end of the paper.

3. Proof of the main theorem

3.1. Construction of the truncated spectral sequence. In this section, we fix the 1-periodic Hamiltonian H and almost complex structure J compatible with ω so that the pair (H, J) is generic (of course, both are in general time-dependent). For simplicity, we will also assume to start that the manifold M is simply-connected but we will see later on that this condition can be dropped with the price that the construction becomes more complicated.

As in [**1**], the truncated spectral sequence we intend to discuss is induced by a natural filtration of an enriched Floer type pseudo-complex. We use the term pseudo-complex here to mean that we will not have a true differential but rather a truncated one. The construction of this pseudo-complex is a refinement of the classical Floer construction in which the coefficient ring is replaced with the ring of cubical chains over the Moore loops on M. Here is this construction in more detail.

3.1.1. Coefficient rings. Let C_* denote the "cubical" chain complex, let ΩX be the Moore loop space over X (the space of loops parameterized by intervals of arbitrary length). Consider the space M' obtained from M by collapsing to a point a simple path γ going through the starting point of each periodic orbit. Notice that $C_*(\Omega M')$ is a differential ring where the product is induced by the concatenation of loops. Finally, our coefficient ring is

$$\mathcal{R}_* = C_*(\Omega M') \otimes \Lambda.$$

This is a (non-abelian) differential ring, and its differential will be denoted by ∂.

The (pseudo)-complex we are interested in is a (left) differential module generated by the contractible periodic orbits of H over this ring,

$$C(H, J) = \oplus_{\tilde{x} \in \tilde{\mathcal{P}}_H} \mathcal{R}_* \; \tilde{x} / \sim,$$

with the identification $\tilde{x}e^{\lambda} \sim \tilde{x}\sharp\lambda$, where $\tilde{x}\sharp\lambda$ stands for the capping of x obtained by gluing a sphere in the class λ to $\tilde{x}$. The grading of an element in $\tilde{x} \in \tilde{\mathcal{P}}_H$ is given by the respective Conley–Zehnder index. There is a natural filtration of this complex which is given by

$$F^r C(H, J) = C_*(\Omega M') < \tilde{x} \in \tilde{\mathcal{P}}_H : \; \mu(\tilde{x}) \leq r > .$$

We will call this the canonical filtration of $C(H, J)$.

3.1.2. Truncated boundary operator. The next step is to introduce a truncated differential on $C(H, J)$. We recall from § 2.1.3 the definition of the moduli spaces $\mathcal{M}(x, y; \lambda)$ of Floer tubes. We will write $\mathcal{M}(\tilde{x}, \tilde{y})$ for the moduli space of Floer tubes which lift to paths inside $\tilde{\mathcal{L}}(M)$ joining $\tilde{x} \in \tilde{\mathcal{P}}_H$ to $\tilde{y} \in \tilde{\mathcal{P}}_H$. With these conventions and — as assumed before — for a generic choice of J and H the moduli spaces $\mathcal{M}(\tilde{x}, \tilde{y})$ are smooth manifolds of dimension $|\tilde{x}|-|\tilde{y}|-1$ when $\tilde{x}\neq\tilde{y}$, and they have a natural compactification involving "breakings" of the tubes on intermediate orbits, or bubbling off of holomorphic spheres. We let $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ be the respective compactification. In our monotone situation, these compactifications are pseudo-cycles with boundary.

To define the truncated boundary operator we proceed as in the usual Floer complex, but we intend to take into consideration the moduli spaces of arbitrary dimensions instead of restricting to the 0-dimensional ones. To associate to the (compactification of the) moduli spaces coefficients in our ring $\mathcal{R}$, we first need to represent them into the loop space $\Omega(M')$, and then choose chains representing them (i.e., defining their fundamental classes relative to their boundary).

Let us start with "interior" trajectories, i.e., elements $v\in\mathcal{M}(\tilde{x},\tilde{y})$. Let $u:\mathbb{R}\times S^1\to M$ be a parametrization of v. Since the value of the action functional

$$\mathfrak{a}_H:\tilde{\mathcal{L}}(M)\to\mathbb{R},\quad \mathfrak{a}_H((\gamma,\Delta))=-\int_{D^2}\Delta^*\omega+\int_{S^1}H(t,\gamma(t))\,dt$$

is strictly decreasing along the $\mathbb{R}$ direction, it can be used to reparametrize u by the domain $[-\mathfrak{a}(\tilde{x}),-\mathfrak{a}(\tilde{y})]\times S^1$, and the restriction of u to the interval $[-\mathfrak{a}(\tilde{x}),-\mathfrak{a}(\tilde{y})]\times\{0\}$ defines a Moore loop in M'. This defines a map

$$\sigma_{\tilde{x},\tilde{y}}:\mathcal{M}(\tilde{x},\tilde{y})\to\Omega(M') \tag{3.1}$$

which is continuous. We will call it the "spine" map.

This map should then be extended to the compactification $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ of $\mathcal{M}(\tilde{x},\tilde{y})$.

It is well known that the objects in $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ are constituted by Floer trajectories possibly broken on some intermediate periodic orbits to which might be attached some J-holomorphic spheres that have bubbled off.

It is easy to see that the map $\sigma_{\tilde{x},\tilde{y}}$ extends continuously over the part of this set where no spheres are attached to some tube in a point belonging to the line $\mathbb{R}\times\{0\}$. Indeed, as in [**1**], except for these types of elements, the spine map is compatible with the breaking of Floer tubes in the sense that the loop associated to a broken trajectory is the product of the loops associated to each "tube" component.

Let $\alpha_{\min}\in\Gamma_0$ be the class so that $c_1(\alpha_{\min})=c_{\min}$ (by our monotonicity assumption there is a single such class). By using again the monotonicity assumption, we see that bubbling off of a sphere in class $\alpha\in\Gamma_0$ can occur in a moduli space $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ with $\tilde{y}\neq\tilde{x}\sharp\alpha$ only if

$$|\tilde{x}|-|\tilde{y}|\geq 2c_1(\alpha)+1.$$

It is also important to note that bubbling of a sphere in the class α is also possible inside the space $\overline{\mathcal{M}}(\tilde{x},\tilde{x}\sharp\alpha)$. In all cases, bubbling of an α sphere is never possible if $|\tilde{x}|-|\tilde{y}|\leq 2c_1(\alpha)-1$.

We summarize this discussion:

Lemma 3.1. *The spine map $\sigma_{\tilde{x},\tilde{y}}$ extends continuously to $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ if*

$$|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1.$$

In case $|\tilde{x}|-|\tilde{y}|=2c_{\min}$ and if σ does not have such a continuous extension to $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$, then $\tilde{y}=\tilde{x}\sharp\alpha_{\min}$.

The spine map obtained in this way satisfies also a compatibility condition which we now make explicit. Given the inclusion $\mathcal{M}(\tilde{x},\tilde{z})\times\mathcal{M}(\tilde{z},\tilde{y})\subset\overline{\mathcal{M}}(\tilde{x},\tilde{y})$, the restriction of $\sigma_{\tilde{x},\tilde{y}}$ on the set on the left of the inclusion equals $m\circ(\sigma_{\tilde{x},\tilde{z}}\times\sigma_{\tilde{z},\tilde{y}})$ where

$$m:\Omega M'\times\Omega M'\rightarrow\Omega M'$$

is loop concatenation.

For pairs $(\tilde{x},\tilde{y})$ with $|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1$, we use the map $\sigma_{\tilde{x},\tilde{y}}$ to represent the moduli spaces $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ inside the loop space $\Omega(M')$. We then choose "chain representatives" $m(\tilde{x},\tilde{y})\in C_*(\Omega M')$, i.e., chains generating the fundamental class of $\sigma(\overline{\mathcal{M}}(\tilde{x},\tilde{y}))$ relative to its boundary, in such a way that

$$\partial m(\tilde{x},\tilde{y})=\sum_{|\tilde{y}|<|\tilde{z}|<|\tilde{x}|}m(\tilde{x},\tilde{z})*m(\tilde{z},\tilde{y}),\tag{3.2}$$

where $*$ is the operation induced on $C_*(\Omega M')$ by the concatenation of loops.

The key point regarding this formula is that, under our assumption $|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1$, the compactified moduli space $\overline{\mathcal{M}}(\tilde{x},\tilde{y})$ is a manifold with boundary. Moreover, its boundary verifies the usual formula valid in the absence of bubbling so that the construction of the $m(-,-)$s is the same as that in the non-bubbling setting. We refer to [**1**] for a complete discussion of this construction.

We now define the boundary operator d by

$$d\tilde{x}=\sum_{1\leq|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1}m(\tilde{x},\tilde{y})\ \tilde{y},\tag{3.3}$$

and extend it to the full complex using the Leibniz rule.

It is easy to check that d has degree -1 with respect to the total grading and that it is compatible with the canonical filtration. Notice first that if

$\gamma\otimes\tilde{x}\in C_*(\Omega M)\otimes\tilde{\mathcal{P}}_H$ we have $d\circ d(\gamma\otimes\tilde{x})=(\gamma\otimes(d\circ d)(\tilde{x}))$. We now compute

$$\begin{aligned} d\circ d(\tilde{x}) &= \sum_{|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1} d(m(\tilde{x},\tilde{y})\ \tilde{y}) \\ &= \sum_{1\leq|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1} \partial m(\tilde{x},\tilde{y})\ \tilde{y}+m(\tilde{x},\tilde{y})\ d\tilde{y} \\ &= \sum_{\substack{1\leq|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1\\ |\tilde{y}|+1\leq|\tilde{z}|\leq|\tilde{x}|-1}} m(\tilde{x},\tilde{z})m(\tilde{z},\tilde{y})\ \tilde{y} \\ &\quad+\sum_{\substack{1\leq|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1\\ 1\leq|\tilde{y}|-|\tilde{z}|\leq 2c_{\min}-1}} m(\tilde{x},\tilde{y})m(\tilde{y},\tilde{z})\ \tilde{z} \\ &= \sum_{\substack{1\leq|\tilde{x}|-|\tilde{y}|\leq 2c_{\min}-1\\ |\tilde{y}|-2c_{\min}+1\leq|\tilde{z}|\leq|\tilde{x}|-2c_{\min}}} m(\tilde{x},\tilde{y})m(\tilde{y},\tilde{z})\ \tilde{z}, \end{aligned}$$

and we see that d^2 drops the filtration index by at least $2c_{\min}$. In the algebraic terms of § 2.1.5 we obtain:

Lemma 3.2. *With the definition above, d is a truncated differential of order $c_{\min}$ with respect to the canonical filtration on $C(H,J)$ and thus it induces a truncated spectral sequence $E^r(H,J)$ of the same order so that*

$$E^2(H,J)\cong H_*(M)\otimes H_*(\Omega M)\otimes\Lambda.$$

The isomorphism in the lemma is obvious because $E^1(H,J)\cong CF_*(H,J)\otimes H_*(\Omega M)$ and as d^1 only involves the 0-dimensional moduli spaces of Floer tubes we obtain that d^1 is just: $d_F\otimes id$ where $(CF_*(H,J),d_F)$ is the usual Floer complex (with coefficients in the Novikov ring Λ). Thus, we have constructed our truncated spectral sequence and have proved property (i) in Theorem 2.2.

Remark 3.3. Without the monotonicity assumption, but still assuming that the moduli spaces in question are regular, there is no way to avoid the bubbling phenomenon, even on low-dimensional moduli spaces. However, on 2-dimensional moduli spaces, the bubbling component is 0-dimensional and hence consists in isolated points: for each of them, the real line $\mathbb{R}\times\{0\}$ can actually be deformed to avoid the point where the bubble is attached. Interpolating between these perturbed real lines with the standard one in a small neighbourhood of the "bubbled" trajectories defines a spine map for the 2-dimensional moduli spaces which satisfies the desired continuity and compatibility conditions.

3.2. Invariance of the truncated spectral sequence. To show invariance, we will proceed along Floer's original proof by first constructing a comparison morphism between the spectral sequences associated to two different sets of generic data $(H_i, J_i)_{i=0,1}$. We will describe the construction of this morphism in more detail below but we only mention here one remarkable fact: despite the fact that in our spectral sequences we might have $d^{c_{\min}} \circ d^{c_{\min}} \neq 0$ it is still true that the morphism $d^{c_{\min}}$ is invariant.

The construction uses a homotopy between (H_0, J_0) and (H_1, J_1). As in the usual Floer case, we consider a generic homotopy between them, $(G, \bar{J})$, and, for $\tilde{x} \in \tilde{\mathcal{P}}_{H_0}$ and $\tilde{x}' \in \tilde{\mathcal{P}}_{H_1}$, we consider the moduli spaces $\mathcal{N}(\tilde{x}, \tilde{x}')$ of tubes $v : \mathbb{R} \times S^1 \to M$ which lift in $\tilde{\mathcal{L}}(M)$ to a path joining $\tilde{x}$ to $\tilde{x}'$ and verify the equation

$$\partial_s u + \bar{J}(s, u(s,t))(\partial_t u - X_G(s, u(s,t))) = 0. \tag{3.4}$$

The moduli space $\mathcal{N}(\tilde{x}, \tilde{x}')$ has properties similar to those of $\mathcal{M}'(-,-)$ except that it has no $\mathbb{R}$-invariance. Its dimension is $|\tilde{x}| - |\tilde{x}'|$. Clearly, bubbling of an α-sphere inside such a moduli space is not possible if $|\tilde{x}| - |\tilde{x}'| \leq 2c_1(\alpha) - 1$. As in § 3.1, sphere bubbling is the only obstruction to extend the spine map. Assuming that $|\tilde{x}| - |\tilde{x}'| \leq 2c_{\min} - 1$ the spine map can therefore be extended over these spaces in a way compatible with the spine maps of (H_0, J_0) and (H_1, J_1) (as in [**1**]).

The chain morphism between the two (truncated)-complexes is defined by a formula similar to (3.3),

$$\Theta(\tilde{x}) = \sum_{0 \leq |\tilde{x}| - |\tilde{x}'| \leq 2c_{\min} - 1} m'(\tilde{x}, \tilde{x}')\tilde{x}',$$

where $m'(\tilde{x}, \tilde{x}')$ is a chain in the loop space representing the moduli space $\mathcal{N}(\tilde{x}, \tilde{x}')$ (as in [**1**]). This morphism clearly respects the canonical filtrations.

We also have:

$$\partial m'(\tilde{x}, \tilde{x}') = \sum_{|\tilde{x}'| \leq |\tilde{y}| \leq |\tilde{x}| - 1} m(\tilde{x}, \tilde{y}) m'(\tilde{y}, \tilde{x}') + \sum_{|\tilde{x}'| + 1 \leq |\tilde{y}'| \leq |\tilde{x}|} m'(\tilde{x}, \tilde{y}') m(\tilde{y}', \tilde{x}').$$

Computing $d\Theta$ and Θd we get:

$$\begin{aligned} d\Theta(\tilde{x}) &= d\left(\sum_{0 \leq |\tilde{x}| - |\tilde{x}'| \leq 2c_{\min} - 1} m'(\tilde{x}, \tilde{x}')\tilde{x}'\right) \\ &= \sum_{\substack{0 \leq |\tilde{x}| - |\tilde{x}'| \leq 2c_{\min} - 1 \\ 1 \leq |\tilde{x}| - |\tilde{y}| \leq 2c_{\min} - 1 \\ 0 \leq |\tilde{y}| - |\tilde{x}'| \leq 2c_{\min} - 1}} m(\tilde{x}, \tilde{y}) m'(\tilde{y}, \tilde{x}')\tilde{x}' \end{aligned}$$

$$+ \sum_{\substack{0 \leq |\tilde{x}|-|\tilde{x}'| \leq 2c_{\min}-1 \\ 0 \leq |\tilde{x}|-|\tilde{y}'| \leq 2c_{\min}-1 \\ 1 \leq |\tilde{y}'|-|\tilde{x}'| \leq 2c_{\min}-1}} m'(\tilde{x}, \tilde{y'}) m(\tilde{y}', \tilde{x}')\tilde{x}'$$

$$+ \sum_{\substack{0 \leq |\tilde{x}|-|\tilde{x}'| \leq 2c_{\min}-1 \\ 1 \leq |\tilde{x}'|-|\tilde{y}'| \leq 2c_{\min}-1}} m'(\tilde{x}, \tilde{x}') m(\tilde{x}', \tilde{y}')\tilde{y}'$$

and

$$\Theta d(\tilde{x}) = \sum_{\substack{1 \leq |\tilde{x}|-|\tilde{y}| \leq 2c_{\min}-1 \\ 0 \leq |\tilde{y}|-|\tilde{x}'| \leq 2c_{\min}-1}} m(\tilde{x}, \tilde{y}) m'(\tilde{y}, \tilde{x}')\tilde{x}'$$

such that

$$d\Theta - \Theta d = \sum_{\substack{0 \leq |\tilde{x}|-|\tilde{x}'| \leq 2c_{\min}-1 \\ |\tilde{x}'|-|\tilde{y}'| \leq 2c_{\min}-1 \\ |\tilde{x}|-|\tilde{y}'| \geq 2c_{\min}}} m'(\tilde{x}, \tilde{x}') m(\tilde{x}', \tilde{y}')\tilde{y}'$$
$$- \sum_{\substack{0 \leq |\tilde{x}|-|\tilde{y}|-1 \leq 2c_{\min}-1 \\ |\tilde{y}|-|\tilde{x}'| \leq 2c_{\min}-1 \\ |\tilde{x}|-|\tilde{x}'| \geq 2c_{\min}}} m(\tilde{x}, \tilde{y}) m'(\tilde{y}, \tilde{x}')\tilde{x}',$$

which is not 0, but has degree at least $-2c_{\min}$ with respect to the Maslov index.

It is easy to see that this implies that Θ induces a morphism of truncated spectral sequences:

$$\bar{\Theta} : E(H_0, J_0) \to E(H_1, J_1).$$

Similarly to the isomorphism in Lemma 3.2 it is easy to see that $E^1(\Theta)$ is identified with:

$$\theta_F \otimes \mathrm{id} : CF(H_0, J_0) \otimes H_*(\Omega M) \to CF(H_1, J_1) \otimes H_*(\Omega M),$$

where θ_F is the Floer comparison morphism. As this morphism induces an isomorphism in homology we deduce that $E^2(\Theta)$, and hence all of $\bar{\Theta}$ are isomorphisms for $r \geq 2$ and this shows the invariance claim in the statement of Theorem 2.2.

Remark 3.4. A morphism of spectral sequences preserves the bi-degree, therefore to show that $\bar{\Theta}$ is a morphism we only need that $d\Theta - \Theta d$ drops the filtration degree by $c_{\min}$. In other words, a considerable part of the geometric information carried by Θ is actually forgotten in the spectral sequence. There are some ways to recover it but as this goes beyond the purpose of the present paper we will not discuss this here.

3.3. Detection of monodromy. The purpose here is to prove Theorem 2.2 (iii) thus we fix a regular pair (H, J) and we assume that $d^{c_{\min}} \circ d^{c_{\min}} \neq 0$.

We start by looking again at the calculation for $d \circ d$ given before Lemma 3.2. We see from that formula that $d^{c_{\min}} \circ d^{c_{\min}}$ is given by a linear combination of terms of the form

$$S(\tilde{x}) = \sum_{\substack{0 \leq |\tilde{x}| - |\tilde{y}| \leq 2c_{\min} - 1 \\ |\tilde{y}| - 2c_{\min} + 1 \leq |\tilde{z}| = |\tilde{x}| - 2c_{\min}}} m(\tilde{x}, \tilde{y}) m(\tilde{y}, \tilde{z}) \ \tilde{z}.$$

For each fixed $\tilde{z}$ with $|\tilde{z}| = |\tilde{x}| - 2c_{\min}$ this last sum can be rewritten as

$$S(\tilde{x}) = \sum_{\tilde{z}} S(\tilde{x}, \tilde{z})$$

with

$$S(\tilde{x}, \tilde{z}) = \sum_{|\tilde{x}| - 1 \geq |\tilde{y}| \geq |\tilde{x}| - 2c_{\min} - 1} m(\tilde{x}, \tilde{y}) m(\tilde{y}, \tilde{z}) \ \tilde{z}.$$

Suppose that $\tilde{z} \neq \tilde{x} \sharp \alpha_{\min}$. In that case, as indicated in Lemma 3.1, the spine map is well defined and continuous on the whole space $\overline{\mathcal{M}}(\tilde{x}, \tilde{z})$ and no bubbling is possible inside this space. But this means that we may find a representing chain $m(\tilde{x}, \tilde{z})$ so that, as in formula (3.2),

$$(\partial m(\tilde{x}, \tilde{z})) \ \tilde{z} = S(\tilde{x}, \tilde{z})$$

which means that $S(\tilde{x}, \tilde{z})$ vanishes in E^r for $r \geq 2$.

Thus, the only terms which count in $d^{c_{\min}} \circ d^{c_{\min}}$ are $S(\tilde{x}, \tilde{x} \sharp \alpha_{\min})$ and if $d^{c_{\min}} \circ d^{c_{\min}} \neq 0$, then at least one such term survives to E^r. To simplify notation we let $\tilde{x} \sharp \alpha_{\min} = \tilde{x}^t$. Notice that the moduli space $\overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$ is only a pseudo-cycle with boundary in the sense that it is a stratified set with three strata:

(i) a co-dimension two stratum

$$\Sigma_{\tilde{x}, \tilde{x}^t} \subset \overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$$

formed by the bubbled configurations,

(ii) a co-dimension one stratum $\partial \mathcal{M} = \cup_{\tilde{z}} \overline{\mathcal{M}}(\tilde{x}, \tilde{z}) \times \overline{\mathcal{M}}(\tilde{z}, \tilde{x}^t)$,

(iii) a co-dimension zero stratum $\mathcal{M}(\tilde{x}, \tilde{x}^t)$.

Fix now some $\tilde{x}$ and, to simplify notation, let $\Sigma = \Sigma_{\tilde{x}, \tilde{x}^t}$ and notice that Σ is a compact manifold. The spine map σ is defined on $\overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$ with the exception of Σ. Notice also that $\Sigma \cap \partial \mathcal{M} = \emptyset$. Suppose that there exists a continuous deformation σ' of σ which agrees with σ with the exception of a neighbourhood of Σ and which extends over Σ. Then, as $\overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$ is a pseudo-cycle, the same argument described above for the case $\tilde{z} \neq \tilde{x} \sharp \alpha_{\min}$ applies also here (the point is that as Σ is of co-dimension 2, the construction of representing cycles is still possible) and it shows that $S(\tilde{x}, \tilde{x}^t)$ does not play any role in E^r for $r \geq 2$. To conclude, our assumption $d^{c_{\min}} \circ d^{c_{\min}} \neq 0$, implies that there exists at least one $\tilde{x}$ so that such a deformation σ' of σ

does not exist. We now want to deduce from this that the first cohomology group of Σ does not vanish.

Given that by definition $c_1(\alpha_{\min}) = c_{\min}$, it follows that each $u \in \Sigma$ is represented by a Floer tube $\mathbb{R} \times S^1$ to which is attached a single sphere in a point $(t_u, a_u) \in \mathbb{R} \times S^1$ so that the tube is mapped in M on the constant orbit $\tilde{x}$ and the sphere is mapped to a pseudo-holomorphic sphere in the class $\alpha_{\min}$. Thus, there is a continuous map

$$\xi : \Sigma \to S^1$$

such that $\xi(u) = a_u$.

To show that $H^1(\Sigma; \mathbb{Z}) \neq 0$ it is enough to show that ξ is not null-homotopic. Assume that $\xi \simeq 0$. Then ξ can be lifted to an application $\tilde{\xi} : \Sigma \to \mathbb{R}$. Fix $\chi : \mathbb{R} \to \mathbb{R}$ a smooth function supported on $[-1, 1]$ and such that $\chi(0) = 1$. For A, s_0 in $\mathbb{R}$ consider the function

$$\chi_{s_0,A} : \mathbb{R} \xrightarrow{A\chi(s-s_0)} \mathbb{R} \to S^1,$$

where the second map in the composition is $t \to e^{it}$.

The graph of this function defines a deformed spine

$$\Delta(s_0, A) = \operatorname{graph}(\chi_{s_0,A})$$

on $\mathbb{R} \times S^1$ with the property that, if $A \neq 2k\pi$, it avoids the point $(s_0, 0)$. For each bubbled curve $u \in \Sigma$, we consider the deformed line on the tube given by $\Delta_u = \Delta(t_u, \tilde{\xi}(u) + \pi)$. This line avoids the point (t_u, a_u) and thus avoids the "bubble." We obtain in this way a continuous spine map: $\sigma' : \Sigma \to \Omega M'$ defined by

$$\sigma'(u) = u(\Delta_u).$$

To conclude our proof, it is enough to show that this spine map extends continuously to $\overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$ without modifying the standard spine map on $\partial\mathcal{M}$. Due to by-now standard gluing results [**8**, **10**], for each point $x \in \Sigma$ there exists a small neighbourhood $U(x) \subset \Sigma$ and an embedding $\phi : \mathbb{C} \times U(x) \to \overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$ such that $\phi(\{0\} \times U(x)) = U(x)$. As Σ is compact, we can cover it with a finite number of such neighbourhoods which we denote by $U_i,\ 1 \leq i \leq k$ with corresponding homeomorphisms ϕ_i. Denote $V_i = \phi_i(U_i)$ and let $p_i : V_i \to U_i$ be the obvious projection. For a point $y \in V(x)$, let $d_i(y) = d(y, p_i(y))$ where $d(-,-)$ is (some) distance in $\overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$. By possibly using smaller neighbourhoods V_i, we may assume that $d_i(y) < 1,\ \forall i, y$. Finally, let $h_i : U_i \to [0, 1],\ 1 \leq i \leq k$, be a partition of the unity. We put $U(\Sigma) = \cup V_i$. We also consider a smooth function $\eta : [0, 1] \to \mathbb{R}$ which is decreasing, supported on $[0, 1/2]$ and so that $\eta(0) = 1$. Let $d'_i : V_i \to \mathbb{R}$ be given by $d'_i(x) = \eta(d_i(x))$.

With these notations we now extend σ' to $U(\Sigma)$: we let

$$\Delta_u = \Delta\left(\sum_i h_i(p_i(u))t_{p_i(u)}, \sum_i h_i(p_i(u))d_i'(u)(\bar{\xi}(p_i(u))+\pi)\right),$$

and put $\sigma'(u) = u(\Delta_u)$. As this map coincides with σ on $\partial U(\Sigma)$ we may extend σ' to a continuous map on all of $\overline{\mathcal{M}}(\tilde{x}, \tilde{x}^t)$ so that it equals σ outside $U(\Sigma)$. This concludes the proof.

3.4. Quantum perturbation of the Serre spectral sequence. The purpose of this subsection is to show point (ii) in Theorem 2.2 and thus conclude the proof of this theorem.

The page E^2 is well defined and invariant, and by Lemma 3.2

$$E^2_{p,q} \cong HF_p(M;\Lambda) \otimes H_q(\Omega(M)).$$

This is also the first page of the (classical) Serre path-loop spectral sequence. However, the second differential, d^2, on this page is, in general, different from the classical one. To interpret d^2 in terms of binary Gromov–Witten invariants, we will use the Piunikin–Salamon–Schwarz construction from [**9**]. To this end, we start with a quantized-Morse version of the spectral sequence constructed before.

3.4.1. The quantized-Morse truncated spectral sequence. To a Morse–Smale pair (f,g) on M, together with a generic almost complex structure J we associate an extended quantized Morse complex $\mathrm{CM}_* = \mathrm{CM}_*(f, M, J)$. This is the free module generated by the critical points Crit (f) over the ring $\mathcal{R}$ together with a differential which will be described below. The degree of a critical point $x \in \mathrm{Crit}(f)$ is given by its index.

Given the almost complex structure J on M, a "quantum-Morse" trajectory from a critical point x to a critical point y in class $\alpha \in \Gamma_0$, is a finite collection $((\gamma_0, \dots, \gamma_k), (S_1, \dots, S_k))$ of paths and spheres in M such that:

(1) each sphere S_i is a J-holomorphic sphere with a marked real line $[p_{i,0}, p_{i,\infty}]$ on it, and $\sum_i [S_i] = \alpha$,
(2) $\forall i$, γ_i is a piece of flow line of $-\nabla_g f$, joining $S_{i-1}(p_{i-1,\infty})$ to $S_i(p_{i,0})$ (with the convention that $S_{-1}(p_{-1,\infty}) = x$ and $S_{k+1}(p_{k+1,0}) = y$).

We denote by $\mathcal{M}_\alpha(x,y)$ the set of all such objects. For a generic choice of (f,g,J), it is a smooth manifold of dimension

$$\dim \mathcal{M}_\alpha(x,y) = |x| - |y| + 2c_1(\alpha) - 1.$$

There is no difficulty to prove this as regularity comes down to the usual transversality of the appropriate evaluation maps [**8**] (in particular, this is much simpler than the relative case discussed, for example, in [**3**]). To verify the dimension formula notice that there are not only two marked points on the spheres, but also a real line joining them.

Such a trajectory defines a path from x to y by concatenation of the flow lines and the marked real lines on the spheres. Notice that each flow line segment can be parameterized by the value of $-f$, while on a holomorphic sphere $u:\mathbb{C}\cup\{\infty\}\to\mathbb{CP}^1\to M$, the map $t\in[0,+\infty)\mapsto\int_{|z|\le t}u^*\omega$ is strictly increasing and defines a parametrization of the marked real line.

These independent parametrizations of the different segments can now be shifted and aligned to produce a parametrization of the full respective path by the segment $[-f(x),-f(y)+\omega(\alpha)]$. We also assume that the path γ used to define M' and associate Moore loops to trajectories goes through all the critical points of f. As a consequence, we obtain a continuous map

$$\sigma:\mathcal{M}(x,y)\to\Omega(M').$$

The space $\mathcal{M}_\alpha(x,y)$ of course has a natural Morse–Gromov compactification $\overline{\mathcal{M}}_\alpha(x,y)$, and the question arises again of extending σ over it. Clearly, σ extends continuously over broken trajectories as long as no bubble components appear (as in § 3.1.2). However, it might fail to extend over trajectories where new spheres bubble off. The arguments used in the discussion of this point for Floer moduli spaces still apply in this situation. Thus, the map σ can be defined with the desired continuity and compatibility conditions whenever, as in Lemma 3.1,

$$|x|-|y|\le 2c_{\min}-1.$$

Choosing the chain representatives $m_\alpha(x,y)$ of $\sigma(\overline{\mathcal{M}}_\alpha(x,y))$, we define a truncated boundary operator on CM_* in the usual way:

$$dx=\sum_{1\le|x|-|y|+2c_1(\alpha)-1\le 2c_{\min}-1} m_\alpha(x,y)\ ye^{\alpha}.$$

The complex (CM_*,d) admits also a differential filtration defined by the degree of the elements in $\mathbb{Z}/2<\mathrm{Crit}(f)>\otimes\Lambda$. This induces a truncated spectral sequence in the same way as before. The second differential, d^2, of this spectral sequence has a natural interpretation in terms of Gromov–Witten invariants.

To see this first notice that if an element $u\in\overline{\mathcal{M}}_\alpha(x,y)$ with $\alpha\neq 0$ is so that it contains k spheres, then the dimension of $\overline{\mathcal{M}}_\alpha(x,y)$ is at least equal to k. Indeed, the choice of the real line on each of the spheres in u gives rise to a full S^1 parametric family of elements in this moduli space. The first consequence of this remark is that the differential d^1 in the spectral sequence is simply $d_{\mathrm{Morse}}\otimes\mathrm{id}$ which is defined on $\mathbb{Z}/2<\mathrm{Crit}(f)>\otimes H_*(\Omega M)\otimes\Lambda$ where d_{Morse} is the usual Morse differential. Indeed, d^1 involves 0-dimensional quantized-Morse moduli spaces and the remark above shows that when $\alpha\neq 0$ these spaces are never 0-dimensional if non-void.

Suppose now, to shorten the discussion, that f is a perfect Morse function (if not, critical points should be replaced by a basis of the Morse homology of M).

In this case, for a critical point x of f the second differential d^2x is defined and is given by $d^2x = \sum_{\alpha,y}[m_\alpha(x,y)]y$, where the sum is taken over all (x,y,α) such that $\overline{\mathcal{M}}_\alpha(x,y)$ is 1-dimensional. Notice that we may associate a homology class in $H_*(\Omega M)$ to each such moduli space in this case because the Morse differential vanishes. We let $[m_\alpha(x,y)]$ be this class.

In the expression for d^2 the sum of the terms where α is trivial, d_0^2, is given by 1-parametric families of Morse trajectories and, as shown in [**1**], this coincides with the second differential in the Serre spectral sequence of the fibration $\Omega M \to PM \to M$. On the other hand, when α is non-trivial, as a second consequence of the remark above, we see that the corresponding moduli space $\overline{\mathcal{M}}_\alpha(x,y)$ is the set of *single* holomorphic spheres in class α with a marked real line $[p_0, p_\infty]$ such that $p_0 \in W^u(x)$ and $p_\infty \in W^s(y)$.

The choice of the real line defines an S^1-action on $\overline{\mathcal{M}}_\alpha(x,y)$, and the quotients are 0-dimensional:

$$S^1 \to \overline{\mathcal{M}}_\alpha(x,y) \to \overline{\mathcal{M}}_\alpha(x,y)/S^1.$$

Moreover, letting

$$\mathrm{GW}_\alpha(x,y) = \mathrm{GW}_\alpha([x],[y])$$

be the Gromov–Witten invariants of holomorphic spheres in class α with two marked points, associated to the homology class of x and the dual class of y, we have:

$$\mathrm{GW}_\alpha(x,y) = \sum \sharp(\overline{\mathcal{M}}_\alpha(x,y)/S^1).$$

In particular, $\mathrm{GW}_\alpha(x,y)$ is the number of components of $\overline{\mathcal{M}}_\alpha(x,y)$. Each component of the space $\overline{\mathcal{M}}_\alpha(x,y)$ defines a loop of loops in M', whose class in $H_1(\Omega M')$ is the image $[\alpha]$ of α under the map $\pi_2(M') \to H_1(\Omega M')$. This image is the same for all the components such that $[m_\alpha(x,y)] = GW_\alpha(x,y)[\alpha] \in H_1(\Omega M)$. Finally, we have

$$d^2x = d_0^2x + \sum_{0 \neq \alpha \in \pi_2(M)} \mathrm{GW}_\alpha(x,y)[\alpha]ye^\alpha.$$

3.4.2. Relating the Morse and Floer spectral sequences. To compare the (truncated) spectral sequences given by the extended Floer and quantized-Morse complexes, we use the technique introduced in [**9**] to compare Floer and Morse homologies. For a generic pair (H,J), we recall the construction of the truncated complex $C(H,J)$ from § 3.1.1.

With the notation in that subsection, consider a critical point x of f and a lift $\tilde{y}$ of a (contractible) periodic orbit y of X_H.

A *hybrid* trajectory from x to $\tilde{y}$ is a quantized-Morse trajectory — as defined in § 3.4.1 — starting at x but now ending with a disk bounded by y.

The definition for a hybrid trajectory is as in § 3.4.1, with the following modifications.

(i) The last sphere S_k is replaced with a disk u with one cylindrical end, so that in polar coordinates and away from 0:

$$\mathbb{C} \xrightarrow{u} M \quad \text{with} \quad \mathbb{C} = \{0\} \cup \{e^{s+it}, (s,t) \in \mathbb{R} \times S^1\}.$$

(ii) The map u satisfies a 'cut off' Floer equation. For a fixed cut-off function χ such that $\chi(s) = 1$ for $s \geq 1$ and $\chi(s) = 0$ for $s \leq 0$ we have

$$\partial_s u + J(u)(\partial_t u - \chi(s) X_H) = 0, \qquad \lim_{s \to +\infty} u(s,t) = y(t). \tag{3.5}$$

(iii) The negative gradient flow arc γ_k ends at $u(0)$.

(iv) The sum of the homotopy class of u with $\sum_{i=1}^{k-1} [S_i]$ defines the capping $\tilde{y}$ of y.

For a generic choice of the data, all the relevant sub-manifolds and evaluation maps can be made transversal, so that the moduli spaces $\mathcal{M}(x, \tilde{y})$ of hybrid trajectories are smooth manifolds of dimension

$$\dim \mathcal{M}(x, \tilde{y}) = |x| - |\tilde{y}|.$$

These moduli spaces admit a natural compactification — we refer to [**9**] for the proof. We only recall here that the key point for showing compactness is to derive a uniform bound

$$E(u) = \iint \left\| \frac{\partial u}{\partial s} \right\|^2 ds\, dt \leq \mathfrak{a}(\tilde{y}) + \|H\|_\infty.$$

for the energy from the "cut off" Floer equation.

To associate to a hybrid trajectory in $\mathcal{M}(x, \tilde{y})$ a path from x to $\tilde{y}(0)$, it is enough to choose a parametrization of the real line $u(\mathbb{R})$ on the terminal disk and for that we use the energy of the curve:

$$E(r) = \int_{(-\infty, r] \times S^1} \left\| \frac{\partial u}{\partial s} \right\|^2 ds\, dt. \tag{3.6}$$

This choice defines a continuous spine map $\sigma : \mathcal{M}(x, \tilde{y}) \to \Omega(M')$, that can again be extended to the natural compactification $\overline{\mathcal{M}}(x, \tilde{y})$ up to dimension $2c_{\min} - 1$. Choosing compatible chain representatives of these spaces, we obtain chains $m(x, \tilde{y}) \in C_*(\Omega(M'))$ such that:

$$\begin{aligned} \partial m(x, \tilde{y}) = & \sum_{\substack{0 \leq |x| - |ze^\alpha| - 1 \leq 2c_{\min} - 1 \\ 0 \leq |ze^\alpha| - |\tilde{y}| \leq 2c_{\min} - 1}} m(x, ze^\alpha) m(ze^\alpha, \tilde{y}) \\ & + \sum_{\substack{0 \leq |x| - |\tilde{z}| \leq 2c_{\min} - 1 \\ 0 \leq |\tilde{z}| - |\tilde{y}| - 1 \leq 2c_{\min} - 1}} m(x, \tilde{z}) m(\tilde{z}, \tilde{y}) \end{aligned} \tag{3.7}$$

(where $m(x, ze^{\alpha}) = m_{\alpha}(x, z)$). Consider now the truncated morphism ϕ given by

$$\phi(x) = \sum_{|x|-|\tilde{y}| \leq 2c_{\min}-1} m(x, \tilde{y})\tilde{y}.$$

As expected, the map $d\phi - \phi d$ fails to vanish in general, but one easily checks that $(d\phi - \phi d)(x)$ is supported on elements $\tilde{y}$ with $|\tilde{y}| \leq |x| - 2c_{\min}$. This means that ϕ induces a morphism Φ between the respective truncated spectral sequences of order $c_{\min}$.

Notice that the Φ^1 coincides with

$$\phi' \otimes id : C_{\mathrm{Morse}}(f, g) \otimes H_*(\Omega M) \to CF_*(H) \otimes H_*(\Omega M),$$

where ϕ' is the usual PSS morphism and $C_{\mathrm{Morse}}(f, g)$ is the Morse complex of (f, g). But, as ϕ' induces an isomorphism in homology, this implies that Φ^2 is an isomorphism which, in particular, proves point (ii) of Theorem 2.2 and concludes the proof of this theorem.

4. Examples, applications and further comments

4.1. Extensions. We recall that the setting considered till now in the paper was that of a closed, simply-connected, monotone manifold for which $c_{\min} \geq 2$. All the constructions described previously in the paper extend much beyond this setting. We will only discuss here a few such possibilities different from those already mentioned in the introduction (use of integer or rational coefficients and the non monotone case).

4.1.1. $\boldsymbol{\pi_1 \neq 0}$. There are two essential ways to perform our constructions in the presence of a non-trivial fundamental group. They both stem from the fact that the only place where the fundamental group of M affects the construction is in the possible dependence of the resulting homology on the path γ which is used to define the quotient

$$M \to M'$$

as described in § 3.1.1. Of course, at the level of the spectral sequences $\pi_1(M) \neq 0$ also plays a role as local coefficients might be necessary.

(A) The first way to deal with the fundamental group consists in enlarging the Novikov ring by tensoring with the group ring $\mathbb{Z}/2\ [\pi_1(M)]$. Geometrically, this can be viewed as performing all the topological constructions on the universal covering, $\widetilde{M}$, of M even though all equations satisfied by the elements in our new moduli spaces take place after projection into M. The covering $\tilde{\mathcal{P}}_H$ is replaced by the covering $\tilde{\mathcal{P}}'_H$ which is the pull-back of $\widetilde{M} \to M$ over $\tilde{\mathcal{P}}_H \to \mathcal{P}_H \to M$. In this case, our truncated complex is isomorphic to:

$$\mathbb{Z}/2 < \mathcal{P}_H > \otimes \Lambda \otimes \mathbb{Z}/2[\pi_1(M)] \otimes C_*(\Omega M).$$

(B) A second possibility is *localization* or change of coefficients. This is maybe even more useful in applications than A and consists in replacing in all the construction the coefficient ring $C_*(\Omega M)$ by $C_*(\Omega X)$, where X is some simply-connected topological space which is endowed with a map:

$$\eta : M \to X.$$

All our moduli spaces are represented inside $\Omega(M')$ and, by composition with the map $\Omega\eta : \Omega(M') \to \Omega(X)$, they are also represented inside $\Omega(X)$. The results in Theorem 2.2 remain true after this change of coefficients except that $H_*(\Omega M)$ is replaced by $H_*(\Omega X)$ and the path loop fibration over M is replaced with the fibration of base M which is obtained by pull-back over the map η from the path-loop fibration over X, $\Omega X \to PX \to X$.

4.1.2. $c_{\min} = 1$. It is easy to see that even if $c_{\min} = 1$ the E^2 term of our spectral sequence is well defined together with the map d^2 (which might not be a differential though) and Theorem 2.2 remains true for the E^2 term. This happens because, to prove the invariance of d^2, only moduli spaces of dimension 2 are needed. In turn, as bubbling is a codimension two phenomenon this means that the bubbling points can be avoided when defining the spine map over these moduli spaces (as also discussed in Remark 3.3).

4.1.3. Non-compactness. Finally, it is obviously possible to extend this theory to the case when M is not compact if it is convex at infinity. In that case, the Hamiltonians used should have the form $H(r, x) = h(e^r)$ near infinity, where r is the $\mathbb{R}$-coordinate in the symplectization of the boundary, and h is a function with $\lim_{r\to\infty} h'(e^r) = \infty$.

4.2. Examples.

4.2.1. $\mathbb{CP}^1$. Take now $M = \mathbb{CP}^1$, and consider the Morse function having only one maximum $a = \infty$ and one minimum $b = 0$ as critical points. Let us choose a simple path from b to a to serve as the base point of M'. As auxiliary data, we can simply stick to the standard metric and complex structure on $\mathbb{CP}^1$: one easily checks that genericity is fulfilled for all the moduli spaces involved in the computations below (namely, only maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ of degree 0 and 1 are used; it is enough to observe that the kernel of the linearization of $\bar\partial$ at a constant map is 1-dimensional, and 3-dimensional at the identity, so that in both cases the cokernel vanishes).

The page 2 of the spectral sequence is simply $H_*(\mathbb{CP}^1) \otimes H_*(\Omega S^2)$. Let α denote the identity map $S^2 \to \mathbb{CP}^1$. Seen as the S^1 family of flow lines going from a down to b, α defines a cycle $[\alpha]$ that generates $H_1(\Omega(S^2))$.

The Novikov ring is generated by the multiples of α:

$$\Lambda = \left\{ \sum_{\lambda_k \in \mathbb{Z}_2} \lambda_k e^{k\alpha}, \right\}.$$

and we have $c_1(\alpha) = 2$.

To make the differential more explicit, we will "unfold" the spectral sequence by removing the Novikov ring from the coefficients, and thinking of $\{ae^{k\alpha}\}$ or $\{be^{k\alpha}\}$ as free families.

To compute the differential d^2, we have to compute all the 1-dimensional moduli spaces. Because of the invariance of the moduli spaces under the action of $\pi_2(S^2)$ on both ends of the trajectories, we can restrict to spaces of the form $\overline{\mathcal{M}}(x, ye^{k\alpha})$ with $x, y \in \{a, b\}$. The dimension of this space is

$$\dim \overline{\mathcal{M}}(x, ye^{k\alpha}) = |x| - |y| + 4k - 1,$$

so there are only two possibilities:

- $k = 0$, $x = a$ and $y = b$,
- $k = 1$, $x = b$ and $y = a$.

The first moduli space consists in classical flow lines only: it contributes to the classical part d^2_0 of d^2, and we have:

$$d^2_0(a) = [\alpha]b, \quad d^2_0(b) = 0,$$

so that the page 2 of the "classical" spectral sequence (tensored by the Novikov ring) has the following form.

$$\begin{tikzcd}
 & \vdots & & H_*(\Omega S^2) \arrow[d, dash] & & \vdots & & \vdots & & \vdots & \\
 & \mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[d, dash] & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \\
 {} \arrow[r, dash] & \mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[d, dash] & & \mathbb{Z}_2 \arrow[llu] & & \mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[llu] & \\
 {} \arrow[r, dash] & \mathbb{Z}_2 \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[llu] \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[llu] \arrow[r, dash] & H_*(\mathbb{CP}^1) \otimes \Lambda \\
 & {\scriptstyle ae^{\alpha}} & & {\scriptstyle b} & & {\scriptstyle a} & & {\scriptstyle be^{-\alpha}} & & {\scriptstyle ae^{-\alpha}} &
\end{tikzcd}$$

The second moduli space, $\overline{\mathcal{M}}(b, ae^{\alpha})$, involves holomorphic spheres of degree 1, and determines the quantum component d^2_Q of d^2. Since there are no flow lines going out of b or into a, it consists in holomorphic spheres of degree 1 with a marked real line from b to a. This is the same cycle as α, but with reversed orientation. as a consequence, we have

$$d^2_Q(a) = 0 \quad \text{and} \quad d^2_Q(b) = ae^{\alpha},$$

and the page 2 of the full spectral sequence has the following form.

$$\begin{tikzcd}
 & \vdots & & H_*(\Omega S^2) \arrow[d, dash] & & \vdots & & \vdots & & \vdots & \\
 & \mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[d, dash] & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \\
{} \arrow[r, dash] & \mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[ull] \arrow[d, dash] & & \mathbb{Z}_2 \arrow[ull] & & \mathbb{Z}_2 \arrow[ull] & & \mathbb{Z}_2 \arrow[ull] & \\
{} \arrow[r, dash] & \mathbb{Z}_2 \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & {} \\
 & ae^{\alpha} & & b & & a & & be^{-\alpha} & & ae^{-\alpha} & H_*(\mathbb{CP}^1) \otimes \Lambda
\end{tikzcd}$$

Notice that $(d^2)^2 a = [\alpha^2] a e^{\alpha} \neq 0$. So some bubbling has to occur in a 3-dimensional moduli space. Indeed, the moduli space $\overline{\mathcal{M}}(a, ae^{\alpha})$ is 3-dimensional and consists of flow lines going out of a down to some point p, and a holomorphic sphere of degree 1 with a marked real line from p to a. When the point p goes to b, the flow line becomes broken, and we see that this space is the one relevant in the computation of $(d^2)^2(a)$. On the other hand, when the point p goes to a, we are left with the constant trajectory from a to itself, with an (unparametrized) holomorphic sphere attached to it. Here, the critical point a is seen as a constant tube with a marked real line: this marked line is responsible for the bubbling monodromy.

It is interesting to note that this bubbling is in fact equivalent to the fact that in the Pontryagin algebra $H_*(\Omega S^2; \mathbb{Z}/2)$ the non-vanishing class in $H_1(\Omega S^2; \mathbb{Z}/2)$ has a non-vanishing square.

4.2.2. $\mathbb{CP}^n$ for $n > 1$. A similar computation can be used for $\mathbb{CP}^n$ when $n > 1$. Notice that the minimal first Chern class is $n+1 \geq 3$ so that the spectral sequence still exists after the second page. It is an easy verification to see that the quantum component of the differential d^2 is given by

$$d^2_Q[pt] = [\Delta] \otimes [\mathbb{CP}^n] e^{\Delta},$$

where Δ is a complex line in $\mathbb{CP}^n$.

$$\begin{tikzcd}
 & H_*(\Omega \mathbb{CP}^n) \arrow[d, dash] & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \mathbb{Z}_2 \arrow[d, dash] & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \cdots & \mathbb{Z}_2 & & \mathbb{Z}_2 & \\
 & \mathbb{Z}_2 \arrow[d, dash] & & \mathbb{Z}_2 \arrow[ull] & & \mathbb{Z}_2 \arrow[ull] & \cdots & \mathbb{Z}_2 & & \mathbb{Z}_2 \arrow[ull] & \\
{} \arrow[r, dash] & \mathbb{Z}_2 \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & \cdots \arrow[r, dash] & \mathbb{Z}_2 \arrow[r, dash] & 0 \arrow[r, dash] & \mathbb{Z}_2 \arrow[ull] \arrow[r, dash] & {} \\
 & [pt] & & & & & & [\mathbb{CP}^n] & & [pt]e^{-\Delta} & H_*(\mathbb{CP}^n) \otimes \Lambda
\end{tikzcd}$$

In particular, the pages of the spectral sequence all vanish after the second page. The contrast between the situation $n = 1$ and $n > 1$ comes from the properties of the Pontryagin product in $H_1(\Omega\mathbb{CP}^n)$. This product appears in the computation of $d^2 \circ d^2$, in particular

$$d^2(d^2([pt])) = [\Delta] * [\Delta] \otimes a_{n-1}e^{\Delta},$$

where a_{n-1} is a generator of $H_{2(n-1)}(\mathbb{CP}^n)$. What is truly remarkable here is that as $c_{\min} = n+1$, when $n > 1$, our construction of the truncated spectral sequence shows that $d^2 \circ d^2 = 0$ which implies $[\Delta] * [\Delta] = 0$ in the Pontryagin ring. Of course, this relation is well known by purely topological methods but it is remarkable that it is a consequence of the existence of the quantized Serre spectral sequence. Moreover, by Theorem 2.2 (ii), d^2 can be expressed in terms of Gromov–Witten invariants together with relative Ganea–Hopf ones (see also Remark 2.3) and our discussion shows that the relations among them in the Pontryagin algebra are not trivial.

4.3. Fibrations over S^2. Given a loop ϕ in $\mathrm{Ham}(M)$, one can construct a fibration E_ϕ over S^2, obtained by gluing two trivial fibrations over the disk via ϕ.

Seidel [**11**] used sections of this fibration to associate an invertible endomorphism on $H_*(M)$ to each such ϕ, deriving strong topological restrictions on elements in $\pi_1(\mathrm{Ham}(M))$. We first give an outline of the construction of this morphism in the context of Morse homology (see also [**7**]) and then explain how it is related to our construction.

Let Ω be a symplectic form on E_ϕ such that its restriction to the fibres is (cohomologous to) ω, and let J_ϕ be an almost complex structure Ω-compatible on E_ϕ. We will identify S^2 with $\mathbb{CP}^1$, and use an almost complex structure for which $d\pi \cdot J_\phi = i \cdot d\pi$ (see [**11**] for a discussion on these choices).

Let $f : M \to \mathbb{R}$ be a Morse function on M, and let $\tilde{f}$ be a Morse function on E_ϕ such that

- $\tilde{f}(z, m) = f(m) + |z^2| + cst$ over a local chart of S^2 near 0;
- $\tilde{f}(\tau, m) = f(m) - |\tau|^2$ over a local chart of S^2 near ∞;
- $\tilde{f}$ has no other critical point than those in the fibres over 0 and ∞.

If x is a critical point of f, we denote by x_+ and x_- the corresponding critical points above ∞ and 0, respectively. We have $i(x_+) = i(x) + 2$ and $i(x_-) = i(x)$.

Roughly speaking, the Seidel morphism is obtained by considering 0-dimensional moduli spaces of flow lines going out of a critical point x_-, hitting a J_ϕ holomorphic section of E_ϕ, followed by a second flow line flowing from the section down to a critical point y_+.

To be able to compare homology classes of sections with homology classes in M we fix a section s_0 of the fibration E_ϕ: the homology classes having degree 1 over the base are then the classes of the form $s_0 + i_*\alpha$, for $\alpha \in H_2(M)$ where i is the inclusion $M = M \times \{0\} \hookrightarrow E_\phi$. The Seidel morphism

Φ will in fact depend on this choice of s_0, or more precisely on the class of s_0 in $\Gamma_0 = \Gamma / \ker \omega$.

Let $\mathcal{M}(x_-, y_+; s_0 + i_*\alpha)$ be the moduli space of J_ϕ-holomorphic sections of E_ϕ in the class $s_0 + i_*\alpha$ intersecting, as described before, the unstable manifold of x_- and the stable manifold of y_+.

For a generic choice of f and J_ϕ, these spaces are manifolds of dimension $i(x) - i(y) + 2c_1(s_0 + i_*\alpha) - 2$, have a natural compactification and are compact when they are 0-dimensional. In this case, let $GW(x_-, y_+; s_0 + i_*\alpha) = \sharp\mathcal{M}(x_-, y_+; s_0 + i_*\alpha)$.

The relation

$$\Phi(x) = \sum_{i(x)-i(y)+2c_1(s_0+i_*\alpha)-2=0} GW(x_-, y_+; s_0 + i_*\alpha) y e^{\alpha}$$

defines a map from the Morse complex of M with Novikov coefficients to itself. It is compatible with the differential, and the Seidel morphism is the map Φ_* induced by Φ at the homology level.

We now discuss how to interpret this morphism as a component of the differential d^2 of the truncated spectral sequence associated to E_ϕ.

We will use the quantized-Morse version of the spectral sequence. With the notation in § 3.4.1, we write the differential of the quantized-Morse complex $\mathrm{CM}(\tilde{f}, E_\phi)$ as $dx = \sum_k d_k(x)$ where

$$d_k = \sum_{\lambda,\ \deg(\lambda)=k} m_\lambda(x, y) y e^{\lambda},$$

with the degree considered over the base.

This decomposition induces an analogous one for the differentials of the associated truncated spectral sequence which we will denote by

$$d^r = \sum d^{r;k}$$

with $d^{r;k}$ induced by d_k.

For $k = 0$, all the moduli spaces involved in $d^{2;0}x_-$ lie in the same fibre as x_-: they are all images of the corresponding moduli spaces in M via the inclusion i of M in E_ϕ as the fibre over 0. At the homology level, we have the following commutative diagram.

$$\begin{tikzcd}
H_*(M) \arrow[r, "d^2"] \arrow[d, "i_*"] & H_*(M) \otimes H_1(\Omega(M)) \arrow[d, "i_*"] \\
H_*(E_\phi) \arrow[r, "d^{2;0}"] & H_*(E_\phi) \otimes H_1(\Omega(E_\phi))
\end{tikzcd}$$

Consider now the case $k = 1$. For dimensional reasons, 1-dimensional moduli spaces of degree 1 quantum trajectories starting at a point x_- have to end in a point of type y_+.

By Theorem 2.2, the differential $d^{2;1}$ applied to the critical point x_- has the following form:

$$d^{2;1}x_- = \sum_{\alpha\in H_2(M)} \mathrm{GW}(x_-, y_+\,; s_0 + i_*\alpha)\ [s_0 + i_*\alpha]\ y_+ e^{s_0 + i_*\alpha}.$$

Using $\pi : E_\phi \to S^2$ to change coefficients and replace $\Omega(E_\phi)$ by $\Omega(S^2)$, and observing that the classes $[s_0 + i_*\alpha]$ are all sent to the generator τ of $H_1(\Omega(S^2))$, we get the following commutative diagram.

$$\begin{tikzcd}
H_*(M) \arrow[r, "\Phi"] \arrow[d, "i_*"] & H_*(M) \arrow[r, hook, "\mathrm{Id}\otimes[\tau]"] & H_*(M)\otimes H_1(\Omega(S^2)) \\
H_*(E_\phi) \arrow[r, "d^{2;1}"] & H_*(E_\phi)\otimes H_1(\Omega(E_\phi)) \arrow[ur, "\cap[M]\otimes\pi"'] &
\end{tikzcd}$$

This relates the Seidel morphism Φ and the $d^{2;1}$ component of the differential of the spectral sequence. From this point of view, when they exist, the higher-dimensional components $d^{r;1}$ can be viewed as higher-dimensional analogues of the Seidel morphism.

More precisely, recall that the choice of a preferred section s_0 induces a map $\Omega M \times \Omega S^2 \xrightarrow{i\times s_0} \Omega E_\phi \times \Omega E_\phi \to \Omega E_\phi$ that is a homotopy equivalence. Using the projection on the first factor we derive a map $p_{s_0} : \Omega E_\phi \to \Omega M$ well defined up to homotopy.

We define a higher-dimensional Seidel morphism by the formula

$$\Phi(x) = \sum_{y,\alpha} p_{s_0}(m(x_-, y_+\,; s_0 + i_*\alpha))\ ye^{\alpha}$$

where the sum runs over the critical points y and the classes α such that $1 \le |x| - |y| + 2c_1(s_0 + i_*\alpha) - 1 \le 2c_{\min} - 1$. It induces maps from the rth page $E^{(r)}(M)$ of the spectral sequence associated to M to itself, like the Seidel morphism does at the homology level. This morphism should have the same functorial properties as the classical Seidel morphism, but its study goes beyond the scope of this paper.

4.4. Non trivial periodic orbits for Morse functions. The construction of the truncated spectral sequence can be used to exhibit extra periodic orbits for Morse functions in some particular situations.

4.4.1. Proof of Corollary 2.4. Let (M, ω) be a monotone symplectic manifold, and consider a perfect Morse function f on M which is self indexed. We may assume that f is as small as we want in C^2 norm as the existence of characteristics is not changed by rescaling. Thus, we now assume that f is small enough so that the only 1-periodic orbits are the critical points of f and their Conley–Zehnder index coincides with the Morse index. From the statement of the Corollary recall that there are two Morse homology classes

x, z, $|z| > |x|$, which are d^r-related. Due to the self-indexing condition each of these classes is represented by a linear combination of critical points with the same critical value, $x = \sum_i x_i$, $z = \sum_i z_i$. We let $f(x) = f(x_i)$. Recall also that we assume that $H_k(M) \otimes \Lambda_q = 0$ for $|ze^\lambda| < k + q < |x|$. We now assume that some interval $[a, b] \subset [f(x), f(z)]$ does not contain any closed characteristic.

For $A \in \mathbb{R}$ large enough, consider a smooth increasing function $\phi_A : \mathbb{R} \to \mathbb{R}$, such that:

- $\phi_A(t) = t$ for $t \leq a + 1/A$,
- $\phi_A(t) = t + A$ for $t \geq b - 1/A$

and $1/A \leq (b - a)/3$.

For any A, the function $f_A = \phi_A \circ f$ has the same critical points as f with the same (unparametrized) flow lines, but the critical levels above b are shifted upward. The critical points of the same index continue to share the same level hypersurface. Of course, the existence of non-trivial characteristics for f and f_A is equivalent.

Given that $[a, b]$ does not contain any closed characteristic, it follows that Floer theory may be applied to the Hamiltonian f_A, the 1-periodic orbits of f_A coinciding with the critical points of f and the Conley–Zehnder index of these critical points still agrees with their Morse index. Moreover, the (not extended) Floer and Morse complexes are then the same (indeed, as the homology of the Floer complex has to be isomorphic with Morse homology it follows in this case that the Floer differential is also trivial). Thus, x and z also give Floer homology classes. We now choose the constant A such that $A \geq \rho(2n + r)/2$ where ρ is the monotonicity constant ($\omega(\alpha) = \rho c_1(\alpha)$).

Consider the truncated spectral sequence associated to the Hamiltonian X_{f_A}. By hypothesis, we know that x is d^r-related to ze^λ. This implies that there is a critical point z_j so that there are Floer trajectories from one of the x_is to $z_j e^\lambda$. Indeed, as $|x| > k + q > |ze^\lambda|$ implies that $H_k(M) \otimes \Lambda_q = 0$ the differential d^r is the first one relating the vertical line through $|x|$ to the one through $|ze^\lambda|$. In other words, letting $p = |ze^\lambda|$, we have $E^r_{p,*}$ is a subgroup of $E^2_{p,*}$ and so, if there are no flow lines relating some x_i to a $z_j e^\lambda$, then x and ze^λ can not be d^r-related.

In view of this we have

$$|x| - r = |ze^\lambda| = |z| - 2c_1(\lambda),$$

which means that $c_1(\lambda) \leq n + r/2$ and also

$$f(x_i) = f_A(x_i) \geq f_A(z_j) - \omega(\lambda) = f(z_j) + A - \omega(\lambda).$$

But, given of our choice of constant A,

$$f(z_j) + A - \omega(\lambda) \geq f(z_j) + A - \rho(2n + r)/2 > f(x_i),$$

which leads to a contradiction and concludes the proof. □

Remark 4.1. (a) This corollary applies easily to the case of $\mathbb{C}P^n$ (see § 4.2.2). In this case x corresponds to the minimum so that $|x| = 0$ and z to the maximum of the Morse function. Moreover, as $c_{\min} = n+1$ we have $|ze^{\Delta}| = -2$ (where Δ is the class of the complex line) and $H_k(\mathbb{C}P^n) \otimes \Lambda_q$ vanishes for $k+q = -1$. As seen in § 4.2.2, x and ze^{Δ} are d^2-related so that the corollary applies with the conclusion that for any perfect, self-indexed, Morse function f on $\mathbb{C}P^n$ the set of values v so that $f^{-1}(v)$ contains a closed characteristic is dense in the interval $[\min(f), \max(f)]$. By inspecting the proof, it is easy to see that, in the case of $\mathbb{C}P^n$, this statement remains true even under a much weaker restriction: it is enough that f be a Morse function so that all its maxima have the same critical value and all its minima have the same critical value. Given that any connected hypersurface in $\mathbb{C}P^n$ can be viewed as a regular hypersurface of such a function and in view of the dynamical stability of Hamiltonian flows in tubular neighbourhoods of contact hypersurfaces, a consequence of this is that any contact hypersurface in $\mathbb{C}P^n$ contains a closed characteristic. While this result is already known by the work of Hofer and Viterbo [**6**] where it is obtained by a different technique, the proof provided here illustrates the power of our method.

(b) The condition that the Morse function f in Corollary 2.4 be perfect seems somewhat artificial. However, there is no immediate way to avoid it in general. The difficulty is that the condition that the homology classes x and ze^{λ} are d^r-related needs to be translated into the existence of a Floer orbit from a critical point x_i of index $|x|$ to a critical point z_j of index $|z|$. If the Morse function is not perfect, the Morse and Floer complexes do not have a vanishing differential. As a consequence, in the spectral sequence, the Floer homology class $[x]$ might not admit as representative a Floer cycle given by a linear combination of critical points of index equal to $|x|$ but rather a sum $\sum x_i e^{\lambda_{x_i}}$ in which some of the λ_{x_i} are non trivial (in essence, there is still a Morse cycle representing x but possibly this Morse cycle has a non-trivial Floer differential). Due to this, the existence of the wanted Floer trajectory can no longer be deduced. By pursuing further this argument, it is, however, possible to see that, if $c_1(\lambda) = c_{\min}$, then this "perfect Morse function" condition can be dropped.

The same technique applies in many other variants of the situation described above. The basic idea is to ensure the existence of a sequence of trajectories, "ending at a higher level than its starting point" (there was just one such trajectory in the case above) in such a way that the relevant intermediate points can be shifted out of the action window (as done before using ϕ_A). For this, besides identifying a chain of differentials which relate a succession of homology classes in the spectral sequence one also needs to be able to choose appropriate chains representing these classes (the self indexing condition and the homological "gap" condition had this purpose above). Here is such a variant valid when $(d^{c_{\min}})^2 \neq 0$.

Corollary 4.1. *If $(d^{c_{\min}})^2(\xi) \neq 0$, for some $\xi \in E^{c_{\min}}_{p,q}$, then for any $\epsilon > 0$, any self-indexed Morse function f on M has infinitely many closed characteristics contained in $f^{-1}([f(\xi) - \epsilon, f(\xi) + \epsilon])$.*

Again, as f is self-indexed the value $f(\xi)$ is well-defined (and equal to the value of f on any of the critical points of index p).

Proof. Fix a self-indexed Morse function $f : M \to \mathbb{R}$. As in the previous proof, we may assume that the critical points of f are nondegenerate periodic orbits of X_f and their Conley–Zehnder index agrees with the Morse index. Thus we may apply our construction of the truncated quantized Serre spectral sequence to the Hamiltonian f (together with a generic time-dependent almost complex structure). For a fixed $\epsilon > 0$, assume that there are no closed characteristics in the set $f^{-1}([f(\xi) - \epsilon, f(\xi) - \epsilon/2] \cup [f(\xi) + \epsilon/2, f(\xi) + \epsilon])$.

Following the same argument as in the previous proof we may also assume, after possibly composing f with an appropriate diffeomorphism $\mathbb{R} \to \mathbb{R}$ which coincides with the identity in the exterior of $[f(\xi) - \epsilon, f(\xi) - \epsilon/2] \cup [f(\xi) + \epsilon/2, f(\xi) + \epsilon]$, that:

* for each critical point $x \in \mathrm{Crit}_p(f)$ the interval $[f(x) - \rho n - \omega_{\min}, f(x) + \rho n + \omega_{\min}]$ does not contain any critical values different from $f(x) = f(\xi)$.

Moreover, all the 1-periodic orbits of X_f are the critical points of f and their Conley–Zehnder index agrees with the Morse index. Here ρ is as before the monotonicity constant so that $\omega_{\min} = \rho c_{\min}$. From the discussion in § 3.3, we see that $(d^{c_{\min}})^2(\xi) \neq 0$ implies that for some critical point $x \in \mathrm{Crit}_p(f)$ we have that the moduli space $\overline{\mathcal{M}}(x, x\#\alpha_{\min})$ is non-void and has a non-void codimension 1 stratum Σ_1 consisting of broken Floer trajectories as well as a non-void codimension 2 stratum Σ_2 consisting of Floer trajectories with some bubble attached.

Assume that among the broken trajectories in Σ_1 there is one which joins x to ye^{α} followed by a second trajectory from ye^{α} to $xe^{\alpha_{\min}}$.

We then have:

$$|x| - |y| + 2c_1(\alpha) - 1 \geq 0, \quad |y| - |x| + 2c_1(\alpha_{\min} - \alpha) - 1 \geq 0.$$

Notice that this implies that $|y| \neq |x|$. Indeed, if $|y| = |x|$, the first inequality implies that $c_1(\alpha) > 0$ and the second that $c_1(\alpha) < c_{\min}$ which is not possible.

There is also an inequality involving the actions:

$$f(x) \geq f(y) - \omega(\alpha) \geq f(x) - \omega_{\min}. \tag{4.1}$$

There are two cases to consider now. If $|y| > |x|$, then $c_1(\alpha) > 0$ so that $c_1(\alpha) \geq c_{\min}$ and we also need to have $2c_1(\alpha) \leq |y| - |x| - 1 + 2c_{\min} < 2n + 2c_{\min}$. By monotonicity, this means $f(y) - \omega(\alpha) > f(y) - \rho(n + c_{\min})$. At the same time, as f is self-indexed and $|y| > |x|$ we have $f(y) > f(x)$ and our assumption $*$ on the critical values of f gives that $f(y) \geq f(x) + \rho(n + c_{\min})$. In other words, $f(y) - \omega(\alpha) > f(y) - \rho(n + c_{\min}) \geq f(x)$ which contradicts the first inequality in (4.1). The second case is $|y| < |x|$. Then $c_{\min} \geq c_1(\alpha) > -n$. This means $f(y) + \rho n > f(y) - \omega(\alpha) \geq f(x) - \omega_{\min}$ which contradicts $*$ and concludes the proof. □

References

[1] J.-F. Barraud and O. Cornea, *Lagrangian intersections and the Serre spectral sequence*, Ann. of Math. **166** (2007), 657–722.

[2] ———, *Homotopic dynamics in symplectic topology*, 'Morse theoretical Methods in Non-Linear Analysis and Symplectic Topology', 109–148, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006.

[3] P. Biran and O. Cornea, *Quantum structures for Lagrangian submanifolds*, preprint, Fall 2007.

[4] O. Cornea, *Homotopical dynamics. IV. Hopf invariants and Hamiltonian flows*, Comm. Pure Appl. Math. **55** (8) (2002), 1033–1088.

[5] O. Cornea, *Homotopical Dynamics II: Hopf invariants, smoothings and the Morse complex*, Ann. Sci. Ec. Norm. Sup. **35** (2002), 549–573.

[6] H. Hofer and C. Viterbo, *The Weinstein conjecture in the presence of holomorphic spheres*. Comm. Pure Appl. Math. **45** (5) (1992), 583–622.

[7] F. Lalonde, D. McDuff and L. Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*, Invent. Math. **135** (2) (1999), 369–385.

[8] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, 52. American Mathematical Society, Providence, RI, 2004, xii+669 pp.

[9] S. Piunikhin, D. Salamon and M. Schwarz, *Symplectic Floer–Donaldson theory and quantum cohomology*, 'Contact and Symplectic Geometry (Cambridge, 1994)', 171–200, Publ. Newton Inst., 8, Cambridge University Press, Cambridge, 1996.

[10] D. Salamon, *Lectures on Floer homology*, 'Symplectic Geometry and Topology (Park City, UT, 1997)', 143–229, IAS/Park City Math. Ser., 7, Amer. Math. Soc., Providence, RI, 1999.

[11] P. Seidel, π_1 *of symplectic automorphism groups and invertibles in quantum homology rings*, Geom. Funct. Anal. **7** (6) (1997), 1046–1095.

Université Lille
Villeneuve D'Ascq
France
E-mail address: Jean-Francois.Barraud@agat.univ-lille1.fr

Département de Mathématiques et Stat
Université de Montréal
Montréal
Canada QC H3T 1J8
E-mail address: cornea@dms.umontreal.ca

Received 12/28/2006, accepted 11/30/2007

It is our great pleasure to submit this paper to the volume of JSG dedicated to Dusa McDuff's 60th birthday. Early in this project, we believed, *a posteriori* without justification, that the monodromy of bubbling is much less relevant in the sense that the representation of the Floer tubes inside the Moore loop space of the ambient manifold can be extended over the codimension 2 stratum containing the bubbled configurations. As discussed in the paper, this is not the case in general. It is one of Dusa's questions which made us reconsider the issue and appreciate the significance of this phenomenon: indeed, it is precisely due to this obstruction that our spectral sequence is not defined, in general, after $c_{\min}$ pages. We thank Ely Kerman for a useful comment which made us realize that the initial statement of Corollary 2.4 was weaker than what our proof implied. Thanks also to the referee for a careful reading of the paper.